THE GEOMETRY OF SUPERGRAVITY TORSION CONSTRAINTS

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1. Introduction

This paper, prepared for the 2001 Park City Research Program in Supergeometry, is an exposition of [4]. References to relevant earlier papers can be found in the bibliography of [4].

As explained in Jim Gates' lectures, an essential ingredient of the superspace formulation of supergravity is a nonzero torsion tensor. Furthermore, there are torsion constraints in the sense that only certain components of the torsion tensor are allowed to be nonzero. These torsion constraints must be stringent enough to give physically relevant solutions, but flexible enough to allow for nonflat solutions. For example, for $N = 1$ supergravity on a Lorentzian 3-manifold, Jim Gates motivated the choice that ${T_{\alpha\beta}}^c = i(\gamma^c)_{\alpha\beta}$, ${T_{\alpha b}}^\gamma$ can be nonzero, ${T_{ab}}^\gamma$ can be nonzero and the other torsion components must be zero.

The existence of a nonzero torsion tensor in supergravity theories, and the constraints on the torsion tensor, are perhaps surprising from the viewpoint of standard differential geometry. In particular, their geometric meaning is not immediately clear.

To give a somewhat analogous situation from conventional geometry, suppose that M^{2n} is an almost complex manifold with a Hermitian metric. Complexifying TM and using notation that will be explained later, suppose that we are told that a desirable set of torsion conditions is

$$T_{ijk} = 0, \quad T_{ij\bar{k}} = T_{\bar{k}ij} - T_{\bar{k}ji}. \tag{1.1}$$

The geometric meaning of (1.1) may also not be immediately clear. If fact, (1.1) holds if and only if M is Kähler. The first equation in (1.1) is the integrability condition for the almost complex structure and the second equation expresses the Kähler condition, in terms of a unitary basis. Now M is Kähler if and only if near each $p \in M$, M has the Hermitian geometry of $\mathbb{C}^{\ltimes}$ to first order, i.e. there exist holomorphic coordinates $\{z^i\}_{i=1}^n$ around p such that the metric tensor takes the form $g_{i\bar{j}} = \delta_{i\bar{j}} + O(|z|^2)$. Thus (1.1) means that M has a first-order flat $U(n)$-geometry at each point.

We wish to give a similar geometric interpretation for the torsion constraints of supergravity, as the first-order flatness of a G-structure for some appropriate Lie group G. In effect we will do reverse engineering, taking the known torsion constraints and trying to find a group G from which they come.

The theory of G-structures goes back to É. Cartan and was extensively developed in the 1960's. It is not well known today, perhaps because much of the literature on G-structures

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is difficult to penetrate. We will only discuss the minimal amount of this theory that is needed for the supergravity torsion constraints.

As our groups G will be super Lie groups, we must first say something about the structure of super Lie groups.

2. Super Lie groups

To start off with an example, what should $GL(p|q)$ mean? Formally,

$$GL(p|q) \text{“=” the set of } (p+q)\times(p+q) \text{ invertible matrices } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (2.1)$$
$$\text{with } A \text{ and } D \text{ even, and } B \text{ and } C \text{ odd.}$$

There is the immediate problem that our ground ring is $\mathbb{R}$ which has no odd elements, so it is not clear how B and C could be nonzero. To get around this problem we add auxiliary odd parameters. Namely, let Λ be a graded-commutative real superalgebra. Then we define $GL(p|q)(\Lambda)$ as in (2.1), where A, B, C and D now have components in Λ. This makes perfect sense, and we can multiply two such matrices to see that $GL(p|q)(\Lambda)$ is an *ordinary* Lie group.

As usual, it is convenient to do formal calculations by implicitly thinking that everything takes value in the unspecified Λ, but one can also work in a Λ-independent setting. To do so, we look for a supermanifold, which we will call $GL(p|q)$, with the property that $GL(p|q)(\Lambda)$ is the set of Λ-points of $GL(p|q)$. To construct $GL(p|q)$, consider the affine superspace $\mathbb{R}^{p^2+q^2|2pq}$. Then we can take

$$GL(p|q) = \mathbb{R}^{p^2+q^2|2pq}\Big|_{GL(p)\times GL(q)}, \quad (2.2)$$

where we think of $GL(p)\times GL(q)$ as a domain in $\mathrm{End}(\mathbb{R}^p)\times\mathrm{End}(\mathbb{R}^q)\cong\mathbb{R}^{p^2+q^2}$. As in John Morgan's talk, there is a group structure on $GL(p|q)$, i.e. a morphism $GL(p|q)\times GL(p|q)\to GL(p|q)$, etc.

To give a concrete description of a general super Lie group G as a supermanifold, imagine starting with a Lie superalgebra $\mathfrak{g}$, the meaning of which is clear. Then imagine exponentiating the even subalgebra $\mathfrak{g}^{\text{even}}$. One can write out the Jacobi identity for $\mathfrak{g}$ in terms of $\mathfrak{g}^{\text{even}}$ and $\mathfrak{g}^{\text{odd}}$, to obtain four equations. Thinking of these as the infinitesimal equations for G, one is led to the following ingredients for a super Lie group:

1. An ordinary Lie group G^{even},
2. A finite-dimensional G^{even}-module V and
3. A G^{even}-equivariant symmetric map $d : V\times V\to\mathfrak{g}^{\text{even}}$ such that

$$d(v_1,v_2)\cdot v_3 + d(v_2,v_3)\cdot v_1 + d(v_3,v_1)\cdot v_2 = 0. \quad (2.3)$$

Given these ingredients, we obtain a supermanifold G with base space $|G| = G^{\text{even}}$ and $C^\infty(G) = C^\infty(G^{\text{even}})\otimes\Lambda^*(V^*)$, and there is a group structure on G.

Example : The superlinear group $G = GL(p|q)$ comes from $G^{\text{even}} = GL(p)\times GL(q)$ and $V = \mathrm{Hom}(\mathbb{R}^p,\mathbb{R}^q)\oplus\mathrm{Hom}(\mathbb{R}^q,\mathbb{R}^p)$.

Example : The superorthogonal group $G = OSp(p|q)$ comes from $G^{\text{even}} = O(p)\times Sp(q)$

and $V = \mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^{2q})$.

Given a supermanifold M, we can define a principal G-bundle on it in terms of transition morphisms $\phi_{\alpha,\beta} : M\big|_{U_\alpha \cap U_\beta} \to G$, where $\{U_\alpha\}$ is an appropriate open covering of $|M|$ and the $\phi_{\alpha,\beta}$'s satisfy a cocycle condition; see John Morgan's lecture for a similar description of vector bundles.

Example : If M is a supermanfold of dimension $(p|q)$ then the frame bundle $FM \to M$ is a principal $GL(p|q)$-bundle whose transition functions are given in terms of local coordinates by $\phi_{\alpha,\beta} = \left(\frac{\partial z}{\partial w}\right)$.

Hereafter we will work somewhat formally, but keeping in mind that super Lie groups should be interpreted as described above. The first ingredient of a supergeometry is a reduction of FM to some principal bundle $P \to M$ with some structure group G. That is, G is a super Lie subgroup of $GL(p|q)$ and there is a G-equivariant embedding $P \subset FM$. One can think of P as giving the preferred sets of frames.

For example, if M is an ordinary n-dimensional manifold then usual Riemannian geometry amounts to a reduction from the $GL(n)$-bundle FM to an $O(n)$-bundle OM. In analogy, if M is a $(p|2q)$-dimensional supermanifold then one's first attempt to define a supergeometry might be to take a reduction of FM from a $GL(p|2q)$-bundle to an $OSp(p|q)$-bundle. This would correspond to having a superRiemannian metric. If this were the correct notion of supergeometry then it would be pretty boring, as it would just be the $\mathbb{Z}_{\not{2}}$-grading of what is done in usual Riemannian geometry. However, as Jim Gates explained, this is the *wrong* notion of supergeometry, at least from the view of supergravity. Instead, in the physicists' description of supergravity, one assumes that $|M|$ is a Spin-manifold and one takes $G = \mathrm{Spin}(p)$, together with some torsion constraints.

In order to interpret the torsion constraints geometrically we will take a slightly different structure group G, but first we must explain the meaning of the torsion tensor.

3. Torsion

In Riemannian geometry courses, what we learn about the torsion tensor is that it's something to be set to zero. While this is true, it's not very illuminating. In a nontrivial sense, the real reason that we set the torsion to zero in Riemannian geometry is because we can. We now discuss what torsion means in general.

Let's first consider an ordinary manifold M^n. To keep things concrete, let's take a local basis $\{E^A\}$ of 1-forms on M. Following Jim Gates' notation, a G-connection can be written as

$$\omega_B{}^A = dz^M \omega_{MB}{}^A = E^C \omega_{CB}{}^A, \tag{3.1}$$

where $\omega_B{}^A$ denotes a 1-form that takes value in the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n)$. The torsion tensor T is given by

$$T^A = dE^A + E^B \wedge \omega_B{}^A. \tag{3.2}$$

As a thought experiment, given $\{E^A\}$, how much of T can we kill by changing ω? From (3.2), if we send ${\omega_B}^A$ to ${\omega_B}^A + \Delta {\omega_B}^A$ then T changes by

$$\begin{aligned} \triangle T^A &= E^B \wedge \triangle {\omega_B}^A \\ &= E^B \wedge E^C \triangle {\omega_{CB}}^A. \end{aligned} \tag{3.3}$$

The latter is the stuff that we can kill.

To say this more formally, let W be our flat space, so $G \subset GL(W)$. The framing $\{E^A\}$ at m gives an isomorphism $T_m M \cong W$. Then we can think of the torsion tensor $T(m)$ at m as an element of $\mathrm{Hom}(W \wedge W, W)$. Equation (3.3) defines a linear map

$$\delta : \mathrm{Hom}(W, \mathfrak{g}) \to \mathrm{Hom}(W \wedge W, W) \tag{3.4}$$

which sends $\triangle \omega$ to $\triangle T$. Note that δ is defined purely algebraically. The part of the torsion that we can kill is $\mathrm{Im}(\delta)$.

Let us define

$$H^{0,2} = \mathrm{Hom}(W \wedge W, W) / \operatorname{Im} \delta. \tag{3.5}$$

In other words, this is the part of the torsion that we cannot kill by changing the connection. Given $T(m) \in \mathrm{Hom}(W \wedge W, W)$, we write its equivalence class in $H^{0,2}$ as $[T(m)]$. While we're at it, let's define $\mathfrak{g}^{(1)}$ to be $\mathrm{Ker}(\delta)$. Given $\{E^A\}$ and T, from (3.2) this is the amount of freedom in the connection ω.

To phrase things in terms of principal bundles, recall that there is the notion of the soldering form τ, a canonically-defined W-valued 1-form on FM. Given the reduction $P \subset FM$, we pullback τ to P and give it the same name. Suppose that we have a local section $s : (U \subset M) \to P$. Then $\{E^A\}$ is just $s^* \tau$.

Let ω be a connection on G, i.e. a G-equivariant $\mathfrak{g}$-valued 1-form on P with the property that $\omega(V_x) = x$, where V_x denotes the vector field on P generated by $x \in \mathfrak{g}$. The torsion tensor is the horizontal W-valued 2-form on P given by $T = d\tau + \tau \wedge \omega$. Using τ, we can also consider the torsion to be a G-equivariant map $T : P \to \mathrm{Hom}(W \wedge W, W)$. Quotienting by $\mathrm{Im}(\delta)$, we obtain a G-equivariant map $[T] : P \to H^{0,2}$. Note that by construction, $[T]$ depends only on the reduction P, i.e. is independent of the choice of connection ω.

What is the significance of $[T]$? It gives us an obstruction for M to be G-flat. By G-flatness, we mean the following. We are given a G-reduction of FM to P. We assume that the flat space W has a canonical reduction of its frame bundle FW to a principal G-bundle $P^{\mathrm{flat}} \subset FW$. Given $m \in M$, consider a diffeomorphic embedding $\phi : (\mathcal{N} \subset W) \to M$, where $\mathcal{N}$ is a neighborhood of $0 \in W$ and $\phi(0) = m$. We can always lift ϕ to an embedding $\phi_* : F\mathcal{N} \to FM$. Here's the geometric question : does ϕ_* send $P^{\mathrm{flat}}\big|_{\mathcal{N}} \subset F\mathcal{N}$ to P? If so then for all practical purposes, M is locally the same as W. We say that M is G-flat if for each $m \in M$, we can find an embedding ϕ so that $\phi(0) = m$ and ϕ_* does send $P^{\mathrm{flat}}\big|_{\mathcal{N}}$ to P.

To see what this has to do with the torsion, let's suppose that M is G-flat. Given $m \in M$, construct $\phi : (\mathcal{N} \subset W) \to M$ as above. Suppose that W has a G-connection ω^{flat} with vanishing torsion. (That is, ω^{flat} is defined on P^{flat}.) Then using the embedding ϕ_*, we can transfer ω^{flat} to obtain a torsion-free connection defined on P over $\phi(\mathcal{N})$. It follows that $[T]$ vanishes in $H^{0,2}$, at least over $\phi(\mathcal{N})$.

Running the logic backwards, we see that a nonvanishing of $[T]$ is an obstruction for M to be G-flat. Again, this is a statement just about the G-reduction P. In fact, one can define a notion of M (or more precisely P) being first-order G-flat and then a precise statement is that M is first-order G-flat at $m \in M$ if and only if $[T(m)]$ vanishes in $H^{0,2}$ [3, Theorem 4.1].

Example : If $G = O(n) \subset GL(n)$ then one computes algebraically that $H^{0,2} = 0$. Thus there is no obstruction to first-order flatness in Riemannian geometry. In Lorentzian geometry, this is a form of the equivalence principle. As $\mathfrak{g}^{(1)} = 0$, there is a unique torsion-free orthogonal connection, the Levi-Civita connection.

Example : If $G = U(n) \subset GL(2n)$ then one finds that $H^{0,2} \neq 0$. In fact, the condition for $[T]$ to be zero becomes exactly the equations in (1.1). Thus M is first-order $U(n)$-flat if and only if M is Kähler. Again $\mathfrak{g}^{(1)} = 0$, so if M is Kähler then there is a unique torsion-free unitary connection.

Now suppose instead that our model space W has a constant nonzero torsion T_0. (We will still consider it to be a flat space, just one with a nonzero torsion.) Although less common in conventional geometry than vanishing torsion, this situation does arise, for example, in CR geometry, the geometry of hypersurfaces in $\mathbb{C}^n$.

If we are to model M by W then we want M to also have a connection with torsion T_0. This doesn't quite make sense as stated, since we still have to take into account the action of G. In terms of the local frame $\{E^A\}$, we want to have a connection whose torsion T differs from T_0 by a G-action, since then we can perform a gauge transformation to make T identically equal to T_0. The residual local symmetry is the subgroup G_0 of G which preserves T_0.

In terms of the principal bundle P, we have the G-equivariant map $[T] : P \to H^{0,2}$. Because of the G-equivariance it doesn't make sense to say that $[T]$ lands on $[T_0]$, but it does make sense to require that for each $p \in P$, $[T](p)$ lies in the G-orbit of $[T_0]$ in $H^{0,2}$. If this is the case then we will say that M (or more precisely P) is first-order G-flat. If M is first-order G-flat, let us choose a G-connection ω whose torsion $T : P \to \mathrm{Hom}(W \wedge W, W)$ takes value in the G-orbit of T_0. Putting $P_0 = T^{-1}(T_0)$, we obtain a reduction of P to a principal G_0-bundle P_0. In fact, once T_0 is nonzero, it is only natural to make such a reduction.

Finally, let $\mathfrak{g}_0$ denote the Lie algebra of G_0 and suppose that we have a G_0-invariant splitting $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. When we pullback ω to $P_0 \subset P$, it decomposes as $\omega = \omega' + \omega''$. Here ω' is a G_0-connection on P_0 and ω'' is a tensor. The torsion equation (3.2) becomes

$$T_0^A - E^B \wedge \omega''_B{}^A = dE^A + E^B \wedge \omega'_B{}^A. \tag{3.6}$$

That is, although we started with a first-order flat G-structure, the induced G_0-structure may not be first-order flat, but instead has the torsion tensor $T_0^A - E^B \wedge \omega''^A_B$.

4. Supergravity torsion constraints

We return to supergeometry. We first consider unextended, i.e. $N = 1$, supergravity theories.

The model flat space W is a superspace $\mathbb{R}^{p|q}$ where $\mathbb{R}^p$ has an inner product of signature (p_+, p_-) and $\mathbb{R}^q$ is a faithful spinor module for $\mathrm{Spin}(p_+, p_-)$. We use the standard notation that lower-case Roman indices are even indices, Greek indices are odd indices and upper-case Roman indices are either. We assume that there is a charge conjugation operator, i.e. a matrix $C \in \mathrm{Aut}(\mathbb{R}^q)$ such that $C\gamma_a C^{-1} = \alpha\gamma_a^T$ and $C^T = \alpha C$, with $\alpha = \pm 1$. Then as discussed in the other lectures, to W one can associate a super Poincaré group, an invariant collection (D_a, D_α) of vector fields and a flat connection whose only nonzero torsion component is $(T_0)_{\alpha\beta}{}^c = (\gamma^c C^{-1})_{\alpha\beta}$.

We now take G to be a super Lie group of the form

$$G = \left\{ \begin{pmatrix} \rho_1(A) & 0 \\ \star & \rho_2(A) \end{pmatrix} : A \in \mathrm{Spin}(p_+, p_-), \star \in \mathcal{S} \right\} \tag{4.1}$$

where $\rho_1 : \mathrm{Spin}(p_+, p_-) \to \mathrm{SO}(p_+, p_-)$ is the orthogonal representation, ρ_2 is the spinor representation and $\mathcal{S}$ is a $\mathrm{Spin}(p_+, p_-)$-invariant subspace of $\mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^q)$. That is, G is the semidirect product $\mathcal{S} \tilde{\times} \mathrm{Spin}(p_+, p_-)$.

Claim : The torsion constraints in supergravity theories of dimension at most six all arise as the first-order flatness of a G-structure.

We will work out in detail the three-dimensional example $(p_+, p_-) = (2, 1)$. First, we make some general remarks. There is some freedom in the choice of subspace $\mathcal{S}$. Of course, because of the $\mathrm{Spin}(p_+, p_-)$-invariance there is only a finite number of possibilities. *A priori*, different choices of $\mathcal{S}$ can give different geometries. In most cases, one can just take $\mathcal{S}$ to be all of $\mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^q)$.

The condition of first-order flatness for a G-structure can be written out in equations, but we will just need the geometric notion. It is easy to see that the subgroup G_0 which preserves T_0 is $\mathrm{Spin}(p_+, p_-)$. In this way we make contact with the physicists' superspace formulation of supergravity. There is an obvious G_0-invariant decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathcal{S}$. After reducing to a G_0-bundle, the structure equations that we derive are exactly (3.6).

The group G preserves the rank-q odd subspace of $\mathbb{R}^{p|q}$. However, if $\mathcal{S} \neq 0$ then it does not preserve $\mathbb{R}^p$.

Geometrically, if we have a first-order flat G-structure on M then we obtain

1. A nonintegrable odd distribution $D^{\mathrm{odd}}M \subset TM$ of rank $(0|q)$,
2. A spinorial $\mathrm{Spin}(p_+, p_-)$-representation on $D^{\mathrm{odd}}M$,
3. An orthogonal $\mathrm{Spin}(p_+, p_-)$-representation on $TM/D^{\mathrm{odd}}M$ and
4. A map $[\cdot, \cdot] : D^{\mathrm{odd}}M \times D^{\mathrm{odd}}M \to TM/D^{\mathrm{odd}}M$, coming from the Lie bracket, which is conjugate to the corresponding flat space map $\mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^p$ described by T_0.

We also endow $TM/D^{\mathrm{odd}}M$ with a compatible inner product. In the case $\mathcal{S} = \mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^q)$, the linear transformations of TM which preserve the above structure give exactly the group G.

The diagonal subgroup G_0 of G preserves the subspaces $\mathbb{R}^p$ and $\mathbb{R}^q$ of $\mathbb{R}^{p|q}$. Hence a reduction to a G_0-structure corresponds to a choice of splitting $TM = (TM/D^{\mathrm{odd}}M) \oplus D^{\mathrm{odd}}M$.

Note that if $\mathcal{S} \neq /$ then the group G is not a subgroup of $OSp\left(p_+, p_- | \frac{q}{2}\right)$. This shows again that the notion of a superRiemannian metric is irrelevant for supergravity theories.

To summarize, to form a supergeometry, suppose that we are given $\mathcal{S}$. Then

1. Pick a reduction P of FM to a G-structure, i.e. a set of framings $\{E^A\}$.
2. Check whether the reduction is first-order flat. If it isn't, throw it out.
3. Choose a G-connection with the correct torsion.

4. Reduce to the subgroup G_0.
5. Write out the structure equations. Analyze their consistency using the Bianchi identities.

The condition of being first-order flat in 2 involves first-derivatives of the frame $\{E^A\}$. It is not at all obvious how to parametrize the set of solutions. One wants to do so in order to find the independent supergravity fields. In the case of three-dimensional supergravity, Jim Gates explained how one can parametrize the independent fields using the spinorial frame $\{E^\alpha\}$. In the four-dimensional case, one encounters prepotentials $\{H^A\}$. It appears that one must do a case-by-case analysis to find the independent fields.

There is generally not a unique choice of connection in 3, as $\mathfrak{g}^{(1)} \neq 0$. However, the ambiguity is mild in the sense that different choices of connection lead to the same structure equations, as we will see in the three-dimensional case.

There is a strong analogy between supergravity theory and CR manifolds. In fact, there is a dictionary

$$\begin{aligned} W &\longleftrightarrow (S^{2N-1} \subset \mathbb{C}^N) \\ D^{\mathrm{odd}} M &\longleftrightarrow D^{\mathrm{complex}} M \\ T_0 &\longleftrightarrow \text{ the Levi form of } S^{2N-1} \subset \mathbb{C}^N \\ \text{superconformal geometry} &\longleftrightarrow \text{ CR geometry (Chern-Moser) [1]} \\ \text{supergravity} &\longleftrightarrow \text{ pseudoHermitian geometry (Webster) [5]} \end{aligned} \tag{4.2}$$

Of course, in CR geometry one does not have odd variables, but the role of the odd distribution $D^{\mathrm{odd}} M$ is played by the complex distribution $D^{\mathrm{complex}} M \subset TM$. It is a historical coincidence that Chern and Moser were analyzing CR manifolds around the same time and in almost the same way that Wess and Zumino were deriving the superspace formulation of supergravity theories [2, 6]. The second half of the Chern-Moser paper looks at the structure equations of a CR manifold, chooses preferred connections and analyzes the consequences of the Bianchi identities in a way that mirrors the Wess-Zumino work, although in a very different language.

To deal with the case of extended supergeometries, let K be a Lie group. We assume that $\mathbb{R}^q$ is the tensor product of representation spaces of $\mathrm{Spin}(p_+, p_-)$ and K. We take $\mathcal{S}$ to be a $(\mathrm{Spin}(p_+, p_-) \times K)$-invariant subspace of $\mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^q)$. Then we put $G = \mathcal{S} \widetilde{\times} (\mathrm{Spin}(p_+, p_-) \times K)$ and proceed as before. Note that the group K is gauged.

Our claim is that the torsion constraints of all supergravity theories of dimension at most six arise from the above procedure. To be precise, this is true for theories with an offshell superspace formulation, i.e. for $q \leq 8$. Also, in four dimensions there are various superspace formulations of the same onshell action and we only pick up the “minimal” superspace formulation.

The claim is shown explicitly in [4] for the following supergravity theories :

$$
\begin{array}{ccccc}
\underline{p} & \underline{p_+} & \underline{p_-} & \underline{q} & \underline{N} \\
 & & & & \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 1 & 1 & (1,0) \\
2 & 1 & 1 & 2 & (1,1) \\
2 & 1 & 1 & 2 & (2,0) \\
2 & 1 & 1 & 3 & (2,1) \\
2 & 2 & 0 & 4 & (2,2) \\
3 & 2 & 1 & 2 & 1 \\
4 & 3 & 1 & 4 & 1 \\
4 & 4 & 0 & 8 & 2 \\
6 & 5 & 1 & 8 & 1
\end{array}
\tag{4.3}
$$

We now work out the three-dimensional case in some detail.

5. Three-dimensional supergravity

We take $p_+ = 2$, $p_- = 1$ and $q = 2$. Then $\mathrm{Spin}(p_+, p_-) = SL(2,\mathbb{R})$ and $\mathbb{R}^q$ has the standard $SL(2,\mathbb{R})$-representation.

We use a notation in which an element $M \in sl(2,\mathbb{R})$ is represented by a traceless matrix $M_\pm{}^\pm$. Using the $SL(2,\mathbb{R})$-invariant symplectic form ϵ on $\mathbb{R}^2$ to raise and lower indices, we can represent M as a symmetric matrix $M_{\pm\pm}$. We also identify $S^2(\mathbb{R}^2)$ with Minkowski 3-space, to write an element $P \in \mathbb{R}^3$ as a symmetric matrix $P^{\pm\pm}$.

Let us first take $\mathcal{S} = \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^2)$. One has $\dim(V) = 5$, $\dim(\mathfrak{g}) = 9$, $\dim(\mathrm{Hom}(V,\mathfrak{g})) = 45$, $\dim(\mathrm{Hom}(V \wedge V, V)) = 60$, $\dim(\mathfrak{g}^{(1)}) = 12$, $\dim(\mathrm{Im}(\delta)) = 33$ and $\dim(H^{0,2}) = 27$. It turns out that G acts trivially on $H^{0,2}$.

Suppose that we have a G-structure. For it to be first-order flat, $[T]$ must equals $[T_0]$ identically. Suppose that this is the case. Then we choose a connection ω and reduce to a G_0-structure, where $G_0 = SL(2,\mathbb{R})$.

The structure equations are given by (3.6). To write these explicitly, we write the $sl(2,\mathbb{R})$-valued connection 1-form ω' in its spinor representation as ω'^{α}_{β} and we write it in its orthogonal representation as $\omega'^{\alpha\beta}_{\epsilon\phi} = \omega'^{\alpha}_{\epsilon}\delta_\phi{}^\beta + \omega'^{\alpha}_{\phi}\delta_\epsilon{}^\beta$. Then (3.6) becomes

$$
\begin{aligned}
- E^\alpha \wedge E^\beta &= dE^{\alpha\beta} + E^{\gamma\beta} \wedge \omega'_\gamma{}^\alpha + E^{\alpha\gamma} \wedge \omega'_\gamma{}^\beta, \\
-E^{\gamma\delta} \wedge \omega''_{\delta\gamma}{}^\alpha &= dE^\alpha + E^\beta \wedge \omega'_\beta{}^\alpha.
\end{aligned}
\tag{5.1}
$$

Let us write the 1-form $\omega''_{\delta\gamma}{}^\alpha$ as

$$
\omega''_{\delta\gamma}{}^\alpha = -E^\epsilon\, T_{\epsilon,\delta\gamma}{}^\alpha - \frac{1}{2} E^{\epsilon\phi}\, T_{\phi\epsilon,\delta\gamma}{}^\alpha. \tag{5.2}
$$

Then (5.1) becomes

$$
\begin{aligned}
dE^{\alpha\beta} + E^{\gamma\beta} \wedge \omega'_\gamma{}^\alpha + E^{\alpha\gamma} \wedge \omega'_\gamma{}^\beta &= -E^\alpha \wedge E^\beta, \\
dE^\alpha + E^\beta \wedge \omega'_\beta{}^\alpha &= E^{\gamma\delta} \wedge E^\epsilon\, T_{\epsilon,\delta\gamma}{}^\alpha + \frac{1}{2} E^{\gamma\delta} \wedge E^{\epsilon\phi}\, T_{\phi\epsilon,\delta\gamma}{}^\alpha.
\end{aligned}
\tag{5.3}
$$

We recognize these as the structure equations for three-dimensional supergravity with structure group $SL(2,\mathbb{R})$ and nonzero torsion components $T_{\epsilon,\delta}{}^{\alpha\beta}$, $T_{\epsilon,\delta\gamma}{}^{\alpha}$ and $T_{\phi\epsilon,\delta\gamma}{}^{\alpha}$, as desired.

Recall that there was an ambiguity of $\mathfrak{g}^{(1)}$ in the choice of the connection. One can check that this amounts to changing $T_{\phi\epsilon,\delta\gamma}{}^{\alpha}$ in (5.2) by something of the form $U_{\phi\epsilon,\delta\gamma}{}^{\alpha}$ with $U_{\phi\epsilon,\delta\gamma}{}^{\alpha} = U_{\delta\gamma,\phi\epsilon}{}^{\alpha}$. However, as $E^{\gamma\delta}\wedge E^{\epsilon\phi} = -E^{\epsilon\phi}\wedge E^{\gamma\delta}$, the structure equation (5.3) remains unchanged.

The Bianchi identities imply that one can express the torsion and curvature in terms of a function R and a tensor $G_{\alpha\beta\gamma}$ which is totally symmetric in its indices. A calculation gives

$$\begin{aligned} T_{\epsilon,\delta\gamma}{}^{\alpha} &= R(\epsilon_{\epsilon\delta}\delta_{\gamma}{}^{\alpha} + \epsilon_{\epsilon\gamma}\delta_{\epsilon}{}^{\alpha}), \\ T_{\phi\epsilon,\delta\gamma}{}^{\alpha} &= \frac{1}{2}(\epsilon_{\phi\delta}\delta_{\epsilon}{}^{\alpha}\nabla_{\gamma}R + \epsilon_{\epsilon\delta}\delta_{\phi}{}^{\alpha}\nabla_{\gamma}R + \epsilon_{\phi\gamma}\delta_{\epsilon}{}^{\alpha}\nabla_{\delta}R + \epsilon_{\epsilon\gamma}\delta_{\phi}{}^{\alpha}\nabla_{\delta}R \\ &\qquad - \epsilon_{\delta\phi}\delta_{\gamma}{}^{\alpha}\nabla_{\epsilon}R - \epsilon_{\gamma\phi}\delta_{\delta}{}^{\alpha}\nabla_{\epsilon}R - \epsilon_{\delta\epsilon}\delta_{\gamma}{}^{\alpha}\nabla_{\phi}R - \epsilon_{\gamma\epsilon}\delta_{\delta}{}^{\alpha}\nabla_{\phi}R \\ &\qquad + G_{\phi\delta}{}^{\alpha}\epsilon_{\epsilon\gamma} + G_{\epsilon\delta}{}^{\alpha}\epsilon_{\phi\gamma} + G_{\phi\gamma}{}^{\alpha}\epsilon_{\epsilon\delta} + G_{\epsilon\gamma}{}^{\alpha}\epsilon_{\phi\delta}), \end{aligned} \tag{5.4}$$

with the constraint

$$\nabla_{\alpha}G_{\beta\gamma}{}^{\alpha} + \nabla_{\beta}\nabla_{\gamma}R + \nabla_{\gamma}\nabla_{\beta}R = 0. \tag{5.5}$$

(We use different conventions than Jim Gates, but the results are equivalent.)

Now suppose that we instead take $\mathcal{S}$ to be the subspace of $\mathrm{Hom}(\mathbb{R}^3,\mathbb{R}^2)$ consisting of maps M that can be written in the form $M_{\delta\gamma}{}^{\alpha} = Z_{\delta}\delta_{\gamma}{}^{\alpha} + Z_{\gamma}\delta_{\delta}{}^{\alpha}$ for some Z. That is, $\mathcal{S}$ consists of the maps $M \in \mathrm{Hom}(S^2(\mathbb{R}^2),\mathbb{R}^2)$ with the property that there exists a $z \in (\mathbb{R}^2)^*$ such that $M(v,w) = z(v)w + z(w)v$. Then it turns out that we obtain the same geometry as if we had taken $\mathcal{S}$ to be all of $\mathrm{Hom}(\mathbb{R}^3,\mathbb{R}^2)$ [4, Proposition 14]. Next, suppose that we take $\mathcal{S}$ to be the subspace of $\mathrm{Hom}(\mathbb{R}^3,\mathbb{R}^2)$ consisting of maps M such that $M_{\alpha\beta}{}^{\beta} = 0$. That is, if we define $M_v : \mathbb{R}^2 \to \mathbb{R}^2$ by $M_v(w) = M(v\otimes w + w\otimes v)$ then $\mathcal{S}$ consists of the maps $M \in \mathrm{Hom}(S^2(\mathbb{R}^2),\mathbb{R}^2)$ such that for all $v \in \mathbb{R}^2$, $\mathrm{Tr}(M_v) = 0$. In this case it turns out that the geometry we obtain is equivalent to that obtained from taking $\mathcal{S}$ to be all of $\mathrm{Hom}(\mathbb{R}^3,\mathbb{R}^2)$, but setting the superfield R to be zero [4, Proposition 13]. Finally, if $\mathcal{S} = 0$ then we only obtain flat solutions.

6. Further topics

6.1. **Higher order obstructions to flatness.** In the theory of G-structures, it is generally not true that first-order flatness implies flatness. It is true when $G = GL(n,\mathbb{C}) \subset GL(2n,\mathbb{R})$, as the first-order flatness amounts to the vanishing of the Nijenhuis tensor and this implies the integrability of the complex structure. It is not true when $G = O(n) \subset GL(n,\mathbb{R})$, as the Riemann curvature tensor is an obstruction to flatness.

There is a algebraic theory of higher order obstructions to flatness, which live in the so-called Spencer cohomology groups $H^{i,2}$. A clear exposition of this theory is in [3]. For example, if $G = O(n)$ then $H^{i,2}$ vanishes if $i \neq 1$, while $H^{1,2}$ consists of the tensors with the symmetry of the Riemann curvature tensor. One of the main issues in the theory of G-structures is to know when the vanishing of all of the algebraic obstructions to flatness actually implies flatness. For example, if $G = GL(n,\mathbb{C}) \subset GL(2n,\mathbb{R})$ then the flatness is the Newlander-Nirenberg theorem.

In the case of the supergeometry group G with $\mathcal{S} = \mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^q)$, it turns out that if $i > 0$ then the only nonvanishing Spencer cohomology group is $H^{1,2}$, which also consists of the tensors with the symmetry of the Riemann curvature tensor for a p-manifold [4, Propositions 18,19]. In other words, there are no formal obstructions to flatness beyond the curvature tensor R_{abcd} with even indices.

6.2. **Superconformal geometry.** In Riemannian geometry, the Weyl tensor fits nicely into the framework of Cartan connections. The latter means that one has Lie groups $H \subset G$, a principal H-bundle $P \to M$ and a $\mathfrak{g}$-valued 1-form ω on P such that
1. ω is H-equivariant,
2. For all $x \in \mathfrak{h}$, $\omega(V_x) = x$, where V_x is the vector field on P generated by x, and
3. For all $p \in P$, ω gives an isomorphism from T_pP to $\mathfrak{g}$.

Define the curvature Ω as usual to be $d\omega + \omega^2$. Suppose that $\Omega = 0$. For simplicity, we assume that P and H are connected. If $p_0 \in P$ is a basepoint then we take the path-ordered integral of ω along paths from p_0. This gives a map from the universal cover $\widetilde{P}$ to G. Taking a quotient, we obtain a $\pi_1(M)$-equivariant map α from $\widetilde{M}$ to G/H. By condition 3, α is a local diffeomorphism. Thus α is a developing map and we have coordinate charts on M modeled by domains in G/H, with each transition map coming from the left-action of an element of G.

Of course, in general we cannot assume that ω is flat. One can write Riemannian geometry in terms of Cartan connections by taking G to be the Euclidean group $\mathbb{R}^n \widetilde{\times} SO(n)$ and $H = SO(n)$. Then the $\mathbb{R}^n$-component of ω can be identified with the soldering form and the $so(n)$-component of ω can be identified with the Riemannian connection. By an appropriate choice of the Riemannian connection, i.e. choosing the Levi-Civita connection, we can kill the $\mathbb{R}^n$-component of Ω, i.e. the torsion. The remaining $so(n)$-component of Ω is the Riemannian curvature, which we see as an obstruction to Euclidean flatness.

In the case of n-dimensional conformal geometry, G is the conformal group $SO(n+1,1)$. It acts on S^n by conformal transformations. Fix a point $\infty \in S^n$ and let H be the stabilizer of ∞. Equivalently, writing $\mathbb{R}^n = S^n - \infty$, H is the conformal group of $\mathbb{R}^n$. By construction, $G/H = S^n$.

Algebraically, $\mathfrak{g}$ is a graded Lie algebra $\mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$, with $\mathfrak{g}^{(-1)} \cong \mathbb{R}^n$, $\mathfrak{g}^{(0)} \cong o(n) \oplus \mathbb{R}$ and $\mathfrak{g}^{(1)} \cong \mathbb{R}^n$. Then $\mathfrak{h} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$. Given a Cartan connection ω, we decompose it as $\omega = \omega^{(-1)} + \omega^{(0)} + \omega^{(1)}$, and similarly for its curvature Ω. We can identify the $\mathbb{R}^n$-valued 1-form $\omega^{(-1)}$ with the soldering form for M.

To see the Weyl curvature as an obstruction to conformal flatness, one assumes that one is given the soldering form, i.e. $\omega^{(-1)}$, and the Levi-Civita connection, i.e. the component of $\omega^{(0)}$ in $o(n)$. Then the question is whether one can extend these components to form a Cartan connection ω which is flat. Taking the $\mathbb{R}$-component of $\omega^{(0)}$ to vanish, one can use the freedom in $\omega^{(1)}$ to make $\Omega^{(0)}$ equal to the Weyl tensor. In this way, one sees that the Weyl curvature is an obstruction to conformal flatness of a Riemannian metric. If $n = 3$ then the Weyl curvature vanishes but $\Omega^{(1)}$ gives its three-dimensional analog.

It is of interest to treat superconformal geometry in terms of Cartan connections. It is a remarkable fact that the superconformal groups G are simple super Lie groups, with graded Lie superalgebra $\mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(-1/2)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1/2)} \oplus \mathfrak{g}^{(1)}$. Here $\mathfrak{g}^{(-1)} \cong \mathbb{R}^p$, $\mathfrak{g}^{(-1/2)} \cong \mathbb{R}^q$,

$\mathfrak{g}^{(0)} \cong o(p_+, p_-) \oplus \mathbb{R} \oplus k$, $\mathfrak{g}^{(1/2)} \cong \mathbb{R}^q$ and $\mathfrak{g}^{(1)} \cong \mathbb{R}^p$, where k is the Lie algebra of an internal symmetry group. We take H to be the subgroup with Lie algebra $\mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1/2)} \oplus \mathfrak{g}^{(1)}$.

We decompose a Cartan connection ω as

$$\omega = \omega^{(-1)} + \omega^{(-1/2)} + \omega^{(0)} + \omega^{(1/2)} + \omega^{(1)}, \tag{6.1}$$

and similarly for its curvature Ω. Now $\omega^{(-1)} + \omega^{(-1/2)}$ can be identified with the soldering form $\{E^a, E^\alpha\}$. Suppose that we are given $\omega^{(0)}$ and that $\Omega^{(-1)}$ vanishes. (Note that $\Omega^{(-1)}$ only depends on $\omega^{(-1)}$, $\omega^{(-1/2)}$ and $\omega^{(0)}$. Its vanishing corresponds to having the flat space expression for the torsion component T^a.) Then the question is whether we can extend these components to a Cartan connection with vanishing curvature. In general one cannot, but one can choose $\omega^{(1/2)}$ and $\omega^{(1)}$ so that certain components of Ω vanish. The remaining components are the obstruction to conformal flatness.

For example, in the case of three dimensions, $G = OSp(1|2)$. It turns out that one can uniquely choose $\omega^{(1/2)}$ and $\omega^{(1)}$ so that $\Omega^{(-1/2)} = \Omega^{(0)} = 0$ [4, Proposition 23]. The remaining curvature components, $\Omega^{(1/2)}$ and $\Omega^{(1)}$, are the supersymmetric analog of the three-dimensional conformal tensor.

6.3. **Onshell theories.** For supergravity theories with N supersymmetries, if N is large enough then it turns out that the superspace torsion constraints already imply the equations of motion, i.e. that the theory is onshell. This is the case for four-dimensional supergravity if $N > 2$. In these cases the torsion constraints do not follow the pattern that we have described above, and we do not know of their geometric interpretation.

6.4. **SuperKähler geometry.** SuperKähler manifolds M whose base space $|M|$ has one complex dimension are well understood. As a purely mathematical question, one can ask about the higher-dimensional situation. Of course, we are not thinking of the approach of defining superKähler forms on a supercomplex manifold, but rather of applying the theory of G-structures.

If $|M|$ has n complex dimensions then there is a natural spinor representation $\rho_2 : U(1) \times SU(n) \to \mathrm{End}(\Lambda^{*,0}(\mathbb{C}^n))$ of real dimension 2^{n+1}. One's first attempt to define a superKähler geometry might be to require first-order flatness of an $\mathcal{S} \widetilde{\times} (U(1) \times SU(n))$-structure, where $\mathcal{S}$ is a $(U(1) \times SU(n))$-invariant subspace of $\mathrm{Hom}\left(\mathbb{R}^{2n}, \mathbb{R}^{2^{n+1}}\right)$. However, one finds that this gives a flat geometry even in the case when $|M|$ has one complex dimension [4, Proposition 27]. Instead, it turns out that one must use the additional "chiral" action of $\mathbb{C}^*$ on $\Lambda^{*,0}(\mathbb{C}^n)$ which multiplies an even form by $z \in \mathbb{C}^*$ and multiplies an odd form by $\overline{z}^{-1}$.

Thus we try taking $G = \mathcal{S} \widetilde{\times} (\mathbb{C}^* \times U(1) \times SU(n))$. If $|M|$ has one complex dimension then one finds that this gives the right answer. In fact, one obtains the same geometry whether one takes $\mathcal{S} = \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^4)$ or $\mathcal{S} = \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}^2)$ [4, Proposition 26].

If $|M|$ has complex dimension two then one finds that having a first-order flat G-structure, with $G = \mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^8) \widetilde{\times} (\mathbb{C}^* \times U(1) \times SU(2))$, implies that $|M|$ is a Hermitian locally symmetric space [4, Proposition 28]. (Note that G is a subgroup of the structure group of a four-dimensional $N = 2$ Riemannian supergeometry.) It may be that this is the best that one can do. In [4, Proposition 30] we explored the consequences of relaxing the torsion constraints, but the results were inconclusive. In any event, we do not have a general understanding of superKähler geometry.

We take this opportunity to correct some mistakes in [4]. Throughout [4] we wrote $\mathrm{End}(\mathbb{R}^p, \mathbb{R}^q)$ when we should have written $\mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^q)$. Equation (146) of [4] should read ${\omega_{\Theta_1}}^{\Theta_1} + {\omega_{\Theta_2}}^{\Theta_2} = {\omega_z}^z$.

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