Problem Set 11 (due November 29)
MATH 110: Linear Algebra

Each problem is worth 10 points.

PART 1

1. Curtis **p. 276** 2.
2. Curtis **p. 276** 4.
3. Curtis **p. 276** 9.
4. Curtis **p. 288** 4.

PART 2

Problem 1(20)

Show that for any $1 \times n$ row matrix X, and any $n \times n$ matrix A, we have that

$$XAX^t = XBX^t$$

where B is the symmetric matrix $B = \frac{1}{2}(A + A^t)$.

Problem 2(10)

Let $T : V \to V$ be a symmetric transformation on a real Euclidean space V with quadratic form $Q(x) = (T(x), x)$. Prove that if Q does not change sign sign on V (i.e. $Q(x) \geq 0\ \forall x \in V$ or $Q(x) \leq 0\ \forall x \in V$) then $Q(x) = 0$ implies that $T(x) = 0$. (Hint: consider the function $p(t) = Q(x + ty)$ and show that $p'(0) = 0$).

Problem 3 (10)

a) Prove that the set of symmetric matrices forms a subspace of dimension $\frac{n(n+1)}{2}$ in the vector space of real $n \times n$ matrices (which has dimension n^2).

b) Prove that the set of skew-symmetric matrices forms a subspace of dimension $\frac{n(n-1)}{2}$ in the vector space of real $n \times n$ matrices (which has dimension n^2).

Problem 4 (10)

Let T be a symmetric linear transformation on a real Euclidean space. Prove that T has a "square root", i.e. there is a symmetric linear transformation S such that

$$T = S^2.$$

Problem 5 (10)

Define the index of a real symmetric matrix A to be the number of strictly positive eigenvalues of A minus the number of strictly negative eigenvalues of A. Let $Q_A(x)$ and $Q_B(x)$ be the quadratic forms associated with A and B, and suppose that

$$Q_A(x) \leq Q_B(x)$$

for all vectors x. Prove that the index of A is less than or equal to the index of B.

Optional Problem

Let $T : V \to V$ be a symmetric linear transformation on a real Euclidean space V and let $Q(x) = (T(x), x)$. Assume that there is an extremum (maximum or minimum) at u for $Q(x)$ among all the values that Q takes on the unit sphere (i.e. $Q(u)$ is either a max or a min amongst all the values $Q(x)$ with $(x, x) = 1$). Show that u is an eigenvector for T.