COLLAPSING AND THE DIFFERENTIAL FORM LAPLACIAN : THE CASE OF A SINGULAR LIMIT SPACE

JOHN LOTT

Abstract. We analyze the limit of the p-form Laplacian under a collapse with bounded sectional curvature and bounded diameter to a singular limit space. As applications, we give results about upper and lower bounds on the j-th eigenvalue of the p-form Laplacian, in terms of sectional curvature and diameter.

1. Introduction

In a previous paper, we analyzed the limit of the p-form Laplacian under a collapse with bounded sectional curvature and bounded diameter to a smooth limit space [19]. In the present paper we extend the analysis of [19] to cover the case of a singular limit space.

We give applications to upper and lower bounds on $\lambda_{p,j}$, the j-th eigenvalue of the p-form Laplacian counted with multiplicity, in terms of sectional curvature and diameter. To do so, we use Gromov's precompactness theorem [17, Chapter 5]. Suppose that M is a smooth connected closed n-dimensional manifold and $\{g_i\}_{i=1}^\infty$ is a sequence of Riemannian metrics on M of uniformly bounded sectional curvature and diameter. Gromov's theorem implies that a subsequence of the metric spaces $\{(M, g_i)\}_{i=1}^\infty$, which we relabel as $\{(M, g_i)\}_{i=1}^\infty$, converges in the Gromov-Hausdorff topology to a compact metric space X. If we can prove that (after passing to a further subsequence) the j-th eigenvalue of the p-form Laplacian on (M, g_i) converges to the j-th eigenvalue of an appropriate operator on X, and if we can effectively analyze the operator on X, then we can obtain analytic results that are valid for all Riemannian metrics on M with given bounds on sectional curvature and diameter.

In [19] we showed that if X happens to be a smooth Riemannian manifold B then the relevant operator on B is the Laplacian associated to a flat degree-1 superconnection on B. In the general case, the limit space X may not be homeomorphic to a manifold. However, Fukaya showed that X is the quotient of a manifold by the action of a compact Lie group [16]. Namely, a Riemannian metric g on M induces a canonical Riemannian metric $\check{g}$ on the orthonormal frame bundle FM [16, Section 1]. The sectional curvature and diameter bounds on $\{g_i\}_{i=1}^\infty$ imply similar bounds on $\{\check{g}_i\}_{i=1}^\infty$. Then $\{(FM, \check{g}_i)\}_{i=1}^\infty$ has a subsequence which converges to a Riemannian manifold $\check{X}$ in the $O(n)$-equivariant Gromov-Hausdorff topology, and $X = \check{X}/O(n)$.

Hence one may hope to construct the relevant operator on X by working equivariantly on $\check{X}$. This is in analogy to what Fukaya did in the case of the function Laplacian [15, Section 7].

Date: February 21, 2002.

1991 *Mathematics Subject Classification.* Primary: 58G25; Secondary: 53C23.

Research supported by NSF grant DMS-0072154.

In this paper we carry out this program, using in part geometric results of Cheeger, Fukaya and Gromov [7]. In effect we construct superconnection Laplacians on the singular space X, and prove convergence and compactness properties of the operators. We now give some of the consequences.

Definition 1. *If M is a connected closed smooth manifold of positive dimension, let $\mathcal{M}(M,K)$ be the set of Riemannian metrics on M with $\mathrm{diam}(M) \leq 1$ and $\parallel R^M \parallel_\infty \leq K$. Let $\mathcal{M}^1(M,K)$ be the subset with $\mathrm{diam}(M) = 1$. Put*

$$a_{p,j,K}(M) = \inf_{\mathcal{M}(M,K)} \{\lambda_{p,j}(M)\}, \qquad a^1_{p,j,K}(M) = \inf_{\mathcal{M}^1(M,K)} \{\lambda_{p,j}(M)\}, \tag{1.1}$$

$$A^1_{p,j,K}(M) = \sup_{\mathcal{M}^1(M,K)} \{\lambda_{p,j}(M)\}.$$

For $n \in \mathbb{Z}^+$, put

$$a^1_{n,p,j,K} = \inf_M \{a^1_{p,j,K}(M)\}, \qquad A^1_{n,p,j,K} = \sup_M \{A^1_{p,j,K}(M)\}, \tag{1.2}$$

where M ranges over connected closed smooth n-dimensional manifolds.

Our first result says that there is a uniform lower bound on $\lambda_{p,j}$ in terms of sectional curvature and diameter. More importantly, we also characterize when there is not a uniform upper bound.

Theorem 1. *1.* $\lim_{j\to\infty} a^1_{n,p,j,K} = \infty$.
2. If $p > 1$ and $A^1_{p,j,K}(M) = \infty$ then M collapses with bounded sectional curvature and bounded diameter to a limit space X satisfying $1 \leq \mathrm{dim}(X) \leq p-1$. Furthermore, there is a (possibly singular) affine fiber bundle $M \to X$ whose generic fiber Z does not admit any nonzero affine-parallel k-forms for $p - \mathrm{dim}(X) \leq k \leq p$.
3. If $p \in \{0,1\}$ then $A^1_{n,p,j,K} < \infty$.

Theorem 1.1 also follows from heat kernel bounds ([2] and references therein), with explicit estimates. Our proof, which just uses collapsing arguments, is given by way of illustration. We remark that the Ricci-analog of Theorem 1.1 is false, as the Betti numbers can be arbitrarily large [1, Theorem 0.4]. Theorem 1.3 also follows from [9], with explicit estimates.

Theorem 1 of [19] gives a partial converse to Theorem 1.2, in the sense that if M is the total space of an affine fiber bundle [19, Definition 1] over a smooth manifold B with $1 \leq \mathrm{dim}(B) \leq p-1$, and the fiber Z does not admit any nonzero affine-parallel k-forms for $p - \mathrm{dim}(B) \leq k \leq p$, then $A^1_{p,j,K}(M) = \infty$.

Theorem 1.2 implies the following characterization of manifolds M which do not have uniform upper bounds on the eigenvalues of the 2-form Laplacian.

Theorem 2. *If $A^1_{2,j,K}(M) = \infty$ then*
1. There is an almost flat manifold Z which does not admit any nonzero affine-parallel 1-forms or 2-forms, and an affine diffeomorphism $\phi \in \mathrm{Aff}(Z)$ such that M is diffeomorphic to the mapping torus of ϕ, or
2. There are almost flat manifolds Z, Z_1, Z_2 which do not admit any nonzero affine-parallel 1-forms or 2-forms, and surjective affine maps $\phi_i : Z \to Z_i$ such that M is homeomorphic to the double mapping cylinder $\mathrm{cyl}(\phi_1) \cup_Z \mathrm{cyl}(\phi_2)$.

The next result says that there are at most $\mathrm{b}_1(M) + \dim(M)$ small eigenvalues of the 1-form Laplacian. It is an extension of [19, Corollary 1].

Theorem 3. *If $a_{1,j,K}(M) = 0$ then $j \leq \mathrm{b}_1(M) + \dim(M)$. More precisely, suppose that $a_{1,j,K}(M) = 0$ and $j > \mathrm{b}_1(M)$. Let X be the limit space coming from the collapsing argument. Then*

$$j \leq \mathrm{b}_1(X) + \dim(M) - \dim(X) \leq \mathrm{b}_1(M) + \dim(M). \tag{1.3}$$

The first inequality in (1.3) is sharp, for example, in the case of the Berger collapse of S^3 to S^2 [10, Example 1.2].

Next, we give a bound on the number of small eigenvalues of the p-form Laplacian for a manifold which is Gromov-Hausdorff close to a codimension-1 space X. It is an extension of [19, Corollary 2] and [11, Théorème 1.17].

Theorem 4. *Let X be a connected closed $(n-1)$-dimensional Riemannian orbifold. Then for any $K \geq 0$, there are $\delta, c > 0$ with the following property : Suppose that M is a connected closed smooth n-dimensional Riemannian manifold with $\| R^M \|_\infty \leq K$ and $d_{GH}(M, X) < \delta$. First, M is the total space of an orbifold circle bundle over X. Let $\mathcal{O}$ be the orientation bundle of $M \to X$, a flat real line bundle on X in the orbifold sense. Then $\lambda_{p,j}(M, g) > c$ for $j = \mathrm{b}_p(X) + \mathrm{b}_{p-1}(X; \mathcal{O}) + 1$.*

In [19, Definition 3] we defined the notion of a collapsing sequence of metrics $\{g_i\}_{i=1}^\infty \subset \mathcal{M}(M, K)$ with a smooth limit space B. This means that there are an affine fiber bundle $M \to B$ and an $\epsilon > 0$ such that each (M, g_i) is ϵ-biLipschitz to a model metric on the total space of the affine fiber bundle. We remark that for a given $\epsilon > 0$, results of Cheeger, Fukaya and Gromov imply that if $(M, g) \in \mathcal{M}(M, K)$ is sufficiently close to B in the Gromov-Hausdorff topology then (M, g) is ϵ-biLipschitz to a model metric on some affine fiber bundle $M \to B$ [7, Proposition 4.9]. Hence the content of our assumption is that there is a single affine fiber bundle involved for all of the g_i's.

There is an extension of the notion of a collapsing sequence to the case of a singular limit space X. Namely, a collapsing sequence consists of a sequence $\{g_i\}_{i=1}^\infty \subset \mathcal{M}(M, K)$ and an $O(n)$-equivariant affine fiber bundle $FM \to \check{X}$ such that $\{\check{g}_i\}_{i=1}^\infty$ is a $O(n)$-equivariant collapsing sequence on FM. Given what is proven in this paper, the results proved in [19] concerning collapsing sequences, with smooth limit space B, extend to results concerning collapsing sequences with limit space X. We state one such result, which is an extension of [19, Theorem 5]. It says that there are three mechanisms to make small positive eigenvalues of the differential form Laplacian in a collapsing sequence. Either the differential form Laplacian on the generic fiber of the map $M \to X$ admits small positive eigenvalues, or the pushforward "cohomology" sheaf on X fails to be semisimple, or the Leray spectral sequence of the map $M \to X$ does not degenerate at the E_2-term.

Theorem 5. *Let $\{(M, g_i)\}_{i=1}^\infty$ be a collapsing sequence with limit X. Suppose that $\lim_{i\to\infty} \lambda_{p,j}(M, g_i) = 0$ for some $j > \mathrm{b}_p(M)$. Write the generic fiber Z of the map $M \to X$ as the quotient of a nilmanifold $\widehat{Z} = \widehat{\Gamma}\backslash N$ by a finite group F. Then*
1. For some $q \in [0, p]$, $\mathrm{b}_q(Z) < \dim\left(\Lambda^q(\mathfrak{n}^*)^F\right)$, or
2. For all $q \in [0, p]$, $\mathrm{b}_q(Z) = \dim\left(\Lambda^q(\mathfrak{n}^*)^F\right)$, and for some $q \in [0, p]$, the "cohomology" sheaf $\mathrm{H}^q(Z; \mathbb{R})$ on X has a subsheaf which is not a direct summand, or

3. For all $q \in [0,p]$, $\mathrm{b}_q(Z) = \dim\left(\Lambda^q(\mathfrak{n}^*)^F\right)$ *and each subsheaf of the "cohomology" sheaf* $\mathrm{H}^q(Z;\mathbb{R})$ *on* X *is a direct summand, and the Leray spectral sequence to compute* $\mathrm{H}^p(M;\mathbb{R})$ *does not degenerate at the* E_2*-term.*

As one consequence of Theorem 5, we obtain a characterization of when the p-form Laplacian has small positive eigenvalues in a collapsing sequence over a codimension-1 space. It is an extension of [19, Corollary 5] and [11, Théorème 1.17].

Theorem 6. *Let* $\{(M,g_i)\}_{i=1}^{\infty}$ *be a collapsing sequence associated to a limit space* X *with* $\dim(X) = \dim(M) - 1$*. Suppose that* $\lim_{i\to\infty} \lambda_{p,j}(M,g_i) = 0$ *for some* $j > \mathrm{b}_p(M)$*. Let* $\mathcal{O}$ *be the orientation bundle of the orbifold circle bundle* $M \to X$*, a flat real line bundle on the orbifold* X *in the orbifold sense. Let* $\chi \in \mathrm{H}^2(X;\mathcal{O})$ *be the Euler class of the orbifold circle bundle* $M \to X$*. Let* $\mathcal{M}_\chi$ *be multiplication by* χ*. Then* $\mathcal{M}_\chi : \mathrm{H}^{p-1}(X;\mathcal{O}) \to \mathrm{H}^{p+1}(X;\mathbb{R})$ *is nonzero or* $\mathcal{M}_\chi : \mathrm{H}^{p-2}(X;\mathcal{O}) \to \mathrm{H}^{p}(X;\mathbb{R})$ *is nonzero.*

The structure of the paper is as follows. Section 2 deals with the construction of the Laplacian associated to a flat degree-1 superconnection on a singular space of the form $X = \check{X}/G$, where $\check{X}$ is a smooth closed Riemannian manifold and G is a closed subgroup of $\mathrm{Isom}(\check{X})$. In Section 3 we prove Theorems 1 and 2. Section 4 uses the compactness results to prove Theorems 3-6. More detailed descriptions appear at the beginnings of the sections

As the present paper is a sequel to [19], we sometimes make reference to the relevant sections of [19]. As for notation in this paper, if G is a group which acts on a set X, we let X^G denote the set of fixed-points. If B is a smooth manifold and E is a smooth vector bundle on B, we let $\Omega(B;E)$ denote the smooth E-valued differential forms on B. If $\mathfrak{n}$ is a nilpotent Lie algebra on which a finite group F acts by automorphisms then $\mathfrak{n}^*$ denotes the dual space, $\Lambda^*(\mathfrak{n}^*)$ denotes the exterior algebra of the dual space and $\Lambda^*(\mathfrak{n}^*)^F$ denotes the F-invariant subspace of the exterior algebra.

2. Basic Laplacian

In this section we construct differential form Laplacians on a certain class of singular spaces, namely those of the form $X = \check{X}/G$ where $\check{X}$ is a smooth closed Riemannian manifold and G is a closed subgroup of $\mathrm{Isom}(\check{X})$. Let $\check{E}$ be a G-equivariant vector bundle on $\check{X}$. We consider the space of basic $\check{E}$-valued forms $\Omega_{basic}(\check{X};\check{E})$. If $\check{A}'$ is a basic flat degree-1 superconnection on $\check{E}$ then we construct the corresponding Laplacian $\triangle^E$ as an operator on $\Omega_{basic}(\check{X};\check{E})$. Although it is not strictly necessary for this paper, we also give a more intrinsic formulation of $\triangle^E$ as an operator on a space $\Omega(X;E)$ of forms on the quotient space X. We then describe a spectral sequence to compute the cohomology of $\check{A}'$.

We remark that the spaces $X = \check{X}/G$ can be quite singular. For example, if G is finite then one finds the well-known orbifold singularities, whereas if G has positive dimension then the singularities of X can be much worse. In view of the well-known difficulties in doing analysis on singular spaces, one may wonder how we can construct reasonable operators on such spaces. The point is that there are special features of the present situation. For example, there is an induced measure on X, the pushforward measure from $\check{X}$, which has the effect of mollifying the singularities.

Let $\check{X}$ be a smooth connected closed Riemannian manifold on which a compact Lie group G acts isometrically on the right. Let $\mathfrak{g}$ denote the Lie algebra of G.

The G-action partitions $\check{X}$ into smooth submanifolds $\check{X}^{[H]}$, where $[H]$ runs over the conjugacy classes of closed subgroups of G, and where $\check{x} \in \check{X}^{[H]}$ if the isotropy subgroup $G_{\check{x}}$ is in the conjugacy class $[H]$. The (connected components of the) $\check{X}^{[H]}$'s induce a stratification of the quotient space $X = \check{X}/G$ as $X = \bigcup_{[H]} X^{[H]}$, where $X^{[H]} = \check{X}^{[H]}/G$. The orbits of G on $\check{X}^{[H]}$ are all diffeomorphic to $H\backslash G$. In fact, $\check{X}^{[H]}$ is a fiber bundle over $X^{[H]}$ with fiber $H\backslash G$. To describe the structure group of this fiber bundle, note that the "internal symmetry group" $(\mathrm{Diff}(H\backslash G))^G$ of a fiber $H\backslash G$ is isomorphic to $H\backslash N_H(G)$, where $N_H(G)$ is the normalizer of H in G. (An element $n \in N_H(G)$ sends Hg to Hng.) Then the structure group of the fiber bundle $\check{X}^{[H]} \to X^{[H]}$ is contained in $H\backslash N_H(G)$. To see this more explicitly, put $\check{X}^{(H)} = \{\check{x} \in \check{X} : G_{\check{x}} = H\}$. Then $H\backslash N_H(G)$ acts freely on $\check{X}^{(H)}$. In fact, $\check{X}^{(H)}$ is a principal $H\backslash N_H(G)$-bundle. Then $\check{X}^{[H]} = \check{X}^{(H)} \times_{H\backslash N_H(G)} (H\backslash G)$ and $X^{[H]} = \check{X}^{(H)}/(H\backslash N_H(G))$.

There is another stratification of X, introduced in [13], which is more convenient for our purposes. It keeps track of both the connected components of $\check{X}^{[H]}$ and their normal bundles in $\check{X}$. We briefly review the setup of [13]. Consider pairs (H, V) where H is a closed subgroup of G and V is a real representation space of H with no trivial subrepresentations, i.e. $V^H = \{0\}$. There is a natural G-action on such pairs and the equivalence classes are called the normal orbit types $[H, V]$. Given a point $\check{x} \in \check{X}$, the differentiable slice theorem says that there is a real representation space $W_{\check{x}}$ of $G_{\check{x}}$ so that a neighborhood of $\check{x} \cdot G$ is G-diffeomorphic to $W_{\check{x}} \times_{G_{\check{x}}} G$. Put $V_{\check{x}} = W_{\check{x}}/W_{\check{x}}^{G_{\check{x}}}$. Then $[G_{\check{x}}, V_{\check{x}}]$ is called the normal orbit type of $\check{x}$. Given a normal orbit type α, put

$$\check{X}_\alpha = \left\{ \check{x} \in \check{X} : [G_{\check{x}}, V_{\check{x}}] = \alpha \right\}. \tag{2.1}$$

Then $\check{X}_\alpha$ is a smooth G-submanifold of $\check{X}$. Let $\mathcal{N}$ be the set of normal orbit types α such that $\check{X}_\alpha$ is nonempty. For $\alpha \in \mathcal{N}$, put $X_\alpha = \check{X}_\alpha/G$, a smooth Riemannian manifold. Then X is stratified by $\{X_\alpha\}_{\alpha \in \mathcal{N}}$. Once again, $\check{X}_\alpha$ is a fiber bundle over X_α with fiber $H\backslash G$.

As G acts isometrically, we may assume that for each normal orbit type $\alpha = [H, V]$, V is given an H-invariant inner product. Let SV denote the corresponding sphere in V.

Given a normal orbit type α, fix a representative (H, V). Consider the diagonal inclusion of H in $O(V) \times G$. Let $N_H(O(V) \times G)$ be the normalizer of H in $O(V) \times G$. Let ν_α be the normal bundle of $\check{X}_\alpha$ in $\check{X}$. The restriction of ν_α to a fiber $H\backslash G$ of $\check{X}_\alpha \to X_\alpha$ is isomorphic to the Euclidean vector bundle $(V \times_H G) \to (H\backslash G)$. The "internal symmetry group" of this vector bundle, i.e. the group of vector bundle automorphisms which commute with the G-action, is $S_\alpha \equiv H\backslash N_H(O(V) \times G)$. Correspondingly, there is a certain principal S_α-bundle P_α such that

$$\nu_\alpha = P_\alpha \times_{S_\alpha} (V \times_H G). \tag{2.2}$$

Using the normal exponential map, there is a neighborhood of $\check{X}_\alpha$ in $\check{X}$ which is G-diffeomorphic to ν_α. In addition,

$$\check{X}_\alpha = P_\alpha \times_{S_\alpha} (H\backslash G), \tag{2.3}$$

$$X_\alpha = P_\alpha / S_\alpha \tag{2.4}$$

and there is a neighborhood N_α of X_α in X whose homeomorphism type is

$$N_\alpha \cong P_\alpha \times_{S_\alpha} (V/H). \tag{2.5}$$

(Note that N_α and V/H are generally not manifolds.)

There is a partial ordering on normal orbit types given by saying that $[H, V] \le [H', V']$ if the G-manifold $V \times_H G$ contains a G-orbit of type $[H', V']$. This induces a partial ordering on the strata of X, with $\alpha \le \alpha'$ if and only if $X_\alpha \subset \overline{X'_\alpha}$.

Let $\check{E} = \oplus_{j=0}^m \check{E}^j$ be a $\mathbb{Z}$-graded real vector bundle on $\check{X}$. We assume that the action of G on $\check{X}$ is covered by an action on $\check{E}$ which preserves the $\mathbb{Z}$-grading. Let $\mathfrak{g}$ be the Lie algebra of G. We say that $\check{E}$ is G-basic if it is equipped with a G-equivariant linear map $\mathfrak{I} : \mathfrak{g} \to C^\infty\left(\check{X}; \mathrm{Hom}(\check{E}^*, \check{E}^{*-1})\right)$ which satisfies $\mathfrak{I}(\mathfrak{x})^2 = 0$ for all $\mathfrak{x} \in \mathfrak{g}$. Given $\mathfrak{x} \in \mathfrak{g}$, we write $\mathfrak{I}_\mathfrak{x}$ for $\mathfrak{I}(\mathfrak{x})$.

Given $\mathfrak{x} \in \mathfrak{g}$, let $\mathfrak{X}$ be the corresponding vector field on $\check{X}$ and let $i_\mathfrak{X}$ be interior multiplication by $\mathfrak{X}$ on $\Omega\left(\check{X}; \check{E}\right)$.

Definition 2. *Put*

$$\Omega_G\left(\check{X}; \check{E}\right) = \{\check{\omega} \in \Omega\left(\check{X}; \check{E}\right) : g \cdot \check{\omega} = \check{\omega} \text{ for all } g \in G\} \tag{2.6}$$

and

$$\Omega_{basic}\left(\check{X}; \check{E}\right) = \{\check{\omega} \in \Omega_G\left(\check{X}; \check{E}\right) : (i_\mathfrak{X} + \mathfrak{I}_\mathfrak{x})\check{\omega} = 0 \text{ for all } \mathfrak{x} \in \mathfrak{g}\}. \tag{2.7}$$

We give $\Omega_{basic}\left(\check{X}; \check{E}\right)$ the total $\mathbb{Z}$-grading coming from the $\mathbb{Z}$-gradings on $\Omega^*(\check{X})$ and $\check{E}$. Similarly, we let $\Omega_{basic}\left(\check{X}; \mathrm{End}(\check{E})\right)$ be the G-invariant elements $\check{\eta}$ of $\Omega\left(\check{X}; \mathrm{End}(\check{E})\right)$ which satisfy $i_\mathfrak{X}\, \check{\eta} + \mathfrak{I}_\mathfrak{x}\, \check{\eta} + (-1)^{|\check{\eta}|}\, \check{\eta}\, \mathfrak{I}_\mathfrak{x} = 0$.

Let $h^{\check{E}}$ be a G-invariant graded Euclidean inner product on $\check{E}$. We obtain L^2-inner products on $\Omega_G\left(\check{X}; \check{E}\right)$ and $\Omega_{basic}\left(\check{X}; \check{E}\right)$. Let $\Omega_{G,L^2}\left(\check{X}; \check{E}\right)$ and $\Omega_{basic,L^2}\left(\check{X}; \check{E}\right)$ be the L^2-completions of $\Omega_G\left(\check{X}; \check{E}\right)$ and $\Omega_{basic}\left(\check{X}; \check{E}\right)$, respectively. Let $P^{hor} : \Omega_{G,L^2}\left(\check{X}; \check{E}\right) \to \Omega_{basic,L^2}\left(\check{X}; \check{E}\right)$ be orthogonal projection.

Definition 3. *Let $\check{E}$ be a real G-basic vector bundle on $\check{X}$. A basic connection on $\check{E}$ is a connection $\nabla^{\check{E}}$ on $\check{E}$ which is G-invariant and satisfies the property that for all $\mathfrak{x} \in \mathfrak{g}$, $\nabla^{\check{E}}(i_\mathfrak{X} + \mathfrak{I}_\mathfrak{x}) + (i_\mathfrak{X} + \mathfrak{I}_\mathfrak{x})\nabla^{\check{E}}$ equals Lie differentiation $\mathcal{L}_\mathfrak{X}$ with respect to $\mathfrak{X}$ on $\Omega\left(\check{X}; \check{E}\right)$.*

If $\nabla^{\check{E}}$ is a basic connection then $\nabla^{\check{E}} i_\mathfrak{X} + i_\mathfrak{X} \nabla^{\check{E}} = \mathcal{L}_\mathfrak{X}$ and $\nabla^{\check{E}} \mathfrak{I}_\mathfrak{x} + \mathfrak{I}_\mathfrak{x} \nabla^{\check{E}} = 0$, i.e. $\nabla^{\check{E}}$ is basic in the usual sense and $\mathfrak{I}_\mathfrak{x}$ is covariantly-constant with respect to $\nabla^{\check{E}}$. A basic flat connection is a connection which is both basic and flat.

We wish to describe the set of basic flat connections on $\check{X}$ in terms of a representation variety. To do so, we need the correct analog of the fundamental group of $\check{X}/G$. Let $\widehat{X}$ be the universal cover of $\check{X}$, with projection $q : \widehat{X} \to \check{X}$. Put

$$\widehat{G} = \{(\phi, g) \in \mathrm{Diff}(\widehat{X}) \times G : q \circ \phi = g \cdot q\}. \tag{2.8}$$

There is an exact sequence of groups

$$1 \longrightarrow \pi_1(\check{X}) \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1. \tag{2.9}$$

The corresponding homotopy exact sequence of spaces gives

$$\ldots \to 1 \to \pi_1(\widehat{G}) \to \pi_1(G) \to \pi_1(\check{X}) \to \pi_0(\widehat{G}) \to \pi_0(G) \to 1. \tag{2.10}$$

The next proposition is implicit in [5, Section 4a].

Proposition 1. *Suppose that for all $\mathfrak{x} \in \mathfrak{g}$, $\mathfrak{I}_{\mathfrak{x}} = 0$. Then there is a bijection between* $\mathrm{Hom}\left(\pi_0(\widehat{G}), \mathrm{GL}(N,\mathbb{R})\right)/\mathrm{GL}(N,\mathbb{R})$ *and the basic flat connections on rank-N G-vector bundles over $\check{X}$, up to G-equivariant gauge equivalence.*

Proof. Given $\rho \in \mathrm{Hom}\left(\pi_0(\widehat{G}), \mathrm{GL}(N,\mathbb{R})\right)$, let $\overline{\rho}$ be the restriction of ρ to $\pi_1(\check{X})$. Let $\rho_0 : \widehat{G} \to \pi_0(\widehat{G}) \xrightarrow{\rho} \mathrm{GL}(N,\mathbb{R})$ be the composite homomorphism. Put $\check{E} = \widehat{X} \times_{\overline{\rho}} \mathbb{R}^N$. Then $\check{E}$ is a flat vector bundle on $\check{X}$. If $\widehat{x} \in \widehat{X}$, $v \in \mathbb{R}^N$ and $\widehat{g} \in \widehat{G}$, put

$$(\widehat{x}, v) \cdot \widehat{g} = (\widehat{x} \cdot \widehat{g}, \rho_0(\widehat{g}^{-1})v). \tag{2.11}$$

Then this action of $\widehat{G}$ on $\widehat{X} \times \mathbb{R}^N$ extends that of the normal subgroup $\pi_1(\check{X})$ of $\widehat{G}$. Hence the quotient group G acts on $\check{E}$. As the representation ρ_0 factorizes through ρ, we see that the flat connection on $\check{E}$ is basic. Conjugate representations ρ give gauge-equivalent basic flat connections.

Conversely, let $\nabla^{\check{E}}$ be a basic flat connection on a rank-N G-vector bundle $\check{E}$. Putting $\widehat{E} = q^* \check{E}$, there is a trivialization $\widehat{E} \cong \widehat{X} \times \mathbb{R}^N$. The action of G on $\check{E}$ lifts to an action of $\widehat{G}$ on $\widehat{E}$. In this way we get a homomorphism $\rho_0 : \widehat{G} \to \mathrm{GL}(N,\mathbb{R})$. As $\nabla^{\check{E}}$ is basic, it follows that ρ_0 factors through a representation $\rho : \pi_0(\widehat{G}) \to \mathrm{GL}(N,\mathbb{R})$. Taking into account the ambiguity in the trivialization of $\widehat{E}$, we obtain a well-defined conjugacy class of ρ. ☐

If $\mathfrak{I}_{\mathfrak{x}}$ is not identically zero then we can describe the basic flat connections as the subset of the connections in Proposition 1 with respect to which $\mathfrak{I}_{\mathfrak{x}}$ is covariantly-constant for all $\mathfrak{x} \in \mathfrak{g}$.

For background information on superconnections, we refer to [3, Chapter 1.4].

Definition 4. *A basic superconnection on $\check{E}$ is a superconnection $\check{A}'$ on $\check{E}$ which is G-invariant and satisfies the property that for all $\mathfrak{x} \in \mathfrak{g}$, $\check{A}'(i_{\mathfrak{X}} + \mathfrak{I}_{\mathfrak{x}}) + (i_{\mathfrak{X}} + \mathfrak{I}_{\mathfrak{x}})\check{A}'$ equals Lie differentiation with respect to $\mathfrak{X}$ on $\Omega\left(\check{X}; \check{E}\right)$.*

A basic superconnection on $\check{E}$ restricts to a superconnection on $\Omega_{basic}\left(\check{X}; \check{E}\right)$. Let $\check{A}'$ be a basic superconnection on $\check{E}$ which is "flat degree-1" in the sense of [5, Section II(a)] and [19, Section 5]. Let $\left(\check{A}'\right)^*$ be its formal adjoint on $\Omega_G\left(\check{X}; \check{E}\right)$, which is the restriction of the formal adjoint on $\Omega\left(\check{X}; \check{E}\right)$.

Lemma 1. *The formal adjoint of $\check{A}'$ on $\Omega_{basic}\left(\check{X}; \check{E}\right)$ is*

$$\left(\check{A}'\right)^*_{basic} = P^{hor}\left(\check{A}'\right)^*. \tag{2.12}$$

Proof. Given $\check{\omega}, \check{\omega}' \in \Omega_{basic}\left(\check{X}; \check{E}\right)$, we have

$$\langle \check{\omega}, \check{A}'\check{\omega}' \rangle = \langle \left(\check{A}'\right)^* \check{\omega}, \check{\omega}' \rangle = \langle \left(\check{A}'\right)^* \check{\omega}, P^{hor}\check{\omega}' \rangle = \langle P^{hor}\left(\check{A}'\right)^* \check{\omega}, \check{\omega}' \rangle. \tag{2.13}$$

As $P^{hor}\left(\check{A}'\right)^* \check{\omega} \in \Omega_{basic,L^2}\left(\check{X}; \check{E}\right)$, the lemma follows. ☐

Put $\Omega_{basic,max}(\check{X}; \check{E}) = \left\{\check{\omega} \in \Omega_{basic,L^2}(\check{X}; \check{E}) : \check{A}'\check{\omega} \in \Omega_{basic,L^2}(\check{X}; \check{E})\right\} \subset \Omega_{basic,L^2}(\check{X}; \check{E})$, where $\check{A}'\check{\omega}$ is originally defined as a distribution.

Lemma 2. *The operator $\check{A}'$ is closed on $\Omega_{basic,max}(\check{X}; \check{E})$.*

Proof. Suppose that $\{\breve{\omega}_i\}_{i=1}^{\infty}$ is a sequence in $\Omega_{basic,max}(\breve{X};\breve{E})$ such that $\lim_{i\to\infty}\breve{\omega}_i = \breve{\omega}$ in $\Omega_{basic,L^2}(\breve{X};\breve{E})$ and $\lim_{i\to\infty}\breve{A}'\breve{\omega}_i = \breve{\eta}$ in $\Omega_{basic,L^2}(\breve{X};\breve{E})$. Given $\breve{\phi}\in\Omega\left(\breve{X};\breve{E}\right)$, we have

$$\langle\breve{\phi},\breve{\eta}\rangle = \lim_{i\to\infty}\langle\breve{\phi},\breve{A}'\breve{\omega}_i\rangle = \lim_{i\to\infty}\langle(\breve{A}')^*\breve{\phi},\breve{\omega}_i\rangle = \langle(\breve{A}')^*\breve{\phi},\breve{\omega}\rangle. \tag{2.14}$$

It follows that $\breve{\omega}\in\Omega_{basic,max}(\breve{X};\breve{E})$ with $\breve{A}'\,\breve{\omega} = \breve{\eta}$. This proves the lemma. □

Let $\left(\breve{A}'\right)^*_{basic}$ be the adjoint operator to $\breve{A}'$, the latter being defined on $\Omega_{basic,max}(\breve{X};\breve{E})$. As $\mathrm{Im}\left(\breve{A}'\right)\subset\mathrm{Ker}\left(\breve{A}'\right)$, a formal result of functional analysis [18, Lemma 4.3] implies that there is an orthogonal decomposition

$$\Omega_{basic,L^2}(\breve{X};\breve{E}) = \left(\mathrm{Ker}(\breve{A}')\cap\mathrm{Ker}((\breve{A}')^*_{basic})\right)\oplus\frac{\Omega_{basic,L^2}(\breve{X};\breve{E})}{\mathrm{Ker}(\breve{A}')}\oplus\frac{\Omega_{basic,L^2}(\breve{X};\breve{E})}{\mathrm{Ker}((\breve{A}')^*_{basic})}. \tag{2.15}$$

Furthermore, $\breve{A}' + (\breve{A}')^*_{basic}$ is self-adjoint on $\Omega_{basic,L^2}(\breve{X};\breve{E})$.

Definition 5. *The basic Laplacian is* $\triangle^{\breve{E}}_{basic} = \left(\breve{A}' + (\breve{A}')^*_{basic})\right)^2$.

In terms of (2.15), the domain of $\triangle^{\breve{E}}_{basic}$ is

$$\begin{aligned}\left(\mathrm{Ker}(\breve{A}')\cap\mathrm{Ker}((\breve{A}')^*_{basic})\right)\oplus\frac{\{\breve{\omega}\in\mathrm{Dom}(\breve{A}')\ :\ \breve{A}'\breve{\omega}\in\mathrm{Dom}((\breve{A}')^*_{basic})\}}{\mathrm{Ker}(\breve{A}')}\\ \oplus\frac{\{\breve{\omega}\in\mathrm{Dom}((\breve{A}')^*_{basic})\ :\ (\breve{A}')^*_{basic}\breve{\omega}\in\mathrm{Dom}(\breve{A}')\}}{\mathrm{Ker}((\breve{A}')^*_{basic})},\end{aligned} \tag{2.16}$$

on which $\triangle^{\breve{E}}_{basic}$ acts by $0 + (\breve{A}')^*_{basic}\breve{A}' + \breve{A}'(\breve{A}')^*_{basic}$. The quadratic form Q associated to $\triangle^{\breve{E}}_{basic}$ has domain

$$\mathrm{Dom}(Q) = \{\breve{\omega}\in\Omega_{basic,L^2}(\breve{X};\breve{E})\ :\ \breve{A}'\breve{\omega}\in\Omega_{basic,L^2}(\breve{X};\breve{E})\text{ and }(\breve{A}')^*_{basic}\breve{\omega}\in\Omega_{basic,L^2}(\breve{X};\breve{E})\}, \tag{2.17}$$

on which it is defined by

$$Q(\breve{\omega}) = \langle\breve{A}'\breve{\omega},\breve{A}'\breve{\omega}\rangle + \langle(\breve{A}')^*_{basic}\breve{\omega},(\breve{A}')^*_{basic}\breve{\omega}\rangle. \tag{2.18}$$

We can also describe $\triangle^{\breve{E}}_{basic}$ when restricted to $\mathrm{Im}(\breve{A}')^\perp$ in terms of a Friedrichs extension. Namely, using Lemma 1, we have that $(\breve{A}')^*_{basic}\breve{A}'$ maps $\Omega_{basic}(\breve{X};\breve{E})$ to $\Omega_{basic,L^2}(\breve{X};\breve{E})$. Then the Friedrichs extension [20, Theorem X.23] of the positive symmetric operator $(\breve{A}')^*_{basic}\breve{A}'$, acting on $\Omega_{basic}(\breve{X};\breve{E})$, is well-defined, coincides with $\triangle^{\breve{E}}_{basic}$ on

$$\mathrm{Im}(\breve{A}')^\perp = \left(\mathrm{Ker}(\breve{A}')\cap\mathrm{Ker}((\breve{A}')^*_{basic})\right)\oplus\frac{\Omega_{basic,L^2}(\breve{X};\breve{E})}{\mathrm{Ker}(\breve{A}')} \tag{2.19}$$

and vanishes on $\frac{\Omega_{basic,L^2}(\breve{X};\breve{E})}{Ker((\breve{A}')^*_{basic})}$.

Using the orthogonal decomposition (2.15), we have that $\mathrm{Ker}(\triangle^{\breve{E}}_{basic}) = \mathrm{Ker}(\breve{A}')/\overline{\mathrm{Im}(\breve{A}')}$. If $\breve{A}'$ has a closed image then $\mathrm{Ker}(\triangle^{\breve{E}}_{basic})$ equals the usual cohomology $\mathrm{Ker}(\breve{A}')/\mathrm{Im}(\breve{A}')$. We do not show that $\breve{A}'$ has a closed image in general, but we will see in Section 4 that it has a closed image in a case that arises in collapsing.

Let us look in more detail at the basic forms when $\breve{X} = H\backslash G$. If $\breve{E}_{He}$ denotes the fiber of $\breve{E}$ at $He\in H\backslash G$ then we can write the homogeneous vector bundle $\breve{E}$ as $\breve{E} = \breve{E}_{He}\times_H G$.

The restriction of $\mathfrak{I}$ to $\check{E}_{He}$ becomes an H-equivariant map $\mathfrak{I} : \mathfrak{g} \rightarrow \mathrm{Hom}\left(\check{E}^*_{He}, \check{E}^{*-1}_{He}\right)$. Let $\mathfrak{h}$ be the Lie algebra of H. Put $\check{K} = \bigcap_{x \in \mathfrak{h}} \mathrm{Ker}\,(\mathfrak{I}_{\mathfrak{x}})$. Let $\{\mathfrak{x}_j\}$ be a basis of $\mathfrak{g}/\mathfrak{h}$ and let $\{\mathfrak{x}^*_j\}$ be the dual basis of $(\mathfrak{g}/\mathfrak{h})^*$. If $\check{\omega}$ is a basic form then as $((i_{\mathfrak{X}} + \mathfrak{I}_x)\check{\omega})\,(He) = 0$ for all $\mathfrak{x} \in \mathfrak{g}$, it follows that $\check{\omega}(He) = \prod_j (1 - e(\mathfrak{x}^*_j)\,\mathfrak{I}_{\mathfrak{x}_j})\,\check{\rho}$ for some $\check{\rho} \in \check{K}^H$. Here $e(\mathfrak{x}^*_j)$ denotes exterior multiplication by $\mathfrak{x}^*_j$ and $\mathfrak{I}_{\mathfrak{x}_j}\check{\rho}$ is well-defined since $\check{\rho} \in \check{K}$. The value of $\check{\omega}$ on the rest of $H\backslash G$ is determined by G-invariance. In this way, we see that $\Omega_{basic}(H\backslash G; \check{E}) \cong \check{K}^H$.

To extend this to general $\check{X}$, suppose that $\check{x} \in \check{X}_\alpha$ has isotropy group H, with Lie algebra $\mathfrak{h}$. Put

$$\check{K}_{\check{x}} = \bigcap_{\mathfrak{x} \in \mathfrak{h}} \mathrm{Ker}\,(\mathfrak{I}_{\mathfrak{x}}(\check{x}))\,. \tag{2.20}$$

A priori $\{\check{K}_{\check{x}}\}_{x \in \check{X}_\alpha}$ may not form a vector bundle on $\check{X}_\alpha$, due to possible jumps in the dimension. Hereafter we make the assumption that for each closed subgroup H of G, $\left\{\bigcap_{\mathfrak{x} \in \mathfrak{h}} \mathrm{Ker}\,(\mathfrak{I}_{\mathfrak{x}}(\check{x}))\right\}_{\check{x} \in \check{X}^H}$ forms a vector bundle on $\check{X}^H$; this is the case that will arise in collapsing. It then follows from G-equivariance that $\{\check{K}_{\check{x}}\}_{x \in \check{X}_\alpha}$ forms a vector bundle $\check{K}_\alpha$ on $\check{X}_\alpha$. By G-equivariance, $\check{K}_\alpha$ gives rise to a $\mathbb{Z}$-graded real vector bundle E_α on X_α such that $C^\infty(X_\alpha; E_\alpha) \cong C^\infty\left(\check{X}_\alpha; \check{K}_\alpha\right)^G$. We will sometimes write E as shorthand for $\{E_\alpha\}_{\alpha \in \mathcal{N}}$. We remark that $\dim(E_\alpha)$ may not be constant in α. However, we will see that if $\alpha < \alpha'$ then $\dim(E_\alpha) \leq \dim(E_{\alpha'})$.

Given $\check{\omega} \in \Omega_{basic}\left(\check{X}; \check{E}\right)$, we obtain a collection of forms $\left\{\check{\omega}_\alpha \in \Omega_{basic}\left(\check{X}_\alpha; \check{E}_\alpha\right)\right\}_{\alpha \in \mathcal{N}}$ by pullback to the $\check{X}_\alpha$'s. Let $\check{\rho}_\alpha$ be the horizontal component of $\check{\omega}_\alpha$ with respect to the fiber bundle $\check{X}_\alpha \rightarrow X_\alpha$. That is, if $\check{x} \in \check{X}_\alpha$ has isotropy group H, let $\{\mathfrak{x}_j\}$ be a basis of $\mathfrak{g}/\mathfrak{h}$ and let $\{\mathfrak{x}^*_j\}$ be the dual basis of $(\mathfrak{g}/\mathfrak{h})^*$. Then

$$\check{\rho}_\alpha(\check{x}) = \prod_j (1 - e(\mathfrak{x}^*_j)\, i_{\mathfrak{X}_j})\, \check{\omega}_\alpha(\check{x}), \tag{2.21}$$

where $e(\mathfrak{x}^*_j)$ denotes exterior multiplication by the 1-form in $T^*_{\check{x}}\check{X}_\alpha$ represented by $\mathfrak{x}^*_j$ which is vertical with respect to the Riemannian fiber bundle $H\backslash G \rightarrow \check{X}_\alpha \rightarrow X_\alpha$. By construction, $\check{\rho}_\alpha$ is a G-invariant element of $\Omega\left(\check{X}_\alpha; \check{K}_\alpha\right)$ and satisfies $i_{\mathfrak{X}}\check{\rho}_\alpha = 0$ for all $\mathfrak{x} \in \mathfrak{g}$.

We obtain a collection of forms $\{\omega_\alpha \in \Omega\,(X_\alpha; E_\alpha)\}_{\alpha \in \mathcal{N}}$ by subsequent pushforward of the $\check{\rho}_\alpha$'s to the X_α's. We define elements of $\Omega\,(X; E)$ to be collections which so arise. Note that given $\check{\rho}_\alpha(\check{x})$, we can recover $\check{\omega}_\alpha(\check{x})$ by

$$\check{\omega}_\alpha(\check{x}) = \prod_j (1 - e(\mathfrak{x}^*_j)\,\mathfrak{I}_{\mathfrak{x}_j})\,\check{\rho}_\alpha(\check{x}). \tag{2.22}$$

In this way, there is an isomorphism between $\Omega_{basic}(\check{X}; \check{E})$ and $\Omega(X; E)$. If $\check{E}$ is the trivial $\mathbb{R}$-bundle on $\check{X}$ then we denote $\Omega(X; E)$ by $\Omega^*(X)$.

Define $v_\alpha : X_\alpha \rightarrow \mathbb{R}^+$ by saying that for $x \in X_\alpha$, $v_\alpha(x)$ is the volume in $\check{X}_\alpha$ of the G-orbit corresponding to x. Then there are inner products $\left\{h^{E_\alpha}\right\}_{\alpha \in \mathcal{N}}$ on the E_α's so that the isomorphism from $\Omega_{basic}(\check{X}_\alpha; \check{E}_\alpha)$ to $\Omega(X_\alpha; E_\alpha)$ becomes an isometry, where the inner product on $\Omega(X_\alpha; E_\alpha)$ is weighted by v_α. If β is the normal orbit type of the principal part of $\check{X}$ then define a new inner product h^E on E_β by $h^E = v_\beta \cdot h^{E_\beta}$. Let $\Omega_{L^2}(X; E)$ be the L^2-closure of $\Omega\,(X; E)$, using the Riemannian metric on X_β and h^E. There is an isometric isomorphism between $\Omega_{basic,L^2}\left(\check{X};\check{E}\right)$ and $\Omega_{L^2}(X;E)$, so we can think of $\triangle^{\check{E}}_{basic}$ as an operator $\triangle^E$ which is densely-defined on $\Omega_{L^2}(X;E)$.

In the special case when α is a normal orbit type with $V = \mathbb{R}$, we will need an additional vector bundle on X_α. (Note that this happens when $\check{X}_\alpha$ has codimension one in $\check{X}$.) If $\alpha = [H,\mathbb{R}]$ then the representation $H \to O(\mathbb{R})(= \mathbb{Z}_2)$ is necessarily onto, with kernel conjugate to H_β. Let ν_α be the normal bundle of $\check{X}_\alpha$ as described in (2.2), a real line bundle, and consider the vector bundle $\check{E}^-_\alpha = \check{E}_\alpha \otimes \nu_\alpha$ on $\check{X}_\alpha$. It is G-invariant and descends to a vector bundle E^-_α on X_α.

Although it is not strictly necessary for this paper, for explicitness we wish to describe the elements of $\Omega(X;E)$ in more conventional terms on the stratified space X. If $\alpha < \alpha'$ then there is a fiber bundle $\pi_{\alpha\alpha'} : F_{\alpha\alpha'} \to X_\alpha$ such that for a small tubular neighborhood $N_\alpha(\epsilon)$ of X_α in X, $X_\alpha \cup (X_{\alpha'} \cap N_\alpha(\epsilon))$ is homeomorphic to the mapping cylinder of $\pi_{\alpha\alpha'} : F_{\alpha\alpha'} \to X_\alpha$. To see this, let $V(\epsilon)$ denote the ϵ-ball in V. From (2.2), a tubular neighborhood $\check{N}_\alpha(\epsilon)$ of $\check{X}_\alpha$ is G-diffeomorphic to

$$\nu_\alpha(\epsilon) = P_\alpha \times_{S_\alpha} (V(\epsilon) \times_H G). \tag{2.23}$$

Let $(V\times_H G)_{\alpha'}$ be the set of points in $V\times_H G$ with normal orbit type α' and put $(V\times_H G)^1_{\alpha'} = ((V \times_H G)_{\alpha'}) \cap (SV \times_H G)$. Then we can take $F_{\alpha\alpha'} = P_\alpha \times_{S_\alpha} (V \times_H G)^1_{\alpha'}/G$.

The vector bundle $E_{\alpha'}$, when restricted to $X_{\alpha'} \cap N_\alpha(\epsilon)$, extends to a vector bundle on $[0,\epsilon) \times F_{\alpha\alpha'}$, where we think of $\{0\} \times F_{\alpha\alpha'}$ as the part which is collapsed to X_α in the above mapping cylinder description. We write $E_{\alpha'}\big|_{F_{\alpha\alpha'}}$ for the restriction of $E_{\alpha'}$ to $\{0\} \times F_{\alpha\alpha'}$.

Lemma 3. *There is an injection $I_{\alpha\alpha'} : \pi^*_{\alpha\alpha'} E_\alpha \to E_{\alpha'}\big|_{F_{\alpha\alpha'}}$.*

Proof. Using (2.2), it is enough to consider the case when $\check{X} = V \times_H G$ and $\check{E}$ is a G-equivariant vector bundle on $\check{X}$. Then there is an H-module W so that $\check{E} = (W\times V)\times_H G$. As $X = \check{X}/G = V/H$, for the purposes of the proof we may assume that $G = H$ and $\check{X} = V$. Taking $\alpha = [H,V]$, we have that $\check{X}_\alpha = 0 \in V$ and X_α is the point $0 \cdot H$ in $X = V/H$. Suppose that $v \in SV \cap \check{X}_{\alpha'}$ has isotropy group $H' \subset H$. Then the fiber of $E_{\alpha'}$ over $vH \in X_{\alpha'}$ is isomorphic to $\check{K}^{H'}_v$. The lemma follows from the injection $\check{K}^H_0 \to \check{K}^{H'}_0$, along with our assumption that $\{\check{K}_{rv}\}_{r>0}$ extends continuously to $r = 0$. □

In the same way, if $\alpha = [H,\mathbb{R}]$ and $\alpha' = [H_\beta, 0]$ then there is an injection $I_{\alpha\alpha'} : \pi^*_{\alpha\alpha'} E^-_\alpha \to E_{\alpha'}\big|_{F_{\alpha\alpha'}}$.

Consider a collection of forms $\{\omega_\alpha \in \Omega(X_\alpha;E_\alpha)\}_{\alpha\in\mathcal{N}}$. If $\alpha < \alpha'$ and $r \in [0,\epsilon)$ is the coordinate in the above mapping cylinder description then we can write $\omega_{\alpha'}$ on $X_{\alpha'} \cap N_\alpha(\epsilon)$ as

$$\omega_{\alpha'} = \omega_1(r) + dr \wedge \omega_2(r), \tag{2.24}$$

where for $r > 0$, we have $\omega_1(r), \omega_2(r) \in \Omega\left(F_{\alpha\alpha'}; E_{\alpha'}\big|_{F_{\alpha\alpha'}}\right)$.

We define a space of forms $\Omega_{str}(X;E)$ which we call the stratified forms.

Definition 6. *The forms $\{\omega_\alpha \in \Omega(X_\alpha;E_\alpha)\}_{\alpha\in\mathcal{N}}$ define an element of $\Omega_{str}(X;E)$ if for $\alpha < \alpha'$, the forms $\omega_1(r)$ and $\omega_2(r)$ of (2.24) are smooth up to $r = 0$, $\omega_1(0) = I_{\alpha\alpha'}(\pi^*_{\alpha\alpha'}\omega_\alpha)$ and* $\omega_2(0) \in \begin{cases} 0 & \text{if } \alpha \neq [H,\mathbb{R}], \\ I_{\alpha\alpha'}(\pi^*_{\alpha\alpha'} E^-_\alpha) & \text{if } \alpha = [H,\mathbb{R}]. \end{cases}$

Example 1 : Let X be a compact manifold-with-boundary. Let $\check{X}$ be the double of X, with $G = \mathbb{Z}_2$ acting on $\check{X}$ so that the generator $\gamma \in \mathbb{Z}_2$ acts by involution. Put $\check{E} = \check{X} \times \mathbb{R}$. If γ acts on $\check{E}$ by sending $(\check{x}, t)$ to $(\check{x}\gamma, t)$ then E is the trivial $\mathbb{R}$-bundle on X and $\Omega_{str}(X; E)$ consists of the smooth forms on X which lie in the quadratic form domain of the differential form Laplacian on X with absolute boundary conditions. If γ acts on $\check{E}$ by sending $(\check{x}, t)$ to $(\check{x}\gamma, -t)$ then the fiber of E is $\mathbb{R}$ over $X - \partial X$ and 0 over ∂X, E^-_α is the trivial $\mathbb{R}$-bundle on ∂X and $\Omega_{str}(X; E)$ consists of the smooth forms on X which lie in the quadratic form domain of the differential form Laplacian on X with relative boundary conditions.

Example 2 : Let Y be a compact manifold-with-boundary. Take $G = S^1$ and $\check{X} = (Y \times S^1) \cup_{\partial Y \times S^1} (\partial Y \times D^2)$. Then the quotient space is the stratified space $X = \check{X}/S^1 = Y$. Let $\check{E}$ be the trivial $\mathbb{R}$-bundle on $\check{X}$, with $\mathfrak{I}_{\mathfrak{x}} \equiv 0$. Then E is the trivial $\mathbb{R}$-bundle on X. One finds that $\Omega_{str}(X; E)$ consists of the smooth forms on Y which lie in the quadratic form domain of the differential form Laplacian on Y with absolute boundary conditions.

Example 3 : Put $\check{X} = \mathbb{C}^2$ and $G = \mathbb{Z}_p \subset S^1$, with the standard action on $\mathbb{C}^2$. (This is a noncompact example, but it illustrates the point.) Then $X = \mathrm{cone}(S^3/\mathbb{Z}_p)$. Let $\rho : \mathbb{Z}_p \to O(N)$ be a representation with $(\mathbb{R}^N)^{\mathbb{Z}_p} = \{0\}$. Put $\check{E} = \mathbb{C}^2 \times \mathbb{R}^N$, with the diagonal $\mathbb{Z}_p$-action and the trivial product connection. Then E vanishes when restricted to the cone point of X, and has fiber $\mathbb{R}^N$ on the rest of X. Putting $\mathcal{W} = S^3 \times_{\mathbb{Z}_p} \mathbb{R}^N$, a flat vector bundle on $S^3/\mathbb{Z}_p$, one finds that $\Omega_{str}(X; E)$ consists of the elements of $\Omega^*([0, \infty)) \widehat{\otimes} \Omega^*(S^3/\mathbb{Z}_p; \mathcal{W})$ which vanish at 0. That is, the entire form vanishes at 0 and not just its pullback.

Lemma 4. *There is an inclusion* $\Omega(X; E) \subset \Omega_{str}(X; E)$.

Proof. As in the proof of Lemma 3, we can reduce to the case $\check{X} = V = V \times_H H$ and $\check{E} = W \times V$, with $\check{X}_\alpha = 0 \in V$. If $\check{\omega} \in \Omega_{basic}(\check{X}; \check{E})$ then write $\check{\omega} = \check{\omega}_1(r) + dr \wedge \check{\omega}_2(r)$, where r is the radial coordinate on V.

Suppose that $\alpha \neq [H, \mathbb{R}]$. As $\check{\omega}$ is smooth and $\dim(V) \geq 2$, we must have that $\check{\omega}_2(0) = 0$ and $\check{\omega}_1(0) \in \Lambda^0(T^*V) \otimes W$. Then as $\check{\omega}$ is basic, we have $\check{\omega}_1(0) \in \Lambda^0(T^*V) \otimes \check{K}_0^H$. Suppose that $v \in SV \cap \check{X}_{\alpha'}$ has isotropy group $H' \subset H$. By the smoothness of $\check{\omega}$, we have

$$\lim_{r \to 0} \check{\omega}_1(rv) \ = \ \check{\omega}_1(0). \tag{2.25}$$

In particular, $\lim_{r \to 0} \check{\omega}_1(rv)$ is a 0-form with value in $\check{K}_0^H \subset \check{K}_0^{H'}$. Translating (2.25) to the quotient space gives $\omega_1(0) = I_{\alpha\alpha'}(\pi^*_{\alpha\alpha'}\omega_\alpha)$.

If $\alpha = [H, \mathbb{R}]$ then the only difference is that $\check{\omega}_2(0)$ can be nonzero. From H-invariance, it must lie in $\check{E}^-_\alpha$. □

Hence we have a space of forms $\Omega_{str}(X; E)$ which is defined intrinsically on the stratified space X and which in some sense is the minimal such space that contains the basic forms $\Omega(X; E)$.

Finally, we describe a spectral sequence to compute the cohomology of $\check{A}'$ acting on $\Omega_{basic}(\check{X}; \check{E})$, which we denote by $\mathrm{H}^*(A')$. In the case when $G = \{e\}$, $\check{X} = B$ and $\check{E} = E$, such a spectral sequence was described in [19, Section 7]. It arises from the filtration of $\Omega(B; E)$ by $F^p = \bigoplus_{q \geq p} \Omega^q(B; E^*)$. In our case, we filter $\Omega_{basic}(\check{X}; \check{E})$ by the form degree on the quotient space X. More precisely, let F^p be the forms $\check{\omega} \in \Omega_{basic}(\check{X};\check{E})$ such that for all $\alpha \in \mathcal{N}$, $\omega_\alpha \in \bigoplus_{q\geq p} \Omega^q(X_\alpha;E_\alpha)$. As usual, the E_1-term $E_1^{p,q}$ of the spectral sequence is the $(p+q)$-degree cohomology of the complex F^p/F^{p+1}. Given an open set $U \subset X$, let $\check{U}$ be its preimage in $\check{X}$. Let $E_1^{p,q}$ also denote the sheaf which assigns to U the space $E_1^{p,q}(U)$ as computed using the basic forms on $\check{U}$. If $\check{x} \in \check{X}$ covers $x \in X$ and has isotropy group H then the stalk of $E_1^{0,*}$ at x is isomorphic to the cohomology of $\check{A}'_{[0]}$ on $\check{K}^H_{\check{x}}$. (The degree-1 component of the equation $\check{A}'\,(i_{\mathfrak{X}} + \mathfrak{I}_{\mathfrak{x}}) + (i_{\mathfrak{X}} + \mathfrak{I}_{\mathfrak{x}})\,\check{A}' = \mathcal{L}_{\mathfrak{X}}$ is $\check{A}'_{[1]}\,i_{\mathfrak{X}} + i_{\mathfrak{X}}\,\check{A}'_{[1]} + \mathfrak{I}_{\mathfrak{x}}\,\check{A}'_{[0]} + \check{A}'_{[0]}\,\mathfrak{I}_{\mathfrak{x}} = \mathcal{L}_{\mathfrak{X}}$. Taking $\mathfrak{x} \in \mathfrak{h}$, one sees that $\check{A}'_{[0]}$ does act on $\check{K}^H_{\check{x}}$.)

Define a $\mathbb{Z}$-graded sheaf $\mathrm{H}^*(A'_{[0]})$ on X which assigns to $U \subset X$ the vector space $\mathrm{Ker}(d_1^{0,*} : E_1^{0,*}(U) \to E_1^{1,*}(U))$. There is a complex of sheaves

$$\mathrm{H}^*(A'_{[0]}) \longrightarrow E_1^{0,*} \xrightarrow{d_1^{0,*}} E_1^{1,*} \xrightarrow{d_1^{1,*}} E_1^{2,*} \xrightarrow{d_1^{2,*}} \dots \tag{2.26}$$

Lemma 5. *The complex (2.26) is a resolution of* $\mathrm{H}^*(A'_{[0]})$ *by fine sheaves.*

Proof. We follow the method of proof of [22], which effectively proves the lemma in the case when $\check{E}$ is the trivial $\mathbb{R}$-bundle on $\check{X}$. Using the ordinary slice theorem, we can reduce to the case that $\check{U} = N \times_H G$ for some representation space N of an isotropy group H. Then we can reduce to the case when $\check{U} = N$ and $G = H$. Now $E_1^{p,*}(U)$ is the cohomology of $\check{A}'_{[0]}$ on the elements of $\Omega_{basic}\left(N;\check{E}\right)$ with p horizontal differentials, with respect to the H-action on N. The degree-1 component of the equation $\left(\check{A}'\right)^2 = 0$ is $\check{A}'_{[1]}\,\check{A}'_{[0]} + \check{A}'_{[0]}\,\check{A}'_{[1]} = 0$, which implies that $\check{A}'_{[1]}$ has an induced action on $E_1^{p,*}(U)$. The degree-2 component of the equation $\left(\check{A}'\right)^2 = 0$ is $\left(\check{A}'_{[1]}\right)^2 + \check{A}'_{[0]}\,\check{A}'_{[2]} + \check{A}'_{[2]}\,\check{A}'_{[0]} = 0$, which implies that $\left(\check{A}'_{[1]}\right)^2$ vanishes on $E_1^{p,*}(U)$. In fact, the action of $\check{A}'_{[1]}$ on $E_1^{p,*}(U)$ is the same as $d_1^{p,*}$. We now use the Poincaré lemma on N, as in [22], to prove the claim. To apply the Poincaré lemma we use a radial trivialization of $\check{A}'_{[1]}$ on N. Thinking of N as the cone over its sphere SN, the H-action on N comes from the H-action on SN. Because of this, it follows that the homotopy operator in the Poincaré lemma does send basic forms to basic forms. The rest of the proof is as in [22]. □

It follows that the E_2-term of the spectral sequence is given by $E_2^{p,q} = \mathrm{H}^p\left(X;\mathrm{H}^q(A'_{[0]})\right)$, where the right-hand-side is the cohomology of the sheaf $\mathrm{H}^q(A'_{[0]})$ on X.

3. Eigenvalue Bounds

In this section we use the results of Section 2 to prove the analogs of [19, Theorems 2 and 3] in the case of a general limit space X. We then prove Theorem 1, giving eigenvalue bounds for the p-form Laplacian in terms of sectional curvature and diameter. The method of proof is to assume that there are no such bounds and use Gromov-Hausdorff convergence in the $O(n)$-equivariant setting, along with our eigenvalue estimates, to get a contradiction. In Theorem 2 we look at the special case of Theorem 1.2 when $p = 2$.

Let M be a smooth connected closed n-dimensional Riemannian manifold and let FM denote its orthonormal frame bundle, the total space of a principal $O(n)$-bundle $\mathfrak{p} : FM \to M$. There is a canonical Riemannian metric on FM, but for the moment we will consider FM

with any $O(n)$-invariant Riemannian metric, and we give M the corresponding quotient metric. Let μ be the (smooth) measure on M given by $\mu(m) = \mathrm{vol}(\mathfrak{p}^{-1}(m))\ d\mathrm{vol}(m)$. Let $\Omega^*_{L^2}(M,\mu)$ denote the completion of $\Omega^*(M)$ with respect to the inner product $\langle \omega, \omega \rangle = \int_M |\omega(m)|^2 d\mu(m)$. Then there is an isometric isomorphism $\Omega^*_{L^2}(M,\mu) \cong \Omega^*_{basic,L^2}(FM)$ coming from pullback. The basic Laplacian $\triangle^{FM}_{basic}$ on $\Omega^*_{basic,L^2}(FM)$ is the Laplacian associated to the complex

$$\Omega^0_{basic}(FM) \longrightarrow \Omega^1_{basic}(FM) \longrightarrow \dots \tag{3.1}$$

It is isomorphic to the weighted Laplacian $\triangle^M_\mu = dd^* + d^*d$ on $\Omega^*_{L^2}(M,\mu)$.

Let $\check{X}$ be a fixed smooth connected closed Riemannian manifold on which $O(n)$ acts isometrically on the right, with quotient $X = \check{X}/O(n)$. We say that a fiber bundle $FM \to \check{X}$ is an equivariant Riemannian affine fiber bundle if it is a Riemannian affine fiber bundle [19, Definition 1] which is $O(n)$-equivariant in the obvious sense. Given $\check{x} \in \check{X}$, let $\check{Z}_{\check{x}}$ be the fiber over $\check{x}$ of the affine fiber bundle. For the applications, it will be sufficient to consider the case when $\check{Z}_{\check{x}}$ is a nilmanifold $\Gamma\backslash N$ [7, (7.2)]. Let $\check{E}$ be the real $\mathbb{Z}$-graded vector bundle on $\check{X}$ whose fibers are isomorphic to the affine-parallel differential forms on $\{\check{Z}_{\check{x}}\}_{\check{x}\in\check{X}}$. It inherits a flat degree-1 superconnection $\check{A}'$ from d^{FM} [19, Section 5]. If $\mathfrak{x}$ is in the Lie algebra $o(n)$ of $O(n)$, let $\mathfrak{X}^{FM}$ be the corresponding vector field on FM and let $\mathfrak{X}_V$ be its vertical component with respect to $FM \to \check{X}$. Then interior multiplication by $\mathfrak{X}_V$ on the fibers $\check{Z}_{\check{x}}$ induces a linear map $\mathfrak{I}_{\mathfrak{x}} \in C^\infty\left(\check{X}; \mathrm{Hom}(\check{E}^*, \check{E}^{*-1})\right)$, which gives $\check{E}$ the structure of an $O(n)$-basic vector bundle. Furthermore, $\check{A}'$ becomes an $O(n)$-basic superconnection. Define $\triangle^{\check{E}}_{basic} \cong \triangle^E$ as in Definition 5. It is the Laplacian associated to the subcomplex $\Omega_{basic}\left(\check{X}; \check{E}\right)$ of (3.1).

The Riemannian metric on FM defines an $O(n)$-invariant family of horizontal planes that are perpendicular to the fibers of $FM \to M$. Let T^{FM} be the corresponding fiber bundle curvature. Let $\check{T}$ be the curvature of the affine fiber bundle $FM \to \check{X}$, let $\check{\Pi}$ be the second fundamental forms of the fibers $\{\check{Z}_{\check{x}}\}_{\check{x}\in\check{X}}$ and let $\mathrm{diam}(\check{Z})$ be the maximum diameter of the fibers.

The next proposition can be considered to be an analog of [19, Theorem 1], in which the nilpotent fiber bundle structure on the total space M of an affine fiber bundle $M \to B$ is replaced by the nilpotent Killing structure on M coming from an $O(n)$-equivariant affine fiber bundle $FM \to \check{X}$ [7].

Proposition 2. *There are positive constants A, A' and C which only depend on* $\dim(M)$ *such that if* $\parallel R^{\check{Z}} \parallel_\infty \mathrm{diam}(\check{Z})^2 \leq A'$ *then*

$$\sigma(\triangle^{FM}_{basic}) \cap \left[0, \left(\sqrt{A\,\mathrm{diam}(\check{Z})^{-2} - C\ \left(\parallel R^{FM} \parallel_\infty + \parallel \check{\Pi} \parallel^2_\infty + \parallel \check{T} \parallel^2_\infty\right)} - C\ \parallel T^{FM} \parallel_\infty\right)^2\right) = \tag{3.2}$$

$$\sigma(\triangle^E) \cap \left[0, \left(\sqrt{A\,\mathrm{diam}(\check{Z})^{-2} - C\ \left(\parallel R^{FM} \parallel_\infty + \parallel \check{\Pi} \parallel^2_\infty + \parallel \check{T} \parallel^2_\infty\right)} - C\ \parallel T^{FM} \parallel_\infty\right)^2\right).$$

Proof. Let $\mathcal{P}^{fib}$ be fiberwise orthogonal projection from $\Omega^*(FM)$ to $\Omega(\check{X};\check{E})$. From the proof of [19, Proposition 1], $\mathcal{P}^{fib}$ amounts to averaging over the nilmanifold fibers $\Gamma\backslash N$ of $FM \to \check{X}$. Hence it preserves $\Omega^*_{basic}(FM)$ and descends to an operator from $\Omega^*_{basic}(FM)$ to

$\Omega_{basic}(\check{X};\check{E}) \cong \Omega(X;E)$, which we also denote by $\mathcal{P}^{fib}$. As in the proof of [19, Theorem 1], it suffices to show that there are constants A, A' and C as above such that $\sigma\left(\triangle^{FM}_{basic}\big|_{Ker(\mathcal{P}^{fib})}\right)$ is bounded below by

$$\left(\sqrt{A\,\mathrm{diam}(\check{Z})^{-2} \,-\, C\,\left(\parallel R^{FM}\parallel_\infty + \parallel \check{\Pi}\parallel_\infty^2 + \parallel \check{T}\parallel_\infty^2\right)} \,-\, C\,\parallel T^{FM}\parallel_\infty\right)^2. \tag{3.3}$$

Let $\triangle^{FM}$ be the Laplacian on differential forms on FM and let $\triangle^{FM}_{O(n)}$ be the Laplacian on $O(n)$-invariant not-necessarily-basic differential forms on FM. Applying [19, Theorem 1] to the Riemannian affine fiber bundle $FM \to \check{X}$, we know that

$$\sigma\left(\triangle^{FM}\big|_{Ker(\mathcal{P}^{fib})}\right) \subset \left[A\,\mathrm{diam}(\check{Z})^{-2} \,-\, C\,\left(\parallel R^{FM}\parallel_\infty + \parallel \check{\Pi}\parallel_\infty^2 + \parallel \check{T}\parallel_\infty^2\right), \infty\right) \tag{3.4}$$

and so

$$\sigma\left(\triangle^{FM}_{O(n)}\big|_{Ker(\mathcal{P}^{fib})}\right) \subset \left[A\,\mathrm{diam}(\check{Z})^{-2} \,-\, C\,\left(\parallel R^{FM}\parallel_\infty + \parallel \check{\Pi}\parallel_\infty^2 + \parallel \check{T}\parallel_\infty^2\right), \infty\right) \tag{3.5}$$

Let $d^*_{O(n)}$ denote the adjoint of d on $\Omega^*_{O(n)}(FM)$. As $\triangle^{FM}_{O(n)} = \left(d + d^*_{O(n)}\right)^2$, it follows from (3.5) that if λ is an eigenvalue of $\left(d + d^*_{O(n)}\right)\big|_{Ker(\mathcal{P}^{fib})}$ then

$$|\lambda| \,\geq\, \sqrt{A\,\mathrm{diam}(\check{Z})^{-2} \,-\, C\,\left(\parallel R^{FM}\parallel_\infty + \parallel \check{\Pi}\parallel_\infty^2 + \parallel \check{T}\parallel_\infty^2\right)}. \tag{3.6}$$

From Lemma 1, the adjoint of d on $\Omega^*_{basic}(FM) \cong \Omega^*(M)$ is $d^*_{basic} = P^{hor}\, d^*_{O(n)}$. With respect to the decomposition $\Omega^*_{O(n)}(FM) = \Omega^*_{basic}(FM) \oplus (\Omega^*_{basic}(FM))^\perp$, we have

$$d + d^*_{O(n)} = \begin{pmatrix} d + d^*_{basic} & P^{hor}\, d\,(I - P^{hor}) \\ (I - P^{hor})\, d^*_{O(n)} P^{hor} & (I - P^{hor})\left(d + d^*_{O(n)}\right)(I - P^{hor}) \end{pmatrix}. \tag{3.7}$$

Using the notation of [19, (5.26)], let $\omega_{i\alpha\beta}$ denote the curvature of the fiber bundle $FM \to M$. A calculation gives that

$$(I - P^{hor})\, d^*_{O(n)}\, P^{hor} = -\sum_{i,\alpha,\beta} \omega_{i\alpha\beta}\, E^i\, I^\alpha\, I^\beta, \tag{3.8}$$

with $P^{hor}\, d\,(I - P^{hor})$ being the adjoint of the right-hand-side of (3.8). Then upon restriction to $\mathrm{Ker}(\mathcal{P}^{fib})$, it follows that there for an appropriate constant $C = C(\dim(M)) > 0$, the diagonal part of the operator in (3.7) has a spectrum which differs from that of $\left(d + d^*_{O(n)}\right)\big|_{Ker(\mathcal{P}^{fib})}$ by at most $C\,\parallel T^{FM}\parallel_\infty$. As the spectrum of $(d + d^*_{basic})\big|_{Ker(\mathcal{P}^{fib})}$ is contained in the spectrum of the diagonal part of (3.7), when restricted to $\mathrm{Ker}(\mathcal{P}^{fib})$, the proposition follows. □

Let $d^{O(n)}_{GH}$ denote the $O(n)$-equivariant Gromov-Hausdorff metric on the space of $O(n)$-equivariant compact metric spaces [7, (2.1.3)]. We say that two nonnegative numbers λ_1, λ_2 are ϵ-close if $e^{-\epsilon}\,\lambda_2 \,\leq\, \lambda_1 \,\leq\, e^{\epsilon}\,\lambda_2$.

Proposition 3. *For $n \in \mathbb{Z}^+$, let $\check{X}$ be a fixed connected closed Riemannian manifold with an isometric $O(n)$-action and quotient space $X = \check{X}/O(n)$. Given $\epsilon > 0$ and $K \geq 0$, there are positive constants $A(n,\epsilon,K)$, $A'(n,\epsilon,K)$ and $C(n,\epsilon,K)$ with the following property : If M^n is an n-dimensional connected closed Riemannian manifold with $\parallel R^M \parallel_\infty \leq K$ and $d_{GH}^{O(n)}(FM,\check{X}) \leq A'(n,\epsilon,K)$ then there are*
1. An $O(n)$-basic $\mathbb{Z}$-graded real vector bundle $\check{E}$ on $\check{X}$,
2. A basic flat degree-1 superconnection $\check{A}'$ on $\check{E}$ and
3. An $O(n)$-invariant Euclidean inner product $h^{\check{E}}$ on $\check{E}$
such that if $\lambda_{p,j}(M)$ is the j-th eigenvalue of the p-form Laplacian on M, $\lambda_{p,j}(X;E)$ is the j-th eigenvalue of $\triangle_p^E$ and

$$\min(\lambda_{p,j}(M),\lambda_{p,j}(X;E)) \leq \left(\sqrt{A(n,\epsilon,K)\, d_{GH}^{O(n)}(FM,\check{X})^{-2} - C(n,\epsilon,K)} - C(n,\epsilon,K)\right)^2 \tag{3.9}$$

then $\lambda_{p,j}(M)$ is ϵ-close to $\lambda_{p,j}(X;E)$.

Proof. For simplicity, we will omit reference to p. Let g_0^M denote the Riemannian metric on M. Let g_0^{FM} denote the canonical induced Riemannian metric on FM. Then $\triangle^M \cong \triangle^{FM}_{basic}$.

From the obvious basic extension of [19, Lemma 3], if an $O(n)$-invariant Riemannian metric g_1^{FM} on FM is ϵ-close to g_0^{FM} in the sense of [19, (5.4)] then the spectrum of $\triangle^{FM}_{basic}$ computed with g_1^{FM} is $J\epsilon$-close to the spectrum computed with g_0^{FM}. We can use the geometric results of [7] to find a $O(n)$-invariant Riemannian metric g_2^{FM} on M which is close to g_0^{FM} and to which we can apply Proposition 2. Note that g_2^{FM} may not be the canonical metric coming from some Riemannian metric on M.

There is an explicit bound for the sectional curvatures of the canonical metric g_0^{FM} in terms of K. Then in the construction of the $O(n)$-invariant metric g_2^{FM}, we may assume that we have bounds on the sectional curvatures of g_2^{FM} and on the sectional curvatures of the quotient metric on M, in terms of ϵ and K (see [7, Proof of Theorem 1.3] and [21, Theorem 2.1]).

We can go through the proof of [19, Theorem 2] working $O(n)$-equivariantly on FM and using Proposition 2. The proofs of [7, Propositions 3.6 and 4.9], giving the equivariant Riemannian affine fiber bundle structure $FM \to \check{X}$, are explicitly phrased in the G-equivariant setting, where G is a compact Lie group. In particular, we have bounds on $\parallel \check{\Pi} \parallel_\infty$ and $\parallel \check{T} \parallel_\infty$. Using O'Neill's formula [4, (9.29c)], we have a bound on $\parallel T^{FM} \parallel_\infty$ in terms of ϵ and K. Then the proof of [19, Theorem 2] goes through to the $O(n)$-equivariant setting, changing the fibration $M \to B$ of [19, Theorem 2] to the $O(n)$-equivariant fibration $FM \to \check{X}$. □

If $\check{E}$ is a real $\mathbb{Z}$-graded $O(n)$-equivariant vector bundle on $\check{X}$, let $\mathcal{I}_{\check{E},basic}$ be the set of $O(n)$-equivariant linear maps $\mathfrak{I} : o(n) \to C^\infty\left(\check{X}, \mathrm{Hom}(\check{E}^*, \check{E}^{*-1})\right)$ which satisfy $\mathfrak{I}(\mathfrak{x})^2 = 0$ for all $\mathfrak{x} \in o(n)$. Let $\mathcal{S}_{\check{E},O(n)}$ be the set of smooth degree-1 $O(n)$-equivariant superconnections on $\check{E}$ and let $\mathcal{S}_{\check{E},basic}$ be the set of pairs $(\mathfrak{I},\check{A}') \in \mathcal{I}_{\check{E},basic} \times \mathcal{S}_{\check{E},O(n)}$ so that $\check{A}'$ is $O(n)$-basic with respect to $\mathfrak{I}$. Let $\mathcal{H}_{\check{E},basic}$ be the set of $O(n)$-invariant Euclidean inner products on $\check{E}$ and let $\mathcal{G}_{\check{E},basic}$ be the group of smooth grading-preserving $O(n)$-equivariant $\mathrm{GL}(\check{E})$-gauge transformations on $\check{E}$ and We equip $\mathcal{I}_{\check{E},basic}$, $\mathcal{S}_{\check{E},O(n)}$ and $\mathcal{H}_{\check{E},basic}$ with the C^∞-topology, $\mathcal{S}_{\check{E},basic}$ with the subspace topology and $(\mathcal{S}_{\check{E},basic} \times \mathcal{H}_{\check{E},basic})/\mathcal{G}_{\check{E},basic}$ with the quotient topology.

Proposition 4. *In Proposition 3, we may assume that* $\check{E}$ *is in one of a finite number of isomorphism classes of real* $\mathbb{Z}$*-graded* $O(n)$*-equivariant topological vector bundles* $\{\check{E}_i\}$ *on* $\check{X}$*. In addition, we may assume that for each* $\alpha \in \mathcal{N}$*, the vector bundle* E_α *is in one of a finite number of isomorphism classes of real* $\mathbb{Z}$*-graded topological vector bundles on* X_α*. Furthermore, there are compact subsets* $C_{\check{E}_i} \subset (\mathcal{S}_{\check{E}_i,basic} \times \mathcal{H}_{\check{E}_i,basic})/\mathcal{G}_{\check{E}_i,basic}$ *depending on* n*,* ϵ *and* K *such that we may assume that the gauge-equivalence class of* $\left(\mathfrak{I}, \check{A}', h^{\check{E}}\right)$ *lies in* $C_{\check{E}}$*.*

Proof. We go through the proof of [19, Theorem 3] equivariantly. In [19, Theorem 3], one obtains the finiteness statement from the fact that there is a finite number of topological types of real vector bundles of a given rank on B which admit a flat connection. This fact follows from the finiteness of the number of connected components of the representation variety of $\pi_1(B, b_0)$. In our case the representation variety of $\pi_0(\widehat{G})$ will have a finite number of connected components. With Proposition 1, this implies that there is a finite number of topological types of $O(n)$-equivariant vector bundles on $\check{X}$ which admit a basic flat connection. The method of proof of [19, Theorem 3] shows that $\check{E}$ admits an $O(n)$-basic flat connection.

The vector bundle E_α on X_α has a flat degree-1 superconnection A'_α induced from $\check{A}'$. From the argument of [19, Theorem 3], there is a finite number of possibilities for the topological type of E_α.

Next, [19, Theorem 3] gives a compactness result for $\left(\check{A}', h^{\check{E}}\right)$ in $(\mathcal{S}_{\check{E}_i,O(n)} \times \mathcal{H}_{\check{E}_i,basic})/\mathcal{G}_{\check{E}_i,basic}$. We recall that the inner product $h^{\check{E}}$ of Proposition 4, when restricted to the fiber $\check{E}_{\check{x}}$ over a point $\check{x} \in \check{X}$, is the L^2-inner product on the affine-parallel forms of the geometric fiber $\check{Z}_{\check{x}}$. As the $O(n)$-orbits on FM, with the canonical metric, are isometric to the standard $O(n)$, from the construction of $\mathfrak{I}_{\mathfrak{r}}$ we have an upper bound on

$$\parallel \mathfrak{I} \parallel^2 = \sup_{\mathfrak{r} \in o(n) \,:\, |\mathfrak{r}|=1} \sup_{\check{x} \in \check{X}} \sup_{\check{e} \in \check{E}_{\check{x}} - 0} \frac{h^{\check{E}}(\mathfrak{I}_{\mathfrak{r}} \check{e}, \mathfrak{I}_{\mathfrak{r}} \check{e})}{h^{\check{E}}(\check{e}, \check{e})} \tag{3.10}$$

that only depends on n, ϵ and K. The component of the equation $\check{A}'(\mathfrak{I}_{\mathfrak{r}} + i_{\mathfrak{X}}) + (\mathfrak{I}_{\mathfrak{r}} + i_{\mathfrak{X}})\check{A}' = \mathcal{L}_{\mathfrak{X}}$ of degree 1 with respect to $\check{X}$ is

$$\check{A}'_{[1]} \, \mathfrak{I}_{\mathfrak{r}} + \mathfrak{I}_{\mathfrak{r}} \, \check{A}'_{[1]} + \check{A}'_{[2]} \, i_{\mathfrak{X}} + i_{\mathfrak{X}} \, \check{A}'_{[2]} = 0. \tag{3.11}$$

As we have C^∞-bounds on $\check{A}'_{[2]}$, we obtain C^∞-bounds on the covariant derivative of $\mathfrak{I}$, and hence on its higher covariant derivatives. Thus we also have precompactness for $\mathfrak{I}$, from which the proposition follows. □

We will need an eigenvalue estimate. Let $\check{E}$ be a real $\mathbb{Z}$-graded $O(n)$-equivariant vector bundle on $\check{X}$ and define $\mathcal{S}_{\check{E},basic}$ and $\mathcal{H}_{\check{E},basic}$ as before. Suppose that we have two triples $\left(\mathfrak{I}_1, \check{A}'_1, h_1^{\check{E}}\right)$ and $\left(\mathfrak{I}_2, \check{A}'_2, h_2^{\check{E}}\right)$ in $\mathcal{S}_{\check{E},basic} \times \mathcal{H}_{\check{E},basic}$. For $i \in \{1, 2\}$, let $\Omega_{basic,i}\left(\check{X}; \check{E}\right)$ denote the basic forms as defined using $\mathfrak{I}_i$. Suppose that $T : \Omega_{basic,1}\left(\check{X}; \check{E}\right) \to \Omega_{basic,2}\left(\check{X}; \check{E}\right)$ is an isomorphism. Let $\triangle_i^E$ denote the basic Laplacian constructed using $\left(\mathfrak{I}_i, \check{A}'_i, h_i^{\check{E}}\right)$.

Given $y \in \Omega_{basic,2}(\check{X}; \mathrm{End}(\check{E}))$, let $\parallel y \parallel$ be the operator norm for the action of y on $\Omega_{basic,2,L^2}(\check{X}; \check{E})$.

Lemma 6. *Suppose that* $\triangle_1^E$ *has a discrete spectrum. If for some* $\epsilon \geq 0$ *we have* $e^{-\epsilon}\,\mathrm{Id.} \leq T^*T \leq e^{\epsilon}\,\mathrm{Id.}$ *then for all* $j \in \mathbb{Z}^+$,

$$|\lambda_j(\triangle_1^E)^{1/2} - \lambda_j(\triangle_2^E)^{1/2}| \leq (2+\sqrt{2}) \parallel T\,\check{A}_1'\,T^{-1} - \check{A}_2' \parallel + (e^{\epsilon} - 1)\,\lambda_j(\triangle_1^E)^{1/2}. \tag{3.12}$$

Proof. We first examine the effect on the eigenvalues of changing from $(\check{A}_2', h_2^{\check{E}})$ to $(T\check{A}_1'T^{-1}, h_2^{\check{E}})$, where in both cases the corresponding superconnection Laplacian acts on $\Omega_{basic,2}\left(\check{X};\check{E}\right)$. From the method of proof of [19, Lemma 4] the change in $\lambda_j^{1/2}$ is bounded by $(2+\sqrt{2}) \parallel T\check{A}_1'T^{-1} - \check{A}_2' \parallel$. Next, we examine the effect of changing from $(T\check{A}_1'T^{-1}, h_2^{\check{E}})$ to $(\check{A}_1', h_1^{\check{E}})$. By naturality, the eigenvalues of the superconnection Laplacian constructed from $(T\check{A}_1'T^{-1}, h_2^{\check{E}})$ can be computed using instead the superconnection $\check{A}_1$ and the inner product on $\Omega_{basic,1}\left(\check{X};\check{E}\right)$ which is pullbacked from $\Omega_{basic,2}\left(\check{X};\check{E}\right)$ via T. The method of proof of [19, Lemma 3] shows that if one compares this with the original inner product on $\Omega_{basic,1}\left(\check{X};\check{E}\right)$ then the eigenvalues differ at most by a multiplicative factor of $e^{2\epsilon}$. The lemma follows. We note that we have implicitly shown that $\triangle_2^E$ also has a discrete spectrum. □

Proof of Theorem 1 :

1. If it is not true that $\lim_{j\to\infty} a^1_{n,p,j,K}$ is always infinite then there are numbers $n \in \mathbb{Z}^+$, $0 \leq p \leq n$, $K \geq 0$, $\Lambda > 0$ and a sequence $\{M_i\}_{i=1}^{\infty}$ of connected closed n-dimensional Riemannian manifolds with $\mathrm{diam}(M_i) = 1$ and $\parallel R^{M_i} \parallel_\infty \leq K$ such that $\lambda_{p,i}(M_i) < \Lambda$. By [7, Theorem 1.12], for any $\epsilon > 0$ there is a sequence $\{A_k(n,\epsilon)\}_{k=0}^{\infty}$ so that we can find a new metric on M_i which is ϵ-close to the old one in the sense of [19, (5.4)], with the new metric satisfying $\parallel \nabla^k R^{M_i} \parallel_\infty \leq A_k(n,\epsilon)$. Fix ϵ to be, say, $\frac{1}{2}$ and consider $\{M_i\}_{i=1}^{\infty}$ with the new metrics. From [14] or [19, Lemma 3], we now have $\lambda_{p,i}(M_i) < e^{J\epsilon}\,\Lambda$ for a fixed integer J. As in [7, III.5], we can apply Gromov's convergence theorem in the equivariant setting to conclude that there are a smooth Riemannian $O(n)$-manifold $\check{X}$ and a subsequence of $\{M_i\}_{i=1}^{\infty}$, which we relabel as $\{M_i\}_{i=1}^{\infty}$, so that $d_{GH}^{O(n)}(FM_i, \check{X}) \leq \frac{1}{i}$. In particular, $X = \check{X}/O(n)$ is not a point. As in the proof of Proposition 3, we slightly perturb the canonical Riemannian metric on FM_i to obtain an $O(n)$-invariant Riemannian metric on FM_i to which we can apply Proposition 2. From Proposition 3, there are

1. $O(n)$-basic $\mathbb{Z}$-graded real vector bundles $\{\check{E}_i\}_{i=1}^{\infty}$ on $\check{X}$,
2. Basic flat degree-1 superconnections $\{\check{A}_i'\}_{i=1}^{\infty}$ on the $\check{E}_i$'s and
3. $O(n)$-invariant Euclidean inner products $\{h^{\check{E}_i}\}_{i=1}^{\infty}$ on the $\check{E}_i$'s

so that for a given j and large i, $\lambda_{p,j}(M_i)$ is ϵ-close to $\lambda_{p,j}(X;E_i)$. From Proposition 4, after taking a subsequence we may assume that the $\check{E}_i$'s are all isomorphic to a single $O(n)$-equivariant $\mathbb{Z}$-graded real vector bundle $\check{E}$, the $E_{i,\alpha}$'s are all isomorphic to a single $\mathbb{Z}$-graded real vector bundle E_α and the triples $\left\{\left(\mathfrak{I}_i, \check{A}_i', h^{\check{E}_i}\right)\right\}_{i=1}^{\infty}$ converge, after gauge transformations, to a triple $\left(\mathfrak{I}, \check{A}', h^{\check{E}}\right)$.

We claim that for $\check{x} \in \check{X}$, $\check{K}_{i,\check{x}}$ does not degenerate as $i \to \infty$. To see this, note first that as the isotropy group $H \subset O(n)$ acts freely and affinely on the nilmanifold fiber $\check{Z}_{i,\check{x}} = \Gamma_i \backslash N_i$ of $FM_i \to \check{X}$, H is virtually abelian and H_0, the connected component of the identity in H, is a subgroup of the torus $C(\Gamma_i)\backslash C(N_i)$. As in the discussion before (3.10), H_0 acts isometrically on $\check{Z}_{i,\check{x}}$, with its orbit isometric to $H_0 \subset O(n)$. Now $\check{E}_{i,\check{x}} \cong \Lambda^*(\mathfrak{n}_i^*)$. As in [19, (6.6)], the Hermitian metric $h^{\check{E}_{i,\check{x}}}$ gives an orthogonal decomposition $\mathfrak{n}_i^* = \mathfrak{c}(\mathfrak{n}_i)^* \oplus (\mathfrak{c}(\mathfrak{n}_i)^*)^\perp$. Then for $\mathfrak{x} \in \mathfrak{h}$, the action of $\mathfrak{I}_\mathfrak{x}$ on $\check{E}_{i,\check{x}} \cong \Lambda^*(\mathfrak{c}(\mathfrak{n}_i)^*) \widehat{\otimes} \Lambda^*((\mathfrak{c}(\mathfrak{n}_i)^*)^\perp)$ is given by interior multiplication by $\mathfrak{x}$ on the first factor. By passing to a subsequence, we may assume that $\dim(\mathfrak{c}(\mathfrak{n}_i))$ is constant in i. Recalling that $h^{\check{E}_{i,\check{x}}}$ comes from the L^2-inner product on the affine-parallel forms on $\check{Z}_{i,\check{x}}$, it follows that the actions of H_0 on $\{\check{E}_{i,\check{x}}\}_{i=1}^\infty$ are related by isomorphisms with norms that are uniformly bounded above and below. Then for all i, $\check{K}_{\check{x}} \cong \check{K}_{i,\check{x}}$.

Hereafter we think of all of the triples $\left\{ \left(\mathfrak{I}_i, \check{A}_i', h^{\check{E}_i} \right) \right\}_{i=1}^\infty$ as living on the same vector bundle $\check{E}$. From the convergence of the $\mathfrak{I}_i$'s to $\mathfrak{I}$, for large i there are $O(n)$-equivariant automorphisms $\mathcal{A}_i$ of $\check{E}$ which converge to the identity in the C^∞-topology so that after conjugating $\left(\mathfrak{I}_i, \check{A}_i', h^{\check{E}_i} \right)$ with $\mathcal{A}_i$, we may assume that $\check{K}_{\check{x}}$ as computed with $\mathfrak{I}_i$ is the same as when computed with $\mathfrak{I}$. Then there is an isomorphism T_i from the basic forms defined using $\mathfrak{I}$ to the basic forms defined using $\mathfrak{I}_i$ which, using the notation of (2.21), is given on $\check{X}_\alpha$ by

$$T_i = \prod_j \left(1 - e(\mathfrak{x}_j^*)\, \mathfrak{I}_{i,\mathfrak{x}_j}\right) \prod_j \left(1 + e(\mathfrak{x}_j^*)\, \mathfrak{I}_{\mathfrak{x}_j}\right). \tag{3.13}$$

Now $T_i^{-1} \check{A}_i' T_i$ is a flat degree-1 superconnection which is basic with respect to $\mathfrak{I}$. For large i, Lemma 6 implies that $\lambda_{p,j}(M_i)$ is 2ϵ-close to $\lambda_{p,j}(X;E)$, the j-th eigenvalue of the Laplacian $\triangle^E$ on $\bigoplus_{a+b=p} \Omega^a(X;E^b)$. In particular, for all $j \geq 0$, $\lambda_{p,j}(X;E) \leq e^{(J+2)\epsilon}\, \Lambda$. However, as a consequence of Lemma 6, $\triangle_p^E$ has a discrete spectrum, which is a contradiction.

2. If $A^1_{p,j,K}(M) = \infty$ then there is a sequence $\{g_i\}_{i=1}^\infty$ of Riemannian metrics on M with $\operatorname{diam}(M, g_i) = 1$ and $\parallel R^M(g_i) \parallel_\infty \leq K$ such that $\lim_{i\to\infty} \lambda_{p,j}(M, g_i) = \infty$. As above, we can find a new metric g_i' on M which is ϵ-close to g_i satisfying $\parallel \nabla^k R^M(g_i') \parallel_\infty \leq A_k(n,\epsilon)$. Fix ϵ to be, say, $\frac{1}{2}$ and relabel the metrics $\{g_i'\}_{i=1}^\infty$ to be $\{g_i\}_{i=1}^\infty$. From [14] or [19, Lemma 3], we again have $\lim_{i\to\infty} \lambda_{p,j}(M, g_i) = \infty$. As above, by taking a subsequence, we can assume that there is an $O(n)$-manifold $\check{X}$ so that $d_{GH}^{O(n)}(FM_i, \check{X}) \leq \frac{1}{i}$. In particular, $\dim(X) > 0$. Also as above, taking a further subsequence, we can assume that there are $\check{E}$, $\mathfrak{I}$, $\check{A}'$ and $h^{\check{E}}$ so that if $\lambda_{p,j}(X;E) < \infty$ then for large i, $\lambda_{p,j}(M, g_i)$ is ϵ-close to $\lambda_{p,j}(X;E)$. Suppose that $\dim(X) \geq p$. As E^0 is the trivial $\mathbb{R}$-bundle on X, there is an inclusion $\Omega^p(X) \subset \bigoplus_{a+b=p} \Omega^a(X;E^b)$. The Laplacian on $\Omega^p(X)$ is unbounded and so $\lambda_{p,j}(X;E) < \infty$. This contradicts the assumption that $\lim_{i\to\infty} \lambda_{p,j}(M, g_i) = \infty$. Thus $\dim(X) < p$. Similarly, if $\Omega^a(X;E^b) \neq 0$ for any a and b satisfying $a + b = p$ then $\triangle_p^E$ is unbounded and we get a contradiction. Using the construction of E in terms of affine-parallel differential forms and the finiteness statement of Proposition 4, the claim follows.

3. If $p \in \{0,1\}$ and $A^1_{n,p,j,K} = \infty$ then there is a sequence $\{M_i\}_{i=1}^\infty$ of connected closed n-dimensional Riemannian manifolds with $\operatorname{diam}(M_i) = 1$ and $\parallel R^{M_i} \parallel_\infty \leq K$ such that $\lim_{i\to\infty} \lambda_{p,j}(M_i) = \infty$. As above, we can assume that there is an $O(n)$-manifold $\check{X}$ so that $d_{GH}^{O(n)}(FM_i, \check{X}) \leq \frac{1}{i}$. Also as above, we can assume that there are $\check{E}$, $\mathfrak{I}$, $\check{A}'$ and $h^{\check{E}}$ so that if $\lambda_{p,j}(X;E) < \infty$ then for large i, $\lambda_{p,j}(M_i)$ is ϵ-close to $\lambda_{p,j}(X;E)$. As E^0 is the trivial $\mathbb{R}$-bundle on X and $\dim(X) \geq 1$, there is an inclusion $0 \neq \Omega^p(X) \subset \bigoplus_{a+b=p} \Omega^a(X;E^b)$. The Laplacian on $\Omega^p(X)$ is unbounded and so $\lambda_{p,j}(X;E) < \infty$. This contradicts the assumption that $\lim_{i\to\infty} \lambda_{p,j}(M_i) = \infty$. $\square$

Proof of Theorem 2 : Consider Theorem 1.2 in the case $p = 2$. Then $\dim(X) = 1$. There are two cases.
1. $X = S^1$. As M is an affine fiber bundle over S^1, it has the claimed structure.
2. $X = [0, 1]$, thought of as a singular space with two strata. With respect to the singular fibration $q : M \to X$, put $Z = q^{-1}(1/2)$ and $Z_i = q^{-1}(i - 1)$. Then M has the claimed structure. □

Remark : The mapping cone is just defined up to homeomorphism, which is why we use the word "homeomorphic" in Theorem 2.2. Writing $Z = \check{Z}/H$ and $Z_i = \check{Z}/H_i$, the fact that M is a manifold implies that H_i/H is a sphere, and hence a circle. Thus ϕ_i defines a circle bundle and so M is the result of gluing the total spaces of two D^2-bundles along their boundaries.

Example 4 [12, Section 8] **:** Let N be the flat 3-manifold $\mathcal{G}_6$ in the notation of [23, p. 122]. It has the rational homology of a 3-sphere. Let h be a flat Riemannian metric on N. Let $C > 0$ be the lowest eigenvalue of $\triangle_1^N$. Put $M = S^1 \times N$. For $t > 0$, give M the (flat) product metric $g_t = d\theta^2 + t^2\, h$. Then one finds that the lowest eigenvalue of $\triangle_2^M$ is $C\, t^{-2}$. Taking $t \to 0$, we obtain that $A^1_{2,j,K}(M) = \infty$ for all $j \in \mathbb{Z}^+$ and $K \geq 0$. This is an example of Theorem 2 in which $X = S^1$ and $Z = N$.

4. Small Positive Eigenvalues

In this section we characterize the manifolds M for which the p-form Laplacian has small positive eigenvalues. We use the compactness result of Section 3 to show that if M has j small eigenvalues of the p-form Laplacian, with $j > \mathrm{b}_p(M)$, then M collapses and there is an associated basic flat degree-1 superconnection $\check{A}'_\infty$ with $\dim(\mathrm{H}^p(A'_\infty)) \geq j$. We then use the spectral sequence of $\check{A}'_\infty$ to characterize when this can happen. In Theorem 3 we give a bound on the number of small eigenvalues of the 1-form Laplacian. In Theorems 4 and 6 we give bounds on the number of small eigenvalues of the p-form Laplacian when one is sufficiently close to a limit space of dimension $\dim(M) - 1$ and characterize when small eigenvalues can occur.

Proposition 5. *If $a_{p,j,K}(M) = 0$ and $j > \mathrm{b}_p(M)$ then there are*
1. An $O(n)$-equivariant affine fiber bundle $FM \to \check{X}$,
2. A corresponding $O(n)$-equivariant $\mathbb{Z}$-graded real vector bundle $\check{E} \to \check{X}$ and
3. A basic flat degree-1 superconnection $\check{A}'_\infty$ on $\check{E}$
such that $\dim(\mathrm{H}^p(A'_\infty)) \geq j$.

Proof. Put $n = \dim(M)$. If $a_{p,j,K}(M) = 0$ then there is a sequence $\{g_i\}_{i=1}^\infty$ in $\mathcal{M}(M, K)$ such that $\lim_{i\to\infty} \lambda_{p,j}(M, g_i) = 0$. Let M_i denote M with the Riemannian metric g_i. As $j > \mathrm{b}_p(M)$, the M_i's must collapse. As in the proof of Theorem 1, after smoothing the metrics and taking a subsequence, we may assume that there is a smooth Riemannian $O(n)$-manifold $\check{X}$ so that $\lim_{i\to\infty} d_{GH}^{O(n)}(FM_i, \check{X}) = 0$. From Proposition 3, there are
1. $O(n)$-basic $\mathbb{Z}$-graded real vector bundles $\{\check{E}_i\}_{i=1}^\infty$ on $\check{X}$,
2. Basic flat degree-1 superconnections $\{\check{A}'_i\}_{i=1}^\infty$ on the $\check{E}_i$'s and
3. $O(n)$-invariant Euclidean inner products $\{h^{\check{E}_i}\}_{i=1}^\infty$ on the $\check{E}_i$'s

so that $\lim_{i\to\infty} \lambda_{p,j}(X;E_i) = 0$. From Proposition 4, after taking a subsequence we may assume that the $\check{E}_i$'s are all isomorphic to a single $O(n)$-equivariant $\mathbb{Z}$-graded real vector bundle $\check{E}$, the $E_{i,\alpha}$'s are all isomorphic to a single $\mathbb{Z}$-graded real vector bundle E_α and the triples $\left\{ \left(\mathfrak{I}_i, \check{A}'_i, h^{\check{E}_i} \right) \right\}_{i=1}^\infty$ lie in a compact subset C of $(\mathcal{S}_{\check{E},basic} \times \mathcal{H}_{\check{E},basic})/\mathcal{G}_{\check{E},basic}$. Following the method of proof of Theorem 1, we obtain a limit triple $\left(\mathfrak{I}_\infty, \check{A}'_\infty, h^{\check{E}_\infty} \right) \in \mathcal{S}_{\check{E},basic} \times \mathcal{H}_{\check{E},basic}$.

Let $g^{T\check{X}_\infty}$ denote the Riemannian metric on $\check{X}$. Let $\triangle^{E_\infty}$ denote the basic Laplacian constructed from $\check{A}'_\infty$, $g^{T\check{X}_\infty}$ and $h^{\check{E}_\infty}$. From Lemma 6 and the discreteness of the spectrum of $\triangle^{M_i}$, it follows that the spectrum of $\triangle^{E_\infty}$ is discrete. From the continuity of $\lambda_{p,j}$ as a function on $\mathcal{S}_{\check{E},basic} \times \mathcal{H}_{\check{E},basic}$, we know that $\dim(\mathrm{Ker}(\triangle_p^{E_\infty})) \geq j$. It remains to show that $\mathrm{Ker}(\triangle_p^{E_\infty}) \cong \mathrm{H}^p(A'_\infty)$. This amounts to a regularity issue.

In what follows, we omit the subscript p. Fix i, with $\check{E}_i$ isomorphic as an $O(n)$-equivariant vector bundle to $\check{E}$. The idea will be to transfer the analysis of the superconnection Laplacian to a more standard analysis on M_i. Let $g^{T\check{X}_i}$ denote the Riemannian metric on $\check{X}$ coming from the Riemannian affine fiber bundle $FM_i \to \check{X}$ and let $h^{\check{E}_i}$ denote the inner product on $\check{E}_i$ induced from the Riemannian affine fiber bundle.

We know that $\mathrm{Ker}(\triangle^{E_\infty}) \cong \mathrm{Ker}(\check{A}'_\infty)/\overline{\mathrm{Im}(\check{A}'_\infty)}$, where $\check{A}'_\infty$ acts on $\Omega_{basic,max}\left(\check{X};\check{E}\right)$. As in the proof of Theorem 1.1, we can conjugate $\left(\mathfrak{I}_\infty, \check{A}'_\infty, h^{\check{E}_\infty} \right)$ by an $O(n)$-equivariant automorphism of $\check{E}$ to make $\check{K}_{\tilde{x}}$ the same whether computed with $\mathfrak{I}_i$ or $\mathfrak{I}_\infty$. There is an isomorphism from the basic forms defined using $\mathfrak{I}_\infty$ to the basic forms defined using $\mathfrak{I}_i$ which, using the notation of (2.21), is given on $\check{X}_\alpha$ by

$$T = \prod_j \left(1 - e(\mathfrak{x}_j^*)\, \mathfrak{I}_{i,\mathfrak{x}_j}\right) \prod_j \left(1 + e(\mathfrak{x}_j^*)\, \mathfrak{I}_{\infty,\mathfrak{x}_j}\right). \tag{4.1}$$

Then $T\, \check{A}'_\infty\, T^{-1}$ is a flat superconnection which is basic with respect to $\mathfrak{I}_i$. Replacing $\check{A}'_\infty$ and $h^{\check{E}_\infty}$ by $T\, \check{A}'_\infty\, T^{-1}$ and the inner product induced by T, we may assume that the basic structure on $\check{E}$ is that of $\mathfrak{I}_i$. We relabel $T\, \check{A}'_\infty\, T^{-1}$ as $\check{A}'_\infty$.

As the underlying topological vector space of the Hilbert space $\Omega_{basic,L^2}\left(\check{X};\check{E}\right)$ is the same whether the Hilbert space is constructed using $\left(g^{T\check{X}_\infty}, h^{\check{E}_\infty}\right)$ or $\left(g^{T\check{X}_i}, h^{\check{E}_i}\right)$, it is equivalent to consider the reduced cohomology of $\check{A}'_\infty$ on $\Omega_{basic,max}\left(\check{X};\check{E}\right)$, where the Hilbert space structure now comes from $\left(g^{T\check{X}_i}, h^{\check{E}_i}\right)$. Let $\triangle^E$ denote the basic Laplacian constructed from $\check{A}'_\infty$, $\mathfrak{I}_i$, $g^{T\check{X}_i}$ and $h^{\check{E}_i}$.

Let us first consider $\triangle^{E_i}$, the basic Laplacian constructed using $\check{A}'_i$, $\mathfrak{I}_i$, $g^{T\check{X}_i}$ and $h^{\check{E}_i}$. As in the proof of Proposition 2, there are isomorphisms

$$\Omega^*(M_i) \cong \Omega^*_{basic}(FM_i) = \Omega_{basic}(\check{X};\check{E}_i) \oplus \mathrm{Ker}(\mathcal{P}^{fib}). \tag{4.2}$$

With respect to the inner products, we have isometric isomorphisms

$$\Omega^*_{L^2}(M_i,\mu_i) \cong \Omega^*_{basic,L^2}(FM_i) = \Omega_{basic,L^2}(\check{X};\check{E}_i) \oplus \mathrm{Ker}(\mathcal{P}^{fib}), \tag{4.3}$$

in terms of which the Laplacians are related by

$$\triangle^{M_i}_{\mu_i} \cong \triangle^{FM_i}_{basic} = \triangle^{E_i} \oplus \triangle^{FM_i}_{basic}\big|_{Ker(\mathcal{P}^{fib})}. \tag{4.4}$$

By standard elliptic theory, $(I + \triangle_{\mu_i}^{M_i})^{-1}$ and $(I + \triangle_{\mu_i}^{M_i})^{-1} d_{M_i}^*$ are compact, and $d_{M_i} (I + \triangle_{\mu_i}^{M_i})^{-1/2}$ is bounded. Then $(I + \triangle^{E_i})^{-1}$ and $(I + \triangle^{E_i})^{-1} \left(\check{A}_i'\right)_{basic}^*$ are compact, and $\check{A}_i' (I + \triangle^{E_i})^{-1/2}$ is bounded.

Now $\triangle^E$ is also well-defined on $\Omega_{basic}(\check{X}; \check{E}_i)$. Hereafter, we will change notation from $\check{E}_i$ to $\check{E}$. Put $y = \check{A}_\infty' - \check{A}_i' \in \Omega_{basic}(\check{X}; \mathrm{End}(\check{E}))$, which in particular is a bounded operator on $\Omega_{basic,L^2}\left(\check{X}; \check{E}\right)$. Then

$$\triangle^E - \triangle^{E_i} = \left(\check{A}_i'\right)_{basic}^* y + y^* \check{A}_i' + y^* y. \tag{4.5}$$

It follows that $(I + \triangle^{E_i})^{-1} \left(\triangle^E - \triangle^{E_i}\right) (I + \triangle^{E_i})^{-1/2}$ is compact. Then from [20, Vol. IV, Chapter 13, Pf. of Corollary 4, p. 116], it follows that $\triangle^E$ has the same essential spectrum as $\triangle^{E_i}$, i.e. the empty set, showing that $\triangle^E$ has a discrete spectrum. Thus $\check{A}_\infty'$ has a closed image on $\Omega_{basic,max}\left(\check{X}; \check{E}\right)$. Hence

$$\mathrm{Ker}\left(\check{A}_\infty'\right) / \overline{\mathrm{Im}\left(\check{A}_\infty'\right)} = \mathrm{Ker}\left(\check{A}_\infty'\right) / \mathrm{Im}\left(\check{A}_\infty'\right), \tag{4.6}$$

where the latter is the (unreduced) cohomology of $\check{A}_\infty'$ on $\Omega_{basic,max}\left(\check{X}; \check{E}\right)$. (This also follows from the discreteness of the spectrum of $\triangle^{E_\infty}$.) It remains to show that this is isomorphic to the cohomology of $\check{A}_\infty'$ on the smooth forms $\Omega_{basic}\left(\check{X}; \check{E}\right)$.

There is an obvious cochain inclusion $\Omega_{basic}\left(\check{X}; \check{E}\right) \rightarrow \Omega_{basic,max}\left(\check{X}; \check{E}\right)$. We will construct a linear map K on $\Omega_{basic,max}\left(\check{X}; \check{E}\right)$ of degree -1 so that $I - \check{A}_\infty' K - K \check{A}_\infty'$ sends $\Omega_{basic,max}\left(\check{X}; \check{E}\right)$ to $\Omega_{basic}\left(\check{X}; \check{E}\right)$. This will give a cochain homotopy equivalence between $\Omega_{basic}\left(\check{X}; \check{E}\right)$ and $\Omega_{basic,max}\left(\check{X}; \check{E}\right)$, showing that the two complexes have isomorphic cohomologies.

To the affine fiber bundle $FM_i \rightarrow \check{X}$ we associate an infinite-dimensional $\mathbb{Z}$-graded $O(n)$-basic vector bundle $\check{W}$ on $\check{X}$, as in [19, Section 5], so that $\Omega\left(\check{X}; \check{W}\right) \cong \Omega^*(FM_i)$. The inclusion of fibers $\check{E}_{\check{x}} \subset \check{W}_{\check{x}}$ is isomorphic to the inclusion $1 \otimes \Lambda^*(\mathfrak{n}^*) \subset C^\infty(\check{Z}_{\check{x}}) \otimes \Lambda^*(\mathfrak{n}^*)$. Then the inclusion $\mathrm{Id} \otimes \mathrm{End}(\Lambda^*(\mathfrak{n}^*)) \subset \mathrm{End}(C^\infty(\check{Z}_{\check{x}}) \otimes \Lambda^*(\mathfrak{n}^*))$ induces an extension $Y \in \Omega\left(\check{X}; \mathrm{End}(\check{W})\right)$ of y. As $y \in \Omega_{basic}\left(\check{X}; \mathrm{End}(\check{E})\right)$, it follows that $Y \in \Omega_{basic}\left(\check{X}; \mathrm{End}(\check{W})\right)$ and that the corresponding map on $\Omega_{basic}\left(\check{X}; \check{W}\right) \cong \Omega^*(M_i)$ is diagonal with respect to (4.2), so that we can write $Y = \begin{pmatrix} y & 0 \\ 0 & * \end{pmatrix}$.

Put $D = d_{M_i} + Y$, a pseudodifferential operator on M_i of order 1. It decomposes with respect to (4.3) as $D = \begin{pmatrix} \check{A}_\infty' & 0 \\ 0 & * \end{pmatrix}$. Then $DD^* + D^*D$ is a elliptic pseudodifferential operator on M_i of order 2, which decomposes as $DD^* + D^*D = \begin{pmatrix} \triangle^E & 0 \\ 0 & * \end{pmatrix}$. Fix $t > 0$ and let K be the restriction of $D^* \frac{I - e^{-t(DD^* + D^*D)}}{DD^* + D^*D}$ to the first factor $\Omega_{basic,L^2}(\check{X}; \check{E})$. Then $K = (\check{A}_\infty')_{basic}^* \frac{I - e^{-t\triangle^E}}{\triangle^E}$, as defined spectrally. We have

$$I - \check{A}_\infty' K - K \check{A}_\infty' = e^{-t\triangle^E}. \tag{4.7}$$

Now $e^{-t\triangle^E}$ is the restriction of $e^{-t(DD^* + D^*D)}$ to $\Omega_{basic,L^2}(\check{X}; \check{E})$. By elliptic theory, $e^{-t(DD^* + D^*D)}$ maps $\Omega_{L^2}^*(M_i, \mu_i)$ to $\Omega^*(M_i)$. Hence $e^{-t\triangle^E}$ maps $\Omega_{basic,L^2}(\check{X}; \check{E})$ to

$$\Omega_{basic,L^2}(\check{X}; \check{E}) \cap \Omega^*(M_i) = \Omega_{basic}(\check{X}; \check{E}). \tag{4.8}$$

This proves the proposition. □

Under the hypotheses of Proposition 5, using the spectral sequence of Section 2 we deduce that

$$j \leq \dim(\mathrm{H}^p(A'_\infty)) \leq \sum_{a+b=p} \dim\left(\mathrm{H}^a(X; \mathrm{H}^b(A'_{\infty,[0]}))\right). \tag{4.9}$$

Given an $O(n)$-equivariant affine fiber bundle $FM \to \check{X}$ with fiber $\check{Z} = \Gamma\backslash N$, there is a spectral sequence to compute $\mathrm{H}^*(M;\mathbb{R})$ as the cohomology of a basic flat degree-1 superconnection $\check{A}'$ as in [19, Example 4], working equivariantly with respect to the $O(n)$-action and using basic forms. With the notation of Section 2, the E_2-term of this spectral sequence is $E_2^{p,q} = \mathrm{H}^p(X; \mathrm{H}^q(A'_{[0]}))$. If $x \in X$ is covered by $\check{x} \in \check{X}$ then the stalk of $\mathrm{H}^q(A'_{[0]})$ at x is given by the cohomology of $\check{A}'_{[0]} = d^{\check{Z}}$ on $\check{K}^H_{\check{x}}$. If $\check{x} \in \check{X}$ has isotropy group $H \subset O(n)$ then for $\mathfrak{x} \in \mathfrak{h}$, $\mathfrak{I}_\mathfrak{x}$ is interior multiplication by the corresponding vector field on $\Omega^*(\check{Z}_{\check{x}})$. We see that $\check{K}^H_{\check{x}}$ consists of the H-basic forms on the fiber $\check{Z}_{\check{x}}$. Then the stalk of $\mathrm{H}^q(A'_{[0]})$ at x is isomorphic to $\mathrm{H}^q(\check{Z}_{\check{x}}/H;\mathbb{R})$; recall that H acts freely on $\check{Z}_{\check{x}}$.

On the other hand, there is a spectral sequence to compute $\mathrm{H}^*(M;\mathbb{R})$ from the map $r : M \to X$ [6, p. 179], with E_2-term $\mathrm{H}^*(X; \mathrm{H}^*(Z;\mathbb{R}))$. Here $\mathrm{H}^*(Z;\mathbb{R})$ is a sheaf on X whose stalk over $x \in X$ is $\mathrm{H}^*(r^{-1}(x);\mathbb{R})$. As $r^{-1}(x) = \check{Z}_{\check{x}}/H$, we see that the two spectral sequences have similar E_2-terms. In fact the two spectral sequences are equivalent, as follows from the construction in [6, p. 179].

To obtain results about small eigenvalues, the idea now is to compare the spectral sequence of $\check{A}'_\infty$ with the spectral sequence of the map $M \to X$.

Example 5 : As an illustration of the methods, we analyze the behavior of the differential form Laplacian under the collapse of S^3 to an interval which is described in [8, Example 1.5]. With respect to the isometric action of $\mathrm{SO}(2)\times\mathrm{SO}(2) \subset \mathrm{SO}(4)$ on the round S^3, we shrink the metric on S^3 in the direction of a subgroup $\mathbb{R} \subset \mathrm{SO}(2)\times\mathrm{SO}(2)$ of irrational slope. The resulting metrics approach the closed interval $X = S^3/(\mathrm{SO}(2)\times\mathrm{SO}(2))$ in the Gromov-Hausdorff topology.

For simplicity, we consider the principal spin bundle $S^3 \times S^3$, with structure group $G = \mathrm{SU}(2)$, instead of the orthonormal frame bundle FM. Then during the collapse we have $\check{X} = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and $\check{Z} = T^2$. Correspondingly, $\check{E}$ is a vector bundle over $\check{X}$ with fiber isomorphic to $\Lambda^*((\mathbb{R}^2)^*)$, on which $\check{A}'_{[0]}$ acts as the zero map. If $\check{x} \in \check{X}$ covers $x \in \mathrm{int}(X)$ then its isotropy group H is trivial, while if $\check{x} \in \check{X}$ covers $x \in \partial X$ then its isotropy group H is isomorphic to $\mathrm{U}(1)$. The sheaf $\mathrm{H}^0(A'_{\infty,[0]})$ has stalk $\mathbb{R}$ over each $x \in X$. The sheaf $\mathrm{H}^1(A'_{\infty,[0]})$ has stalk $\mathbb{R}$ over $x \in \partial X$ and stalk $\mathbb{R}^2$ over $x \in \mathrm{int}(X)$, where we can think of the $\mathbb{R}$ over one component of ∂X as corresponding to $\mathbb{R}\oplus 0$ and the $\mathbb{R}$ over the other component of ∂X as corresponding to $0 \oplus \mathbb{R}$. The sheaf $\mathrm{H}^2(A'_{\infty,[0]})$ has stalk 0 over $x \in \partial X$ and stalk $\mathbb{R}$ over $x \in \mathrm{int}(X)$. One finds that the only nonzero components of the E_2-term of the spectral sequence for $\check{A}'_\infty$ are $\mathrm{H}^0\left(X; \mathrm{H}^0(A'_{\infty,[0]})\right) = \mathrm{H}^1\left(X; \mathrm{H}^2(A'_{\infty,[0]})\right) = \mathbb{R}$. (These correspond to the zero-eigenvalues of the differential form Laplacian on S^3.)

Let $r : S^3 \to X$ be the quotient map. If $x \in \mathrm{int}(X)$ then $r^{-1}(x) = T^2$, while if $x \in \partial X$ then $r^{-1}(x) = S^1$. One finds that the Leray spectral sequence to compute $\mathrm{H}^*(S^3;\mathbb{R})$ from the map r coincides with the spectral sequence to compute the cohomology of $\breve{A}'_\infty$. The conclusion is that there are no small positive eigenvalues in the collapse. This is in contrast to what happens in the Berger collapse of S^3 to S^2 [10, Example 1.2].

Corollary 1. *For any $K \geq 0$, $a_{0,2,K}(M) > 0$. That is, there are no small positive eigenvalues of the Laplacian on functions.*

Proof. Suppose that $a_{0,2,K}(M) = 0$. Using Proposition 5 and equation (4.9) in the case $p = 0$, we conclude that $2 \leq \dim\left(\mathrm{H}^0(X;\mathrm{H}^0(A'_{\infty,[0]}))\right)$. However, E^0 is the trivial $\mathbb{R}$-bundle on X, on which $A'_{\infty,[0]}$ acts by zero. Then $\mathrm{H}^0(A'_{\infty,[0]})$ is the trivial $\mathbb{R}$-bundle on X and so $\dim(\mathrm{H}^0(X;\mathrm{H}^0(A'_{\infty,[0]}))) = 1$, which is a contradiction. □

Remark : Corollary 1 is true under the weaker assumption of a lower bound on the Ricci curvature ([2] and references therein).

Corollary 2. *Suppose that $a_{p,j,K}(M) = 0$ and $j > \mathrm{b}_p(M)$. Let X be the limit space coming from the above argument. If $\dim(X) = 0$, write the almost flat manifold M topologically as the quotient of a nilmanifold $\widehat{\Gamma}\backslash N$ by a finite group F. Let $\mathfrak{n}$ denote the Lie algebra of the nilpotent Lie group N and let $\cdot^F$ denote F-invariants. Then $j \leq \dim(\Lambda^p(\mathfrak{n}^*)^F)$.*

Proof. With reference to Proposition 5, in this case X is a point and $E = \Lambda^*(\mathfrak{n}^*)^F$. Then for any superconnection A'_∞ on E, i.e. differential $A'_{\infty,[0]}$ on E, we have $\dim(\mathrm{H}^p(A'_\infty)) \leq \dim(E^p)$. □

Remark : It follows from [19, Theorem 6] that in fact $a_{p,j,K}(M) = 0$ for $j = \dim(\Lambda^p(\mathfrak{n}^*)^F)$.

Proof of Theorem 3 : With reference to Proposition 5 and equation (4.9),

$$j \leq \dim\left(\mathrm{H}^1(X;\mathrm{H}^0(A'_{\infty,[0]}))\right) + \dim\left(\mathrm{H}^0(X;\mathrm{H}^1(A'_{\infty,[0]}))\right). \tag{4.10}$$

As $\mathrm{H}^0(A'_{\infty,[0]})$ is the trivial $\mathbb{R}$-bundle on X, $\dim\left(\mathrm{H}^1(X;\mathrm{H}^0(A'_{\infty,[0]}))\right) = \mathrm{b}_1(X)$. As $A'_{\infty,[0]}$ acts by zero on E^0, there is an injection $\mathrm{H}^1(A'_{\infty,[0]}) \to E^1$. Let $\dim\left(\mathrm{H}^1(A'_{\infty,[0]})\right)$ denote the dimension of the generic stalk of the sheaf $\mathrm{H}^1(A'_{\infty,[0]})$ and put $\dim(E^1) = \dim(E^1_\beta)$, where β again denotes the principal normal orbit type. Then

$$\dim\left(\mathrm{H}^0(X;\mathrm{H}^1(A'_{\infty,[0]}))\right) \leq \dim\left(\mathrm{H}^1(A'_{\infty,[0]})\right) \leq \dim\left(E^1\right) \leq \dim(M) - \dim(X). \tag{4.11}$$

Thus $j \leq \mathrm{b}_1(X) + \dim(M) - \dim(X)$. On the other hand, the spectral sequence for $\mathrm{H}^*(M;\mathbb{R})$ gives

$$\mathrm{H}^1(M;\mathbb{R}) = \mathrm{H}^1(X;\mathbb{R}) \oplus \mathrm{Ker}\left(\mathrm{H}^0(X;\mathrm{H}^1(Z;\mathbb{R})) \to \mathrm{H}^2(X;\mathbb{R})\right). \tag{4.12}$$

In particular, $\mathrm{b}_1(X) \leq \mathrm{b}_1(M)$. The corollary follows. □

Remark : Using heat equation methods [2] one can show that there is an increasing function f such that if $\mathrm{Ric}(M) \geq -(n-1)\lambda^2$ and $\mathrm{diam}(M) \leq D$ then the number of small eigenvalues of the 1-form Laplacian is bounded above by $f(\lambda D)$. This result is weaker than Theorem 3 when applied to manifolds with sectional curvature bounds, but is more general in that it applies to manifolds with just a lower Ricci curvature bound.

Proof of Theorem 4 : In general, if X is a compact metric space of Hausdorff dimension $n-1$, and a manifold M^n with $\parallel R^M \parallel_\infty \leq K$ is sufficiently Gromov-Hausdorff close to X, then the description of the local geometry of M in [7, Theorem 1.3] says that M has the following structure. Given $m \in M$, there are a neighborhood U_m of m, a finite regular covering $\widehat{U}_m$ of U_m with covering group F and a locally free S^1-action on $\widehat{U}_m$ which is F-equivariant with respect to a homomorphism $\eta : F \rightarrow \mathrm{Aut}(S^1) (\cong \mathbb{Z}_2)$. (In fact, we can take F to be $\{e\}$ or $\mathbb{Z}_2$.) Then we can take X so that locally, it is the quotient $\widehat{U}_m/(F \widetilde{\times} S^1)$. Hence X is an orbifold and $M \rightarrow X$ is an orbifold circle bundle, with its orientation bundle $\mathcal{O}$ given locally as $\widehat{U}_m \times_{F \widetilde{\times} S^1} \mathbb{R} \rightarrow \widehat{U}_m/(F \widetilde{\times} S^1)$. Here $F \widetilde{\times} S^1$ acts on $\mathbb{R}$ through F.

Suppose that the claim of the corollary is not true. Then there is a sequence of connected closed n-dimensional Riemannian manifolds $\{(M_i, g_i)\}_{i=1}^\infty$ with $\parallel R^{M_i}(g_i) \parallel_\infty \leq K$ and $\lim_{i\rightarrow\infty} M_i = X$ which provides a counterexample. As there is a finite number of isomorphism classes of flat real line bundles on X, after passing to a subsequence we may assume that each M_i is an orbifold circle bundle over X with a fixed orientation bundle $\mathcal{O}$ and that $\lim_{i\rightarrow\infty} \lambda_{p,j}(M_i, g_i) = 0$ for $j = \mathrm{b}_p(X) + \mathrm{b}_{p-1}(X;\mathcal{O}) + 1$. As in the proofs of Theorem 1 and Proposition 5, we obtain $E = E^0 \oplus E^1$ on X, with E^0 a trivial $\mathbb{R}$-bundle and $E^1 = \mathcal{O}$, and a limit superconnection A'_∞ on E with $A'_{\infty,[0]} = 0$ and $A'_{\infty,[1]} = \nabla^E$, the canonical flat connection. Then as in (4.9), we obtain

$$j \leq \dim(\mathrm{H}^p(A'_\infty)) \leq \mathrm{b}_p(X) + \mathrm{b}_{p-1}(X;\mathcal{O}), \tag{4.13}$$

which is a contradiction.

Proof of Theorem 5 : The proof of the theorem is along the same lines as that of [19, Theorem 5], replacing [19, (7.8)] by (4.9). We omit the details.

Proof of Theorem 6 : The E_2-term of the spectral sequence to compute $\mathrm{H}^*(M;\mathbb{R})$ consists of $E_2^{p,0} = \mathrm{H}^p(X;\mathbb{R})$ and $E_2^{p,1} = \mathrm{H}^p(X;\mathcal{O})$. The differential is $\mathcal{M}_\chi$. The corollary now follows from Theorem 5. □

References

[1] M. Anderson, "Hausdorff Perturbations of Ricci-Flat Manifolds and the Splitting Theorem", Duke Math. J. 68, p. 67-82 (1992)

[2] P. Bérard, "From Vanishing Theorems to Estimating Theorems: the Bochner Technique Revisited", Bull. Amer. Math. Soc. 19, p. 371-406 (1988)

[3] N. Berline, E. Getzler and M. Vergne, Heat Kernels and the Dirac Operator, Grundl. der Math. Wiss. 298, Springer, Berlin-Heidelberg-New York (1992)

[4] A. Besse, Einstein Manifolds, Springer-Verlag, New York (1987)

[5] J.-M. Bismut and J. Lott, "Flat Vector Bundles, Direct Images and Higher Real Analytic Torsion", J. Amer. Math. Soc. 8, p. 291-363 (1995)

[6] R. Bott and L. Tu, Differential Forms in Algebraic Topology, Springer-Verlag, New York (1982)

[7] J. Cheeger, K. Fukaya and M. Gromov, "Nilpotent Structures and Invariant Metrics on Collapsed Manifolds", J. Amer. Math. Soc. 5, p. 327-372 (1992)

[8] J. Cheeger and M. Gromov, "Collapsing Riemannian Manifolds While Keeping Their Curvature Bounded I", J. Diff. Geom. 23, p. 309-346 (1986)

[9] S.-Y. Cheng, "Eigenvalue Comparison Theorems and its Geometric Applications", Math. Z. 143, p. 289-297 (1975)

[10] B. Colbois and G. Courtois, "A Note on the First Nonzero Eigenvalue of the Laplacian Acting on p-Forms", Manuscripta Math. 68, p. 143-160 (1990)

[11] B. Colbois and G. Courtois, "Petites Valeurs Propres et Classe d'Euler des S^1-Fibrés", Ann. Sci. École Norm. Sup. 33, p. 611-645 (2000)

[12] B. Colbois, P. Ghanaat and E. Ruh, "Curvature and Gradient Expansions for Eigenforms of the Laplacean", preprint (1999)

[13] M. Davis, "Smooth G-Manifolds as Collections of Fiber Bundles", Pac. J. Math. 77, p. 315-363 (1978)

[14] J. Dodziuk, "Eigenvalues of the Laplacian on Forms", Proc. Amer. Math. Soc. 85, p. 437-443 (1982)

[15] K. Fukaya, "Collapsing of Riemannian Manifolds and Eigenvalues of Laplace Operator", Inv. Math. 87, p. 517-547 (1987)

[16] K. Fukaya, "A Boundary of the Set of the Riemannian Manifolds with Bounded Curvatures and Diameters", J. Diff. Geom. 28, p. 1-21 (1988)

[17] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Progress in Mathematics 152, Birkhäuser, Boston (1999)

[18] M. Hilsum, "Signature Operator on Lipschitz Manifolds and Unbounded Kasparov Bimodules", in Operator Algebras and their Connections with Topology and Ergodic Theory, Lecture Notes in Math. 1132, Springer, New York, p. 254-288 (1985)

[19] J. Lott, "Collapsing and the Differential Form Laplacian : The Case of a Smooth Limit Space", to appear, Duke Math. J., http://www.math.lsa.umich.edu/~lott

[20] M. Reed and B. Simon, Methods of Mathematical Physics, Academic Press, New York (1978)

[21] X. Rong, "On the Fundamental Groups of Manifolds of Positive Sectional Curvature", Ann. Math. 143, p. 397-411 (1996)

[22] A. Verona, "A de Rham Type Theorem for Orbit Spaces", Proc. Amer. Math. Soc. 104, p. 300-302 (1988)

[23] J. Wolf, Spaces of Constant Curvature, McGraw Hill, New York (1967)

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109, USA
E-mail address: lott@math.lsa.umich.edu