Long-time behavior in geometric flows

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Outline of the talk

1. Homogeneous spaces and the geometrization conjecture
2. The geometrization conjecture and Ricci flow
3. Finiteness of the number of surgeries
4. Long-time behavior of Ricci flow
5. The Einstein flow

Long-time behavior

Homogeneous spaces and the geometrization conjecture

Geometrization conjecture and Ricci flow

Finiteness of the number of surgeries

Long-time behavior of Ricci flow

Einstein flow

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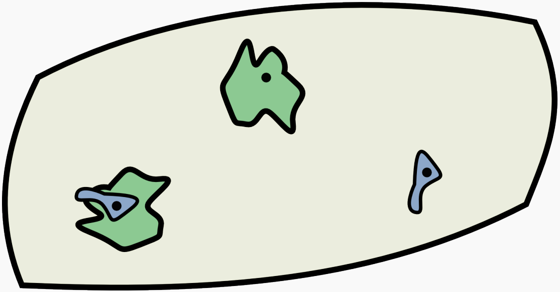
First, how do we understand three dimensional spaces?

In terms of homogeneous spaces.

Locally homogeneous metric spaces

A metric space X is *locally homogeneous* if all $x, y \in X$, there are neighbourhoods U and V of x and y and an isometric isomorphism $(U, x) \to (V, y)$.



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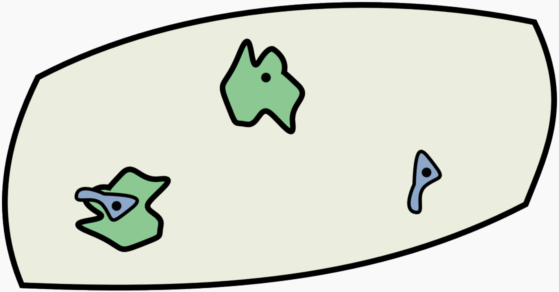
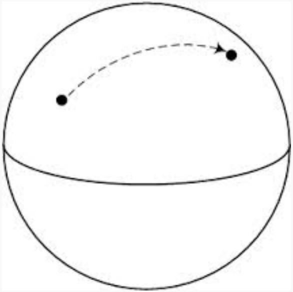


The metric space X is *globally homogeneous* if for all $x, y \in X$, there is an isometric isomorphism $\phi : X \to X$ that $\phi(x) = y$.

Locally homogeneous Riemannian manifolds

Any Riemannian manifold M gets a metric space structure.



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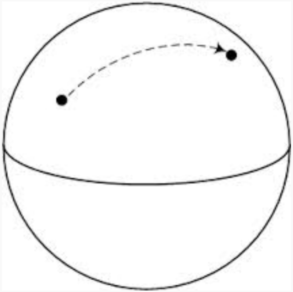

Theorem

(Singer 1960) If M is a complete, simply connected Riemannian manifold which is locally homogeneous, then M is globally homogeneous.

From “The Geometries of 3-Manifolds” by Peter Scott

We will say that a smooth manifold M *admits a geometric structure* if M admits a complete, locally homogeneous Riemannian metric.

It is a theorem of Singer that such a metric on a simply connected manifold X must be homogeneous, i.e. the isometry group of X must act transitively.

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Thus we can regard the universal cover X of M, together with its isometry group, as a geometry in the sense of Klein, and we can sensibly say that M admits a geometric structure modelled on X. Thurston has classified the 3-dimensional geometries and there are eight of them.

Two-dimensional geometries

Globally homogeneous S^2,

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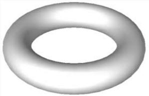
Globally homogeneous $\mathbb{R}^2$, locally homogeneous 

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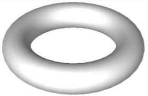
Globally homogeneous $\mathbb{R}^2$, locally homogeneous 

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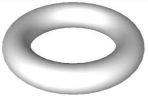
Globally homogeneous $\mathbb{R}^2$, locally homogeneous 

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Three-dimensional Thurston geometries

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Warning : Unlike in two dimensions, not every compact three-dimensional manifold admits a geometric structure, i.e. admits a locally homogeneous Riemannian metric.

Geometrization conjecture

If M is a compact orientable 3-manifold then there is a way to split M into canonical pieces, using certain embedded 2-spheres and 2-tori. (The collection of 2-spheres and 2-tori could be empty.)

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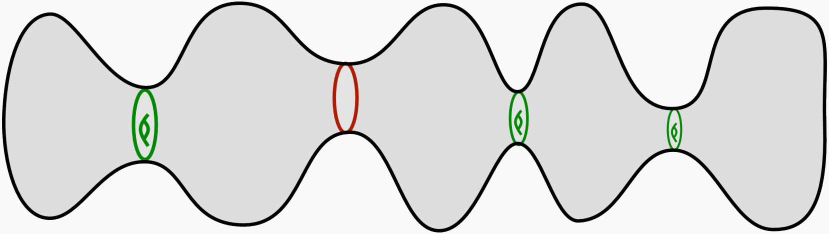

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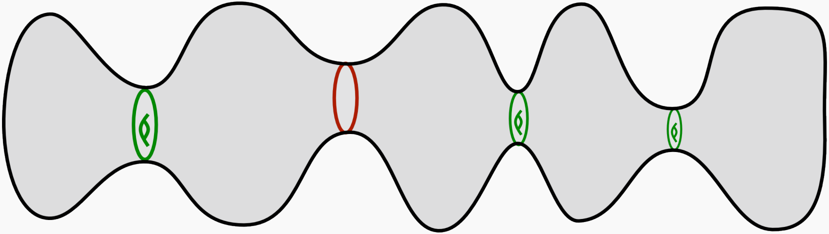

Conjecture (Thurston, 1982)

The ensuing pieces have geometric structures, i.e. admit locally homogeneous metrics

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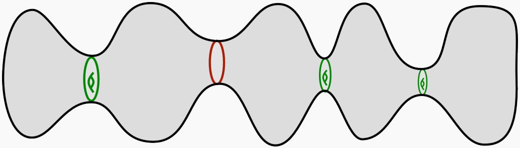

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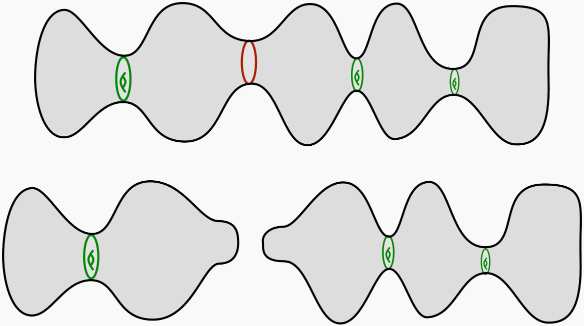

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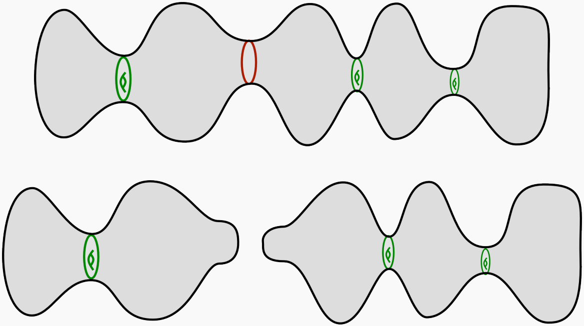

Cut along the 2-tori. The interiors of the ensuing pieces should admit complete locally homogeneous metrics.

Long-time behavior

Ricci flow approach to geometrization

Hamilton's Ricci flow equation

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Maybe the Ricci flow will evolve an initial Riemannian metric into something homogeneous.

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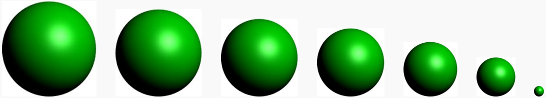

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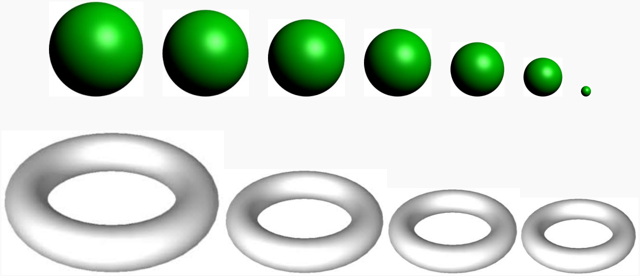

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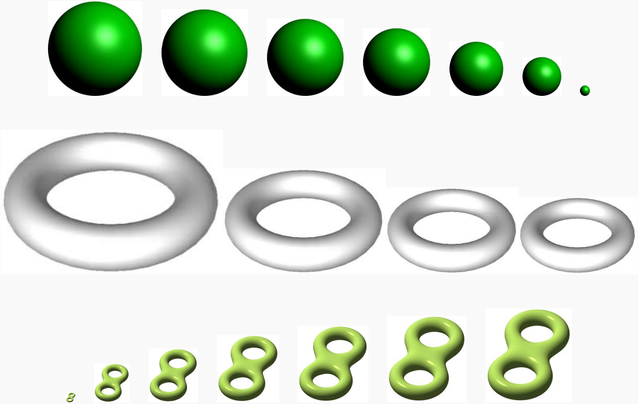
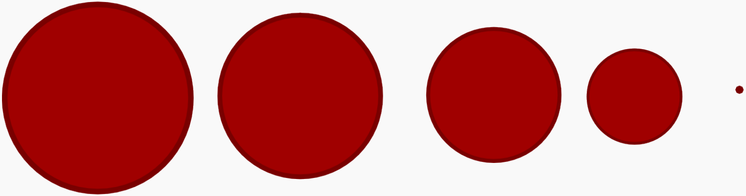


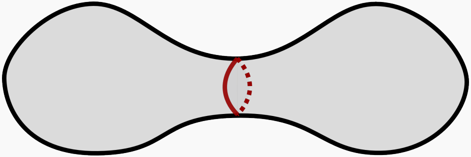
Singularities in 3D Ricci flow

Some components may disappear, e.g. a round shrinking 3-sphere.



Neckpinch

A 2-sphere pinches off. (Drawn one dimension down.)



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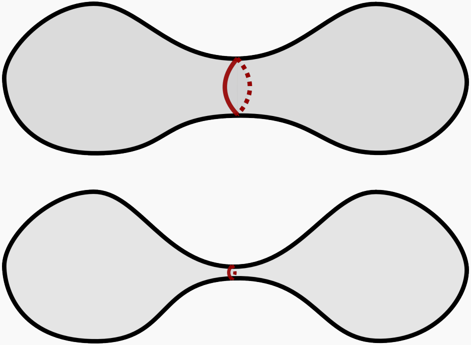

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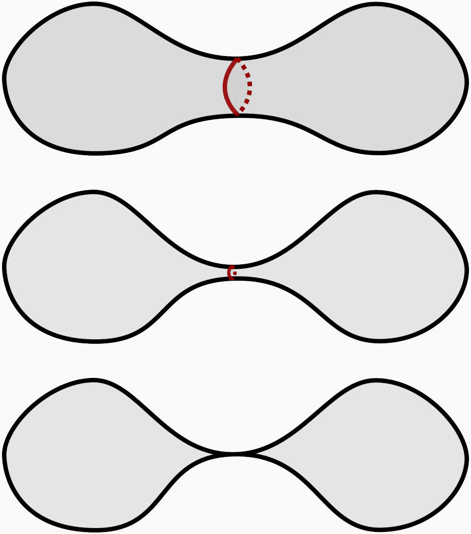
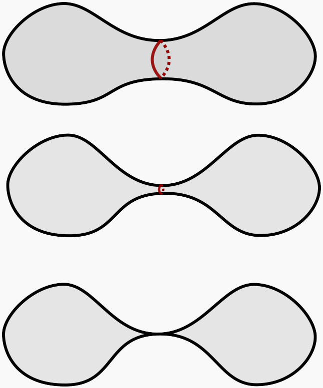


Hamilton's idea of surgery



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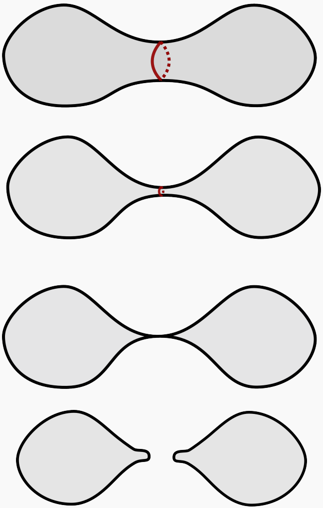

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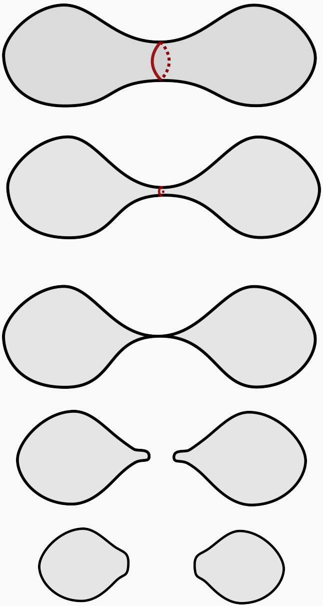
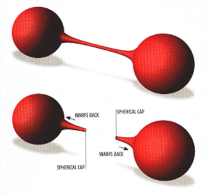


Role of singularities



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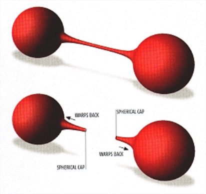

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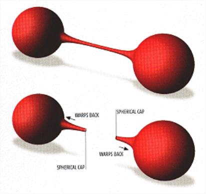

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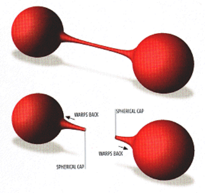

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Remark : the surgeries are done on 2-spheres, not 2-tori.

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(Relevant geometries : $\mathbb{R}^3$, H^3, $H^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, Sol, Nil.)

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From Perelman's first Ricci flow paper : *Moreover, it can be shown ... that the solution is smooth (if nonempty) from some finite time on.*

From Perelman's second Ricci flow paper : *This is a technical paper, which is a continuation of [I]. Here we verify most of the assertions, made in [I, §13]; the exceptions are ... the smoothness of the solution from some time on, which turned out to be unjustified, and, on the other hand, irrelevant for the other conclusions.*

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2. The thin part is a “graph manifold”. (This doesn’t use Ricci flow. Stated by Perelman, proofs by Shioya-Yamaguchi, Morgan-Tian, Bessières-Besson-Boileau-Maillot-Porti and Kleiner-L.)

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Graph manifolds were known to have a geometric decomposition. Along with knowledge of the topological effects of surgeries, this proved the geometrization conjecture.

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Remark : Answering these questions has no topological implication. We already know that the geometrization conjecture holds. Rather, they are *analytic* questions about the Ricci flow.

Long-time behavior

Finiteness of the number of surgeries

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The statement is that if this parameter is small enough (which can always be achieved) then there is a finite number of surgeries.

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Relevance of the second statement :

In Ricci flow, the Riemannian metric has engineering dimension *length*^{2} and time has engineering dimension *length*^{2}.

So the scale-invariant time-t metric is $\widehat{g}(t) = \frac{g(t)}{t}$.

The statement is that for large time, the rescaled metrics $\{\widehat{g}(t)\}$ have *uniformly* bounded sectional curvatures.

This is good because we know lots about metrics with bounded sectional curvature (Cheeger-Fukaya-Gromov).

Ingredients of the proof

Bamler's proof uses all of Perelman's work, and more. Some of the new ingredients :

1. Localizing Perelman's estimates and applying them to local covers of the manifold.

2. Use of minimal surfaces to control the geometry of the thin part.

3. Use of minimal embedded 2-complexes.

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The only case that we completely understand is when M admits *some* hyperbolic metric. Then from Perelman's work, for *any* initial metric on M, as $t \to \infty$ the rescaled Riemannian metric $\widehat{g}(t)$ approaches the metric on M of constant sectional curvature $-\frac{1}{4}$.

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Question : if M doesn't admit a hyperbolic metric, what are the candidate geometries for the long-time behavior?

Quasistatic solutions

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Apparent paradox : What happens to the Ricci flow if our 3-manifold doesn't admit a constant curvature metric?

Nil geometry

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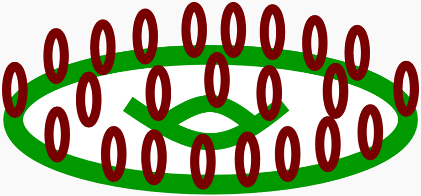

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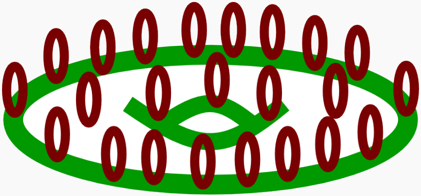

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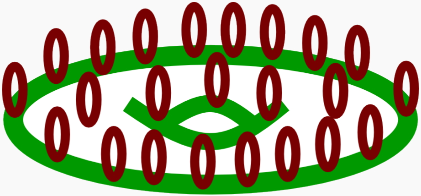

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With the rescaled metric $\widehat{g}(t) = \frac{g(t)}{t}$, $(M, \widehat{g}(t))$ shrinks to a point.

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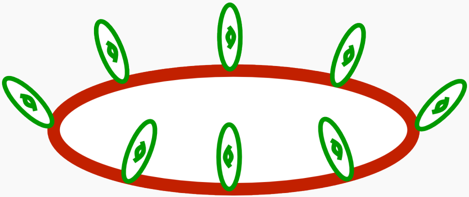
M fibers over a circle with 2-torus fibers. The monodromy is a hyperbolic element of $\mathrm{SL}(2,\mathbb{Z})$.



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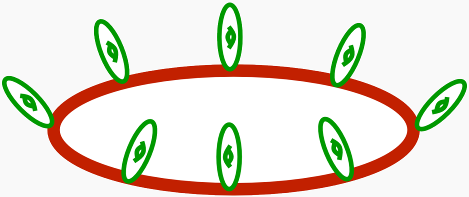

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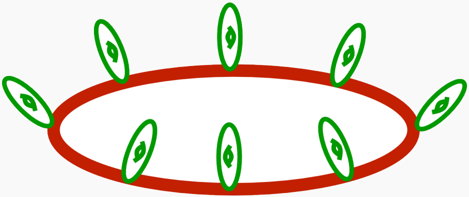

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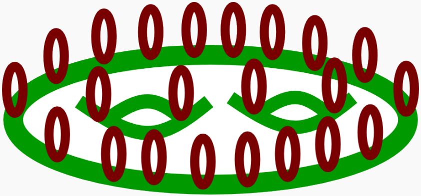
Suppose that M is the unit tangent bundle of a hyperbolic surface Σ.



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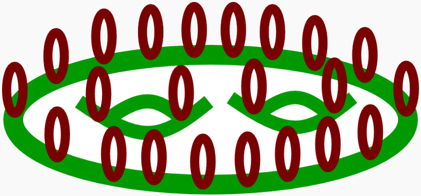

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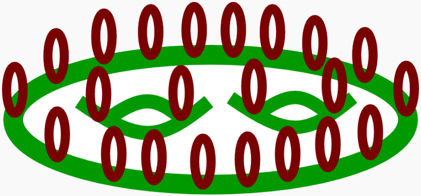

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With the rescaled metric, $(M, \widehat{g}(t))$ approaches the hyperbolic surface Σ. As the fibers shrink, the local geometry of the total space becomes more product-like.

Is there a common pattern?

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To undo the collapsing, let's pass to the universal cover. That is, we are looking at the Ricci flow on a Thurston geometry of type $\mathbb{R}^3, H^3, H^2 \times \mathbb{R}, \mathrm{Sol}, \mathrm{Nil}$ or $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

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(L. 2007) For any initial globally homogeneous metric on such a Thurston geometry, there is a limiting (blowdown) Ricci flow solution, which is an expanding soliton. There is one such soliton for each homogeneity type. It is a universal attractor.

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The limiting solitons

Thurston type	Expanding soliton
H^3	$4\,t\,g_{H^3}$
$H^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}(2,\mathbb{R})}$	$2\,t\,g_{H^2} + g_{\mathbb{R}}$
Sol	$e^{-2z}\,dx^2 + e^{2z}\,dy^2 + 4\,t\,dz^2$
Nil	$\frac{1}{3t^{\frac{1}{3}}}\left(dx + \frac{1}{2}ydz - \frac{1}{2}zdy\right)^2 + t^{\frac{1}{3}}\left(dy^2 + dz^2\right)$
$\mathbb{R}^3$	$g_{\mathbb{R}^3}$

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(L. 2010) Suppose that $(M, g(t))$ is a Ricci flow on a compact three-dimensional manifold, that exists for $t \in [0, \infty)$. Suppose that the sectional curvatures are $O\left(t^{-1}\right)$ in magnitude, and the diameter is $O(\sqrt{t})$. Then the pullback of the Ricci flow to $\widetilde{M}$ approaches one of the homogeneous expanding solitons.

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Conjecture

For a long-time 3D Ricci flow, the diameter is $O(\sqrt{t})$ if and only if M admits a locally homogeneous metric.

Some of the tools

1. A compactness result for possibly-collapsing Ricci flow solutions.

2. Monotonic quantities for Ricci flow coupled to harmonic map flow and Yang-Mills flow (extensions of the Feldman-Ilmanen-Ni $\mathcal{W}_+$-functional).

3. Local stability results for certain expanding Ricci solitons (due to Dan Knopf).

Long-time behavior

Homogeneous spaces and the geometrization conjecture

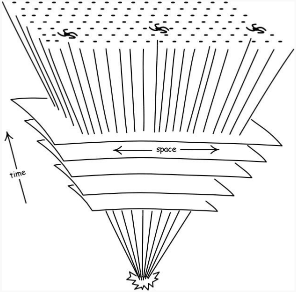
Geometrization conjecture and Ricci flow

Finiteness of the number of surgeries

Long-time behavior of Ricci flow

Einstein flow

The setup



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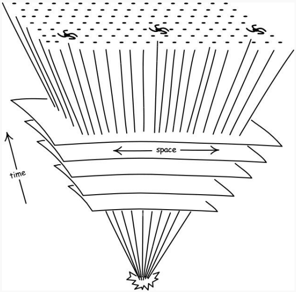

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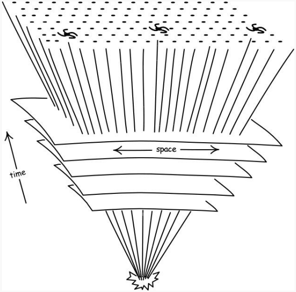

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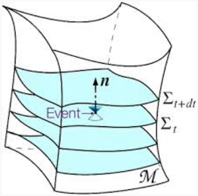
The spacetime has a Ricci-flat Lorentzian metric g.

What is time?

Suppose that we have a foliation of the spacetime by compact hypersurfaces.



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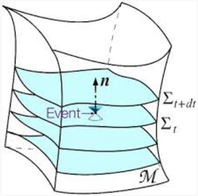

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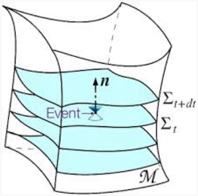

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Let's *assume* that along any given hypersurface, the expansion factor is constant. This defines a constant mean curvature (CMC) foliation.

Einstein flow

Using the foliation, the metric takes the form

$$g = -L^2 dt^2 + h(t),$$

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$$\frac{\partial h_{ij}}{\partial t} = -2LK_{ij} \tag{3}$$

and

$$\frac{\partial K_{ij}}{\partial t} = LHK_{ij} - 2L\sum_{k,l} h^{kl} K_{ik} K_{lj} - L_{;ij} + LR_{ij}, \tag{4}$$

along with certain time-independent "constraint" equations. Here the mean curvature $H = \sum_{i,j} h^{ij} K_{ij}$ is spatially constant.

Monotonicity

With our conventions, *expanding* solutions have $H < 0$. There's a corresponding time parameter, the Hubble time $t = -\frac{3}{H}$.

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The analogous statement in Ricci flow is that $t^{-\frac{3}{2}}\,\mathrm{vol}(X, h(t))$ is monotonically nonincreasing.

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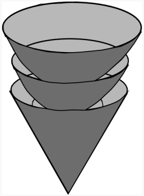

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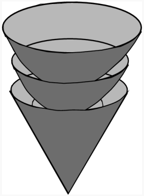

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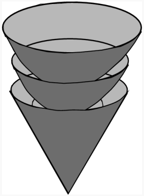

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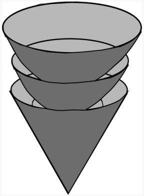

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2. The Bianchi-III flat spacetime is $\mathbb{R}$ times the interior of a forward lightcone in $\mathbb{R}^{2,1}$.

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Here

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.$$

Convergence result

The scale-invariant curvature condition is that $\| \mathrm{Rm}_g \| = O(t^{-2})$ as $t \to \infty$. (This is the analog of a type-III solution in Ricci flow.)

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(If there is a lower volume bound $\mathrm{vol}(h(t)) \geq \text{const.}\, t^3$ then the model space is the Milne spacetime. This case is due to Mike Anderson.)

Type-II solutions

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It turns out to be flat.

An apparent paradox

In the blowdown analysis, we rescale so that $\| \operatorname{Rm}_g(x_i, t_i)\| = 1$. How can the limit be flat?

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In effect, there are increasing curvature fluctuations that average out the curvature to zero.

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But new techniques need to be developed.