Problem Set 1 Solutions
MATH 110: Linear Algebra

PART 1

The following problems are each worth 5 points..

1. Curtis **p. 14** 1: a) The base case $n = 1$ is trivial. Suppose the theorem is true for some $n-1$. Then $1+3+\ldots+(2n-1)+(2n+1) = n^2+2n+1 = (2n+1)^2$. b) Again, the base case is trivial. The inductive step involves a bit of algebra, namely $1^2 + \ldots + n^2 + (n+1)^2$ is equal, by induction, to $\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$.

2. Curtis **p. 15** 4: The closure axioms are satisfied because of the way the operations are defined (using a table). The "inverse element" axiom is satisfied because $1^{-1} = 1$. The operations are commutative because they are symmetric about the diagonal. The rest of the properties simply involve verification by examining the tables.

3. Curtis **p. 25** 3. First of all, notice that the process of checking for R_n and F_n involves only checking F_n, since R_n is a special case of F_n (where the field is R). The important thing is that all the axioms for a vector space behave the same on a tuple as on a singleton. The zero element in the vector space is $< 0, 0, \ldots, 0 >$. The function space is a bit harder to check. It is important to recognize that each function is an *element* in the vector space. For example, showing that $(f + g)(x) = (g + f)(x)$ involves understanding that $(f+g)$ and $(g+f)$ are each functions themselves. Now it is a fact that for every x, $(f+g)(x) = f(x)+g(x)$ (this is by the very definition of the operation "+"). $f(x), g(x)$ are both some real numbers (depending on x). Because of the field axioms for the reals, it follows that $f(x)+g(x) = g(x)+f(x)$. Once again, by the definition of "+" this is $(g + f)(x)$.

4. Curtis **p. 33** 3. The answers for this exercise are all at the back of the book. Notice that the answer for part (h) is that the solutions are a subspace only if g is the zero function. This is because if it is not, then the sum of two solutions, when plugged into the differential equation (LHS) equal $2g$, thus sums do not satisfy the solution.

5. Curtis **p. 37** I will assume in questions such as this one that you know your basic calculus (if you don't, review!). So you should know that if the derivative of a function exists and $df/dt = 0$ then the function is of the form $f(x) = c$ for some constant c. *In terms of vector spaces* this is the same as saying that f consists of the subspace generated by the element $f(x) = 1$,

where $f(x) = 1$ is the function that is 1 for every x. You could just as well take $f(x) = 2$, or any other constant. The important point is that this one function is a basis for the subspace which consists of all the solutions of the differential equation. Thus the dimension is 1. The general case involves solving the diffeq $d^{(n)} f/dt = 0$. The solutions, you should know, are the space P_{n-1} defined in problem 3 on the same page. Its dimension is n. This is because a basis for P_{n-1} is $1, x, x^2, \ldots, x^{n-1}$. Why? Certainly they span the space. They are independent because suppose

$$\sum_{k=0}^{n} c_k x^k = 0$$

for all real x. Substituting $x = 0$ we get that $c_0 = 0$. We can take the derivative and then repeat, setting $x = 0$ and getting $c_1 = 0$. Thus, we can deduce that all the c_i's are 0 and indeed the functions are independent.

PART 2

Remember that the starred problem was non collaborative.

Problem 1 (5)

Let $V = R^+$, the set of positive real numbers. Define the "sum" of two elements x, y in V to be their product xy (in the usual sense), and define "multiplication of an element x in V by a scalar c to be x^c. Prove that V is a real linear space with 1 as the zero element.

Proof: Addition in the space is commutative because multiplication is commutative in the reals. Same for associativity. The zero element is 1. "Adding" 1 to any element in the vector space leaves it unchanged because in the reals 1 is an identity for multiplication. There is also a "negative" for every x in the space. Take "$-x$" to be $\frac{1}{x}$. Once again, by the properties of multiplication in the reals we see that $x + (-x) = 0$. The "multiplication" is associative because $(x^a)^b = x^{ab} = x^{ba}$. $(a +_R b)x = x^{a +_R b} = x^a x^b = ax +_V bx$. The rest of the verifications work the same.

Problem 2 (10)

Let V be the vector space of all real valued functions defined on the real line ($F(R)$). Consider the n exponential functions:

$$u_1(x) = e^{a_1 x}, \ldots, u_n(x) = e^{a_n x}$$

where $a_1, a_2, \ldots, a_n$ are distinct real numbers. Show that these n functions are independent.

Proof: We need to show that if

$$\sum_{k=1}^{n} c_k e^{a_k x} = 0$$

for all x then the c_i's are all zero. Let a_M be the largest of the a_i's. Multiplying both sides of the above equation by $e^{-a_M x}$, we obtain

$$\sum_{k=1}^{n} c_k e^{(a_k - a_M)x} = 0.$$

If $k \neq M$, then the number $a_k - a_M$ is negative, which means that as $x \to \infty$, each term with $k \neq M$ goes to 0 and we find that $c_M = 0$. Deleting the Mth term, and applying the induction hypothesis we find that each of the remaining coefficients is zero. Note: This proof is known as a "direct" proof. Another method, would be "contradiction". This would entail assuming that there is a c_i that is nonzero and then explicitly finding an x that renders the sum nonzero.

Problem 3*(5)

Curtis **p. 26** 10.

In order to do this problem you will have to read some definitions and theorems in Curtis (3.7, 3.9, 3.10 and 3.11).

Proof: Consider a quadrilateral A, B, C, D. These are vectors in R_2. First note that the midpoints are exactly the vectors

$$\frac{1}{2}(A+B), \frac{1}{2}(B+D), \frac{1}{2}(C+D), \frac{1}{2}(A+C).$$

This follows from Theorem 3.10. Showing that they form a parallelogram is, by definition, the process of showing that $M_1M_2 = M_3M_4$ where the M_i's are the midpoints. This, again by definition, is equivalent to showing that the vector $M_2 - M_1$ is equal to the vector $M_4 - M_3$. This is clear using elementary algebra of vectors.

Problem 4 (15)

Let V be a finite dimensional vector space and let S be a subspace of V. Prove each of the following statements:

a) S is finite dimensional and $dim\ S \leq dim\ V$.
b) $dim\ S = dim\ V$ if and only if $S = V$.

c) Every basis for S is part of a basis for V.
d) A basis for V need not contain a basis for S.
e) Is the union of two subspaces always a subspace? Explain.

Proofs: a) There is a basis for B for V. If S is spanned by a part of it, we are done. Suppose not. Place all the elements in B that are in a S in a set B'. There must be a b' not a linear combination of elements in B that is in S, take it and add it to B'. Repeat the process. It must terminate before B' has $dimV$ elements by definition of a basis for V. b) Suppose $S = V$. Then clearly $dimV = dimS$. Now suppose $dimV = dimS$. Consider an element $x \in V$. If x is a linear combination of elements in the basis of S then $x \in S$. Suppose not, then x can be added to the basis of S, and x together with the previous elements are independent. This contradicts $dimS = dimV$. So x must be in S. Thus every element of V is in S as well, and they are equal. c) This follows from theorems in class. d) Consider V=R_2 and S consists of all vectors where the two components of the tuple are equal (check that S is a subspace). A basis for V is the standard coordinate basis ($< 1, 0 >$ and $< 0, 1 >$). Neither of these vectors even lie in S. e) No. Consider $V = R_2$ as before, with the subspaces S_1 defined as above (both coordinates equal) and S_2 which is the subspace where the second coordinate is the negative of the first. Their union is not a vector space (it fails the closure axioms, for example).

PART 3 - Optional Problems

1. Recall that the **cross product** of two vectors A and B in R_3 is defined by

$$A \times B = < a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 >$$

where $A = < a_1, a_2, a_3 >$ and $B = < b_1, b_2, b_3 >$.

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit coordinate vectors in R_3. Notice that the cross product is not associative. For example,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j} \quad \textit{but} \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0 \times \mathbf{j} = 0.$$

Thus, given k vectors $v_1, v_2, \ldots, v_k$ in R_3, in order to make the expression

$$v_1 \times v_2 \times \cdots \times v_k$$

well defined, it is necessary to insert parentheses to indicate the order of evaluation. We will define an **association** to be an insertion of $k-2$ parentheses so that the order of evaluation is determined. For example,

$$(v_1 \times v_2) \times (v_3 \times v_4) \; and \; ((v_1 \times v_2) \times v_3) \times v_4$$

are two different associations.

Prove that if two associations of $v_1 \times v_2 \times \cdots \times v_k$ are given, there exists an assignment of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ to $v_1, v_2, \ldots, v_k$ such that when evaluated with the assignment, the two associations are equal and nonzero (this means that you cannot set $v_1 = v_2 = \cdots = v_k$).

Comments: This question is actually equivalent to the famous four color theorem. This theorem asserts, that every map can be colored with four colors, so that any two bordering countries have different colors. Sounds easy? I don't think so...

It was unsolved for almost a hundred years before Aapel and Haken gave a proof in the late 70's using over 1400 hours of computer verifications. It remains a controversial theorem to this day, with no known proof that can be verified by humans (without computers). If you want to learn more, or see why the two theorems are equivalent (which is actually not that hard), come see me.

The vector question is due to Kauffman (1990).