On Ricci-pinched 3-manifolds

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Ricci-pinched

Statement of results

Steps in the proof

Details of the proof

Pinching in Riemannian geometry

Quarter pinching conjecture: Let (M, g) be a compact Riemannian manifold. Choose $c \in (\frac{1}{4}, 1]$. Suppose that the sectional curvatures lie between c and 1. Then (M, g) is diffeomorphic to a spherical space form.

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Theorem

(Brendle-Schoen) Let (M, g) be a compact Riemannian manifold with positive sectional curvature. Choose $c \in (\frac{1}{4}, 1]$. Suppose that for each $m \in M$ and any two 2-planes $\pi_1, \pi_2 \subset T_m M$, we have $K(\pi_1) \geq c\, K(\pi_2)$. Then M is diffeomorphic to a spherical space form.

Ricci pinching

Suppose that (M, g) is an n-dimensional Riemannian manifold with nonnegative Ricci curvature. At each $m \in M$, using the metric to turn the Ricci curvature into a self-adjoint operator on T_mM, one can diagonalize it to get eigenvalues

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Definition

Given $c > 0$, the metric is c-Ricci pinched if for all $m \in M$, we have

$$r_1 \geq c r_n.$$

The conjecture

Let (M, g) be a complete Riemannian 3-manifold with bounded sectional curvature.

Conjecture A: Suppose that (M, g) has nonnegative Ricci curvature, and is c-Ricci pinched for some $c > 0$. Then (M, g) is flat or M is compact.

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Compare with Myers' theorem: If $\mathrm{Ric} \geq (n-1)k^2 g$ then $\mathrm{diam}(M, g) \leq \frac{\pi}{k}$.

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Conjectures A and B are equivalent. First, A implies B. To see that B implies A, if (M, g) isn't flat then after running the Ricci flow, $\mathrm{Ric} > 0$.

Extensions

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One motivation for the conjectures:

Theorem

(Hamilton (1994)) Let M^n be a smooth strictly convex complete hypersurface bounding a region in $\mathbb{R}^{n+1}$. Suppose that its second fundamental form is c-pinched. Then M is compact.

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Remark : the case $K \geq 0$ was claimed by Chen-Zhu (Inv. Math. (2000)).

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Convergence part: This has nothing to do with Ricci flow. We look at a noncompact Riemannian 3-manifold with c-pinched positive Ricci curvature, cubic volume growth and $K(m) \geq -\frac{\text{const.}}{d(m,m_0)^2}$.

By a rescaling argument, we get a contradiction.

Ricci-pinched

Ricci flow I

Theorem

Let (M, g_0) be a complete noncompact Riemannian 3-manifold having bounded curvature and c-pinched positive Ricci curvature.

The ensuing Ricci flow solution $(M, g(\cdot))$ exists for all $t \geq 0$ and satisfies

$$\| \operatorname{Rm}(g(t)) \|_\infty \leq \frac{\text{const.}}{t}.$$

Ricci flow II

Theorem

Let $(M, g(\cdot))$ be a Ricci flow on a noncompact Riemannian 3-manifold that exists for all $t \geq 0$, with complete time slices.

Suppose that $\mathrm{Ric} > 0$ *and*

$$\| \mathrm{Rm}(g(t))\|_\infty \leq \frac{\mathrm{const.}}{t}.$$

Then $(M, g(\cdot))$ is noncollapsing for large time. That is,

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This result does not need c-pinching. Examples of such flows come from asymptotically conical expanding Ricci solitons, which exist in abundance (Deruelle).

Corollary

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Then (M, g) has cubic volume growth, i.e.

$$\liminf_{r \to \infty} r^{-3} \operatorname{vol}(B(m_0, r)) > 0.$$

Proof when $K \geq 0$

Suppose that (M, g) has nonnegative sectional curvature and is c-Ricci pinched. For $s \geq 1$, put $g_s(u) = s^{-1} g(su)$. Let $g_\infty(u) = \lim_{j \to \infty} g_{s_j}(u)$ be a pointed blowdown limit.

Proof when $K \geq 0$

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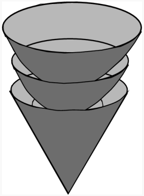
It will be a Ricci flow solution coming out of a cone, namely the tangent cone at infinity $T_\infty M = \lim_{j \to \infty} (M, m_0, s_j^{-\frac{1}{2}} d)$.



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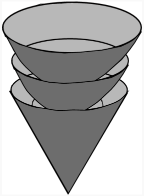

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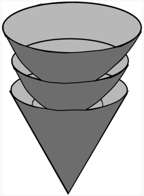

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Lemma

A three dimensional expanding gradient soliton that is c-Ricci pinched must be flat.

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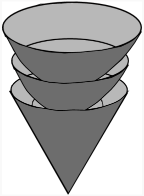

Now the tangent cone at infinity of $(M_\infty, g_\infty(u))$ is also equal to $T_\infty M$. The first is flat and the second is three dimensional, so the latter must be $\mathbb{R}^3$.

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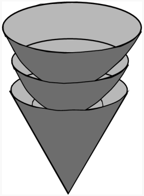

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Theorem

(Colding) If a complete Riemannian n-manifold (M, g) has $\mathrm{Ric} \geq 0$, and a tangent cone at infinity isometric to $\mathbb{R}^n$, then (M, g) is isometric to $\mathbb{R}^n$.

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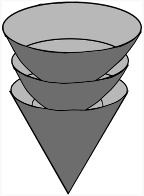

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Thus (M, g) is flat, which contradicts our assumption that $\mathrm{Ric} > 0$.

Quadratic decay of negative curvature

Theorem

There is no complete noncompact Riemannian 3-manifold (M, g) with c-pinched positive Ricci curvature and cubic volume growth, that satisfies

$$K(m) \geq -\frac{\text{const.}}{d(m, m_0)^2}.$$

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Noncollapsing for large time

Theorem

Let $(M, g(\cdot))$ be a Ricci flow on a noncompact Riemannian 3-manifold that exists for all $t \geq 0$, with complete time slices.

Suppose that $\mathrm{Ric} > 0$ *and*

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Idea: Put $g_s(u) = s^{-1} g(su)$. It's enough to show that there is a sequence $s_i \to \infty$ so that $\{(M, m_0, g_{s_i}(1))\}_{i=1}^{\infty}$ has a three dimensional pointed Gromov-Hausdorff limit. Suppose not.

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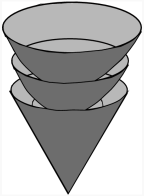
One gets larger and larger Seifert-fibered regions in (M, g_0). One shows that they can be fitted together to get a Seifert fibering of M. But by Schoen-Yau, M is diffeomorphic to $\mathbb{R}^3$. Contradiction.

Approaches to the general conjecture I



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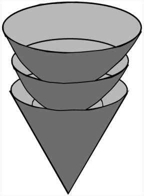

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If so, this would prove the general conjecture.

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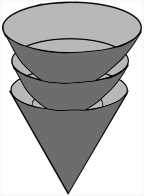

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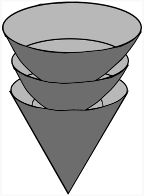

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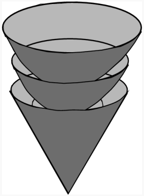

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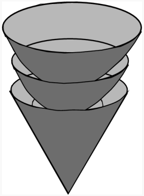

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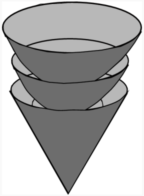

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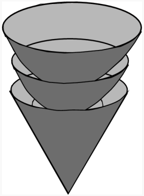

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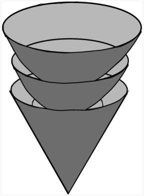

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If so, from the previous result it must be flat, so $T_\infty M = \mathbb{R}^3$, so $M = \mathbb{R}^3$, contradiction.

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If so, we can use this instead of the Lebedeva-Petrunin results to prove the general conjecture.

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$$\| \mathrm{Rm}(g(t)) \|_\infty \leq \frac{\text{const.}}{t}.$$

Put $\sigma = \left(\frac{c}{2+c}\right)^2 \in (0, \frac{1}{9}]$ and

$$f = R^{\sigma - 2} \Big| \mathrm{Ric} - \frac{1}{3} R g \Big|^2 .$$

Ricci flow I

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One finds that

$$\left(\frac{\partial}{\partial t} - \triangle\right) f^{\frac{1}{\sigma}} \leq -\frac{2}{3} f^{\frac{2}{\sigma}}.$$

Ricci flow I

From the weak maximum principle,

$$\sup_{m \in M} f^{\frac{1}{\sigma}}(m,t) \leq \frac{3}{2t},$$

so

$$R^{-2}\left|\mathrm{Ric} - \frac{1}{3}Rg\right|^2 \leq \left(\frac{3}{2tR}\right)^{\sigma}. \qquad (1)$$

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Suppose that there is a singularity at time $T < \infty$. There is a sequence $\{t_i\}_{i=1}^{\infty}$ of times increasing to T, and points $\{m_i\}_{i=1}^{\infty}$ in M so that $\lim_{i\to\infty} |\operatorname{Rm}(x_i, t_i)| = \infty$ and $|\operatorname{Rm}(m_i, t_i)| \geq \frac{1}{2} \sup_{(m,t)\in M\times[0,t_i]} |\operatorname{Rm}(m,t)|$.

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Put $Q_i = |\operatorname{Rm}(m_i,t_i)|$ and $g_i(x,u) = Q_i g(x, t_i + Q_i^{-1}u)$. Then g_i is a Ricci flow solution with curvature norm equal to one at $(m_i, 0\ )$, and curvature norm uniformly bounded above by two for $u \in [-Q_i t_i, 0]$.

Ricci flow I

Suppose first that for some $i_0 > 0$ and all i, we have $Q_i \operatorname{inj}_{g(t_i)}(m_i)^2 \geq i_0$. After passing to a subsequence, there is a pointed Cheeger-Hamilton limit

$$\lim_{i\to\infty}(M, g_i(\cdot), m_i) = (M_\infty, g_\infty(\cdot), m_\infty),$$

where $g_\infty(u)$ is defined for $u \in (-\infty, 0]$.

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The property of having nonnegative Ricci curvature passes to the limit. By construction, g_∞ has curvature norm one at $(m_\infty, 0)$. Hence g_∞ has positive scalar curvature at $(m_\infty, 0)$. By the strong maximum principle, it follows that g_∞ has positive scalar curvature everywhere.

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Given $m' \in M_\infty$, the point $(m', 0)$ is the limit of a sequence of points $\{(m'_i, 0)\}_{i=1}^\infty$ with $\lim_{i \to \infty} R_{g_i}(m'_i, 0) = R_{g_\infty}(m', 0) > 0$. As $\lim_{i \to \infty} Q_i = \infty$, after undoing the rescaling it follows that $\lim_{i \to \infty} R_g(m'_i, t_i) = \infty$. As $\lim_{i \to \infty} t_i = T$, we also have $\lim_{i \to \infty} t_i R_g(m'_i, t_i) = \infty$.

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Equation (1) implies that the metric $g_\infty(0)$ satisfies $\mathrm{Ric} - \frac{1}{3} R g_\infty(0) = 0$. As $g_\infty(0)$ has positive scalar curvature at $(m_\infty, 0)$, it follows that M_∞ is a spherical space form. Then M is compact, which is a contradiction.

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Even if there is no uniform lower bound on $Q_i \, \mathrm{inj}_{g(t_i)}(m_i)^2$, after passing to a subsequence we can take a limit to get a Ricci flow on an étale groupoid. By the same argument, the metric $g_\infty(0)$ on the unit space of the groupoid has constant positive sectional curvature. Then by a Bonnet-Myers argument, the orbit space of the groupoid is compact. It follows that M is compact, which is a contradiction.

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We claim now that there is some $C < \infty$ so that for all $t > 0$, we have $\| \operatorname{Rm}(g(t)) \|_\infty \leq \frac{C}{t}$.

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Suppose not. After doing a type-II point picking, there are points (m_i, t_i) so that $\lim_{i\to\infty} t_i|\operatorname{Rm}(m_i, t_i)| = \infty$ and $|\operatorname{Rm}| \leq 2|\operatorname{Rm}(m_i, t_i)|$ on $M \times [a_i, b_i]$, with $\lim_{i\to\infty} |\operatorname{Rm}(m_i, t_i)|(t_i - a_i) = \lim_{i\to\infty} |\operatorname{Rm}(m_i, t_i)|(b_i - t_i) = \infty$. Put $Q_i = |\operatorname{Rm}(m_i, t_i)|$ and $g_i(x, u) = Q_i g(x, t_i + Q_i^{-1} u)$.

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Suppose first that for some $i_0 > 0$ and all i, we have $Q_i \operatorname{inj}_{g(t_i)}(m_i)^2 \geq i_0$. After passing to a subsequence, we get a limiting Ricci flow solution $\lim_{i\to\infty} (M, g_i(\cdot), m_i) = (M_\infty, g_\infty(\cdot), m_\infty)$ defined for times $u \in \mathbb{R}$. Here M_∞ is a 3-manifold and $|\operatorname{Rm}(m_\infty, 0)| = 1$. As before, for each $m' \in M_\infty$, the point $(m', 0)$ is the limit of a sequence of points $(m'_i, 0)$ with $\lim_{i\to\infty} t_i R_g(m'_i, t_i) = \infty$, where the latter statement now comes from the type-II rescaling.

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From (1), we get $\operatorname{Ric} - \frac{1}{3} R g_\infty = 0$. Then (M_∞, g_∞) has constant positive curvature time slices, which implies that M_∞ is compact. Then M is also compact, which is a contradiction.

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If $\liminf_{i\to\infty} Q_i \operatorname{inj}_{g(t_i)}(m_i)^2 = 0$, we can still take a limit in the sense of étale groupoids. As before, we conclude that M is compact, which is a contradiction.

Distance distortion estimates

Let $d_t : M \times M \to \mathbb{R}$ be the distance function on M with respect to the Riemannian metric $g(t)$. In particular, d_0 be the distance function with respect to g_0.

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Lemma

There is some $C' < \infty$ so that whenever $0 \le t_1 \le t_2 < \infty$, we have

$$d_{t_1} - C'\left(\sqrt{t_2} - \sqrt{t_1}\right) \le d_{t_2} \le d_{t_1}\,. \tag{2}$$

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Fix $m_0 \in M$. Given $s > 0$, put $g_s(u) = s^{-1} g(su)$. Its distance function at time u is $\widehat{d}_{s,u} = s^{-\frac{1}{2}} d_{su}$. From (2), we have

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Also, $\| \operatorname{Rm}(g_s(u)) \| \le \frac{C}{u}$.

Ricci flow II

Theorem

Let $(M, g(\cdot))$ *be a Ricci flow on a noncompact Riemannian 3-manifold that exists for all* $t \geq 0$*, with complete time slices.*

Suppose that $\operatorname{Ric} > 0$ *and*

$$\|\operatorname{Rm}(g(t))\|_\infty \leq \frac{\text{const.}}{t}.$$

Then $(M, g(\cdot))$ *is noncollapsing for large time. That is,*

$$\operatorname{vol}(B_{g(t)}(m_0, \sqrt{t})) \geq \text{const.}\, t^{\frac{3}{2}}.$$

Ricci flow II

Given a sequence $\{s_i\}_{i=1}^{\infty}$ tending to infinity, after passing to a subsequence we can assume that there is a pointed Gromov-Hausdorff limit $\lim_{i \to \infty}(M, g_{s_i}(1), m_0) = (X_\infty, d_{X_\infty}, x_\infty)$.

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We want to show that there is some sequence with a three dimensional Gromov-Hausdorff limit. Suppose not. Then for large s, $(M, g_s(1), m_0)$ is almost one dimensional or almost two dimensional. We will eventually get a contradiction to the fact that M is diffeomorphic to $\mathbb{R}^3$ (Schoen-Yau).

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Suppose first that there is some sequence $\{s_i\}_{i=1}^{\infty}$ so that there is a one dimensional limit. We will show that this leads to a contradiction.

Ricci flow II

If there is some one dimensional limit then there is some $s_0 > 1$ so that $(M, \widehat{d}_{s_0,1}, m_0)$ is very close to a line or a ray in the pointed Gromov-Hausdorff topology. Using the theory of bounded curvature collapse, given $L < \infty$, we can assume that there is a pointed possibly-singular fibration $\pi : B(m_0, L) \to B(x_\infty, L)$ so that

- The generic fiber is T^2,
- $\mathcal{C} = \pi^{-1}(x_\infty)$ is S^1 or T^2, with small diameter, and
- The inclusion $\mathcal{C} \to B(m_0, L)$ induces a nonzero map on π_1.

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Since M is diffeomorphic to $\mathbb{R}^3$, there is some $\sigma < \infty$ so that the inclusion $\mathcal{C} \to B_{d_0}(m_0, \sigma)$ vanishes on π_1. Let Δ be the infimum of such σ's.

Ricci flow II

Let $\mu(s)$ be the infimum of the numbers l so that the inclusion $\mathcal{C} \to B_{\widehat{d}_{s,1}}(m_0, l)$ vanishes on π_1.

Lemma

μ is continuous in s.

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Lemma

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We have $\mu(s_0) \geq L$. From the distance distortion estimate, if s is sufficiently large then $\mu(s) \leq \frac{1}{2}$. Let s_1 be the smallest $s \geq s_0$ so that $\mu(s) = 1$.

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The space $(M, \widehat{d}_{s_1,1}, m_0)$ must be almost two dimensional. There is some $r << 1$ (which can be chosen uniformly) so that $B_{\widehat{d}_{s_1,1}}(m_0, r)$ is a solid torus, and $\mathcal{C} \subset B_{\widehat{d}_{s_1,1}}(m_0, r)$. Since $\mu(s_1) = 1$, the inclusion $\mathcal{C} \subset B_{\widehat{d}_{s_1,1}}(m_0, r)$ must be nontrivial on π_1.

Ricci flow II

However, $B_{\widehat{d}_{s_1},1}(m_0, 2)$ is the total space of a Seifert fibration on a noncompact base, so π_1 of the regular fiber injects. This contradicts that $\mu(s_1) = 1$.

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Thus for all large s, $(M, \widehat{d}_{s,1}, m_0)$ is almost two dimensional.

Given $\rho > 0$, there is a Seifert fibering of $B_{\widehat{d}_{s,1}}(m_0, \rho)$. Using the distance distortion estimates, this gives a Seifert fibering of a region in the time-zero manifold(M, g) that is close to a ball of radius comparable to $\rho\sqrt{s}$. In itself, this does not contradict that M is diffeomorphic to $\mathbb{R}^3$. However, we can fit these Seifert fiberings together, as s varies, to get a Seifert fibering of $\mathbb{R}^3$, which is a contradiction.

Ricci flow III

Corollary

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Then (M, g) has cubic volume growth, i.e.

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Then (M, g) has cubic volume growth, i.e.

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This follows from the fact that the blowdown limit is three dimensional, along with the distance distortion estimates.

Quadratic decay of negative curvature

Theorem

There is no complete noncompact Riemannian 3-manifold (M, g) with c-pinched positive Ricci curvature and cubic volume growth, that satisfies

$$K(m) \geq -\frac{\text{const.}}{d(m, m_0)^2}.$$

Results of Lebedeva-Petrunin

Given an n-dimensional Riemannian manifold (M, g), let Riem be the curvature operator of M and let $\star_M : \Lambda^{n-2}(TM) \to \Lambda^2(TM)$ be Hodge duality.

Results of Lebedeva-Petrunin

Given an n-dimensional Riemannian manifold (M, g), let Riem be the curvature operator of M and let $\star_M : \Lambda^{n-2}(TM) \to \Lambda^2(TM)$ be Hodge duality.

Given C^1-functions $\{f_j\}_{j=1}^{n-2}$ on M, put

$$\sigma = \star_M(\nabla f_1 \wedge \nabla f_2 \wedge \ldots \wedge \nabla f_{n-2}) \qquad (4)$$

and define

$$r_M(f_1, \ldots, f_{n-2}) = \langle \sigma, \mathrm{Riem}(\sigma) \rangle \ \mathrm{dvol}_M, \qquad (5)$$

a measure on M.

Results of Lebedeva-Petrunin

Suppose that $\{M_i, g_i\}_{i=1}^{\infty}$ is a sequence of compact n-dimensional pointed Riemannian manifolds with sectional curvatures uniformly bounded below, that converges to a compact n-dimensional pointed Alexandrov space X_∞ in the Gromov-Hausdorff topology.

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Given C^1-functions $\{f_i\}_{i=1}^{\infty}$, there is a notion of the sequence C^1-converging to a function f_∞ on X_∞ [LP]. A function f_∞ on X_∞ is called Alexandrov smooth if it arises as the limit of such a sequence. Averaged distance functions are Alexandrov smooth.

Results of Lebedeva-Petrunin

The main result of [LP] is the following. Suppose that for each i, $\{f_{i,j}\}_{1 \le j \le n-2}$ is a collection of C^1-functions on M_i. Suppose that for each j, there is a C^1-limit $\lim_{i \to \infty} f_{i,j} = f_{\infty,j}$, where $f_{\infty,j}$ is a function on X_∞. Then there is a weak limit

$$\lim_{i \to \infty} r_{M_i}(f_{i,1}, \ldots, f_{i,n-2}) = r_{X_\infty}(f_{\infty,1}, \ldots, f_{\infty,n-2}). \tag{6}$$

Furthermore, the measure $r_{X_\infty}(f_{\infty,1}, \ldots, f_{\infty,n-2})$ is intrinsic to X_∞. It vanishes on the strata of X_∞ with codimension greater than two, and has descriptions on the codimension-two stratum and the set of regular points.

Quadratic decay of negative curvature

Let X_∞ be a tangent cone at infinity of (M, g), with link Y. The latter is a two dimensional length space with Alexandrov curvature bounded from below, because of the curvature decay assumption.

Lemma

Let ∂_r denote the radial vector field on X_∞. Then

$$r_{X_\infty}(f) = (\partial_r f)^2 dr \wedge (d\omega_Y - \mathrm{dvol}_Y), \tag{7}$$

where $d\omega_Y$ is the curvature measure of the Alexandrov surface Y and dvol_Y is the two-dimensional Hausdorff measure of Y.

Quadratic decay of negative curvature

Using the c-Ricci pinching and the weak convergence of the curvature measures, one shows that $d\omega_Y = \mathrm{dvol}_Y$. Then one shows that this implies that Y is a round S^2. Hence $X_\infty = \mathbb{R}^3$.

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By Colding, (M, g) isometric to the flat $\mathbb{R}^3$, which is a contradiction.