Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman) and some applications

Natasa Sesum, Gang Tian

Abstract

In this short note we present a result of Perelman with detailed proof. The result states that if $g(t)$ is a Kähler Ricci flow on a compact, Kähler manifold M with $c_1(M) > 0$, the scalar curvature and diameter of $(M, g(t))$ stay uniformly bounded along the flow, for $t \in [0, \infty)$. We learned about this result and its proof from G.Perelman when he was visiting MIT in the spring of 2003. This may be helpful to people studying the Kähler Ricci flow.

1 Introduction

We will consider a Kähler Ricci flow,

$$\frac{d}{dt} g_{i\bar{j}} = g_{i\bar{j}} - R_{i\bar{j}} = \partial_i \partial_{\bar{j}} u, \tag{1}$$

on a compact, Kähler manifold M, with $c_1(M) > 0$, of an arbitrary complex dimension n. Cao proved in ([1]) that (1) has a solution for all time t. One of the most important questions regarding the Kähler Ricci flow is whether it develops singularities at infinity, that is whether the curvature of $g(t)$ blows up as $t \to \infty$. This question was only answered in the case curvature operator or bisectional curvature is nonnegative (cf. [8], [7], [2]). In 2003, Perelman made a surprising claim that the scalar curvature of $g(t)$ does not blow up as $t \to \infty$. He also showed us a sketched proof.This result of Perelman strengthens the belief that the Kähler Ricci flow converges to a Kähler Ricci soliton as t tends to infinity, at least outside a subvariety of complex codimension 2. Partial progress was already made ([**?**]).

The goal of this paper is to give a detailed proof of Perelman's bound on a scalar curvature and a diameter.

Theorem 1 (Perelman). *Let $g(t)$ be a Kähler Ricci flow (1) on a compact, Kähler manifold M of complex dimension n, with $c_1(M) > 0$. There exists a uniform constant C so that*

- $|R(g(t))| \leq C$,
- $\mathrm{diam}(M, g(t)) \leq C$,
- $|u|_{C^1} \leq C$.

The outline of the main steps of the proof of Theorem 1 is as follows:

1. Getting a uniform lower bound on Ricci potential $u(t)$.
2. Bounding $|\nabla u(t)|$ and a scalar curvature $R(t)$ by C^0 norm of Ricci potential $u(t)$. This can be achieved by considering the evolution equations for $\frac{|\nabla u|^2}{u+2B}$ and $\frac{-\Delta u}{u+2B}$, where B is a uniform constant such that $u + B > 0$, whose existence is guaranteed by step 1.
3. Step 2 tells us that $\sqrt{u + 2B}$ is a Lipshitz function and that it is enough to bound $\mathrm{diam}(M, g(t))$ in order to have uniform bounds on $|u(t)|_{C^1}$ and scalar curvature $R(t)$.
4. To show that the diamters are uniformly bounded along the flow, we will argue by contradiction. We will assume that the diameters are unbounded as we approach infinity. Using that, we will show that the integral of the scalar curvature over some large annulus is bounded by $C \cdot V$, where C is a uniform constant and V is a volume of a slightly larger annulus than the one we started with. We can find such an annulus at every time t in the sequence of times for which diamters go to infinity. By choosing similar cut off functions as in the proof of Perelman's noncollapsing theorem in [12] we will show that we get a contradiction if the diameters are unbounded as we approach infinity.

The organization of the paper is as follows. In section 2 we will give the proof of Theorem 1. In section 3 we will discuss the convergence of the normalized Kähler Ricci flow, using Perelman's results.

Acknowledgements: We would like to thank Perelman for his generousity for telling us about his results on the Kähler Ricci flow. The first author would like to thank H.D.Cao for numerous useful discussions and helpful suggestions.

2 Ricci potential $u(t)$

In this section we will show that there is a uniform lower bound on $u(x,t)$. We will also show that it is enough to bound diameters of $(M, g(t))$ in order to have Theorem 1.

By taking the trace of (1) we get $\Delta u = n - R$. Let $\phi(t)$ be a metric potential, that is,

$$g_{i\bar{j}}(t) = g_{i\bar{j}}(0) + \partial_i \partial_{\bar{j}} \phi.$$

Then we can take $u(t) = \frac{d}{dt}\phi(t)$. Normalize so that

$$\int_M e^{-u} = (2\pi)^n.$$

Define

$$\mu(g,\tau) = \inf_{\{f | \int_M e^{-f}(4\pi\tau)^{-n} = 1\}} (4\pi\tau)^{-n} \int_M e^{-f}\{2\tau(R + |\nabla f|) + f - 2n\} dV,$$

to be Perelman's functional for $g(t)$ as in [12]. Perelman has proved that $\mu(g,\tau)$ is achieved. Take $f = u$ and $\tau = 1/2$. Then by monotonicity of $\mu(g(t))$ along the Kähler Ricci flow,

$$\begin{aligned}
A &= \mu(g(0), 1/2) \le \mu(g(t), 1/2) \\
&\le \int_M (2\pi)^{-n} e^{-u}(R + |\nabla u|^2 + u - n) \\
&= \int_M (2\pi)^{-n} e^{-u}(-\Delta u + |\nabla u|^2 - n + u) \\
&= \int_M (2\pi)^{-n} \Delta e^{-u} - n + (2\pi)^{-n} \int_M e^{-u} u \\
&= -n + (2\pi)^{-n} \int_M e^{-u} u. \qquad (2)
\end{aligned}$$

We have just proved the following lemma.

Lemma 2. *There is a uniform constant $C_1 = C_1(A)$ so that $\int e^{-u} u \ge C_1$.*

Define $a = -(2\pi)^{-n} \int_M u e^{-u} dV$. In the following claim we will prove a lower bound on a.

Claim 3. *Moreover, there is a uniform constant $C_2 > 0$ so that $a \ge -C_2$.*

Proof. Let $u_- = \min\{u, 0\}$ and $u_+ = \max\{0, u\}$. Then we have

$$\begin{aligned} a &= -(2\pi)^{-n}\int_M ue^{-u}dV = -(2\pi)^{-n}\int_M u_-e^{-u_-}dV - (2\pi)^{-n}\int_M u_+e^{-u_+}dV \\ &\geq -(2\pi)^{-n}\int_M u_+e^{-u_+}dV \geq -C_2, \end{aligned}$$

for some constant $C_2 \geq 0$, since $f(x) = xe^{-x}$ is a bounded function for $x \geq 0$. □

Remark 4. *It is pretty well known that the scalar curvature is uniformly bounded from below along the flow. We may assume* $R > 0$.

Proof. Function $u(t)$ satisfies,

$$\begin{aligned} \partial_i\partial_{\bar{j}}u_t &= g_{\bar{j}} - R_{i\bar{j}} + \frac{d}{dt}\partial_i\partial_{\bar{j}}\ln\det(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\phi)^n \\ &= \partial_i\partial_{\bar{j}}(u + \Delta u), \end{aligned}$$

which implies

$$\frac{d}{dt}u = \Delta u + u + a, \tag{3}$$

where we can choose $a = -\int ue^{-u}(2\pi)^n \leq C$, unifomly bounded from above by the previous lemma.

Lemma 5. *Function $u(t)$ is uniformly bounded from below.*

If the Ricci potential u is very negative for some time t_0, say $u(t_0) \leq -2(n + C_1)$, from (3), by Lemma 2 and Remark 4 we have

$$\frac{d}{dt}u = n - R + u + a \leq n + C_1 + u < 0, \tag{4}$$

at $t = t_0$, which implies that $u(t)$ stays very negative for $t \geq t_0$. In other words, if for some y_0 we have $u(y_0) << 0$ at some time t_0, $u(y) << 0$ for all y in some neighbourhood U of y_0, at time t_0. By (4), $u(y) << 0$ continues holding in U, for all $t \geq t_0$. Then $\frac{d}{dt}u \leq C + u$ implies

$$u(t)(z) \leq e^{t-t_0}(C + u(t_0)) \leq -\tilde{C}e^t, \tag{5}$$

for $t \geq t_0$, for all $z \in U$. Then $\dot{\phi} = u$ yields

$$\phi(t)(z) \leq \phi(t_0)(z) - \tilde{C}e^{t-t_0} \leq -C_1e^t, \tag{6}$$

for big enough t and all $z \in U$. On the other hand, $\int_M e^{-u(t)} = (2\pi)^n$, which tells us $u(t)$ can not be very negative everywhere on M, that is, there is a uniform constant C_2 such that $u(x_t, t) = \max_M u(t) \geq -C_2$ and $\mathrm{Vol}_{g(0)}(U) < \mathrm{Vol}_{g(0)}(M)$. Since $\dot{\phi}(t) = u(t)$, from (3) we get

$$u(x_t, t) - \phi(x_t, t) \leq \max_M (u(\cdot, 0) - \phi(\cdot, 0)) + \tilde{C}t,$$

which implies,

$$\frac{d}{dt}(u(t) - \phi(t)) = n - R + a \leq \tilde{C},$$

by Lemma (2) and Remark 4. This implies

$$\max_M \phi(t) \geq -C_3 - \tilde{C}t, \tag{7}$$

for a uniform constant C_3.

By taking a trace of $g(t) = g(0) + \partial\bar{\partial}\phi(t)$ at time $t = 0$, we get

$$-\Delta_0 \phi(t) = -\mathrm{tr}_{g(0)} g(t) + n \leq n.$$

Consider a fixed metric $g(0)$. By Green's formula applied to $\phi(t)$ we have

$$\begin{aligned}
\phi(x_t, t) &= \frac{1}{\mathrm{Vol}_0(M)} \int_M \phi(y, t) dV_0 - \frac{1}{\mathrm{Vol}(M)} \int_M \Delta_0 \phi(y, t) G_0(x_t, y) dV_0 \\
&\leq \frac{\mathrm{Vol}_0(M \backslash U)}{\mathrm{Vol}_0(M)} \sup_M \phi(\cdot, t) + \frac{\mathrm{Vol}_0(U)}{\mathrm{Vol}_0(M)} \int_U \phi(y, t) dV_0 + C \\
&\leq \frac{\mathrm{Vol}_0(M \backslash U)}{\mathrm{Vol}_0(M)} \sup_M \phi(\cdot, t) - C_4 e^t + C,
\end{aligned}$$

for $t \geq t_0$, where $\phi(x_t, t) = \max_M \phi(y, t)$ and G_0 is Green's function associated with metric $g(0)$ (recall that $\int_M G_0(x_t, y) dV_y = const$). Since $\frac{\mathrm{Vol}_0(M \backslash U)}{\mathrm{Vol}_0(M)} < 1$, we get

$$\max_M \phi(\cdot, t) \leq -C_5 e^t + C_6,$$

for some uniform constants C_5, C_6. This together with (7) yields a contradiction for big values of t. Therefore, there exists a uniform lower bound on $u(t)$, that is, there is some constant B such that $u(x, t) \geq B$ for all $(x, t) \in M \times [0, \infty)$. $\square$

Standard computation gives the following evolution equations for Δu and $|\nabla u|^2$.

$$\Box(\Delta u) = \frac{d}{dt}\Delta u - \Delta^2 u = -|\nabla\bar{\nabla}u|^2 + \Delta u, \tag{8}$$

$$\Box(|\nabla u|^2) = \frac{d}{dt}|\nabla u|^2 - \Delta|\nabla u|^2 = -|\nabla\nabla u|^2 - |\nabla\bar{\nabla}u|^2 + |\nabla u|^2. \tag{9}$$

Proposition 6. *There is a uniform constant C so that*

$$|\nabla u|^2 \leq C(u + C), \tag{10}$$

$$R \leq C(u + C). \tag{11}$$

Proof. We will first prove an estimate (10) which we will need in the proof of Lemma 11. By Lemma 5 we may assume $u(x,t) > -B$. The proof resembles the arguments in [19] and [**?**]. If $H = \frac{|\nabla u|^2}{u+2B}$, by (8) and (9) we get,

$$\begin{aligned}
\Box H &= \frac{-|\nabla\bar{\nabla}u|^2 - |\nabla\nabla u|^2}{u+2B} + \frac{|\nabla u|^2(2B-a)}{(u+2B)^2} + \\
&+ \frac{2\bar{\nabla}u\cdot\nabla|\nabla u|^2}{(u+2B)^2} - \frac{2|\nabla u|^4}{(u+2B)^3}.
\end{aligned} \tag{12}$$

We can write

$$\frac{2\bar{\nabla}u\cdot\nabla|\nabla u|^2}{(u+2B)^2} - \frac{2|\nabla u|^4}{(u+2B)^3} = (2-\epsilon)\frac{\bar{\nabla}u\cdot\nabla H}{u+2B} + \epsilon\frac{\bar{\nabla}u\cdot\nabla|\nabla u|^2}{(u+2B)^2} - \epsilon\frac{|\nabla u|^4}{(u+2B)^3}, \tag{13}$$

for some small $\epsilon > 0$. Since

$$\begin{aligned}
\nabla_{\bar{i}}u\nabla_i(\nabla_j u\nabla_{\bar{j}}u) &= \nabla_{\bar{i}}u\nabla_i\nabla_j u\nabla_{\bar{j}}u + \nabla_{\bar{i}}u\nabla_j u\nabla_i\nabla_{\bar{j}}u \\
&\leq |\nabla u|^2(|\nabla\nabla u| + |\nabla\bar{\nabla}u|),
\end{aligned}$$

by Cauchy Schwartz inequality,

$$\begin{aligned}
\epsilon\frac{|\nabla u\cdot\nabla|\nabla u|^2|}{(u+2B)^2} &\leq C\epsilon\frac{|\nabla u|^2(|\nabla\nabla u| + |\nabla\bar{\nabla}u|)|}{(u+2B)^{3/2}(u+2B)^{1/2}} \\
&\leq \frac{\epsilon}{2}\frac{|\nabla u|^4}{(u+2B)^3} + \frac{2C^2\epsilon(|\nabla\nabla u|^2 + |\nabla\bar{\nabla}u|^2)}{u+2B}.
\end{aligned} \tag{14}$$

Choose ϵ small so that $2\sqrt{C}\epsilon < 1/2$. Combining (12), (13) and (14) yields

$$\Box H \leq \frac{|\nabla u|^2(2B-a)}{(u+2B)^2} + (2-\epsilon)\frac{\bar{\nabla}u\cdot\nabla H}{u+2B} - \frac{\epsilon}{2}\frac{|\nabla u|^4}{(u+2B)^3}. \tag{15}$$

At a point at which H achieves its maximum we have that ∇H vanishes and therefore by maximum principle, an estimate (15) reduces to

$$0 \leq \frac{d}{dt}H_{\max} \leq \frac{|\nabla u|^2}{(u+2B)^2}(2B - a - \frac{\epsilon}{2}\frac{|\nabla u|^2}{u+2B}). \tag{16}$$

If we assume that

$$|\nabla u|^2 >> u + 2B, \tag{17}$$

then a term on the right hand side of (16) becomes negative for large t and we have that $H_{\max}$ is decreasing for big values of t, which gives that $|\nabla u|^2 \leq C(u+2B)$ for some constant C. This contradicts (17) and therefore we have (10).

Our next goal is to prove that $-\Delta u$ is bounded by $C(u+C)$ which yields (11), since $\Delta u = n - R$. Let $K = \frac{-\Delta u}{u+2B}$, where B is a uniform constant as above. Similar computation as before gives that

$$\Box(-\frac{\Delta u}{u+2B}) = \frac{|\nabla\bar{\nabla}u|^2}{u+2B} + \frac{(-\Delta u)(2B-a)}{(u+2B)^2} + 2\frac{\bar{\nabla}u\cdot\nabla K}{u+2B}.$$

Take $b > 1$. Then

$$\begin{aligned}\Box(\frac{-\Delta u + b|\nabla u|^2}{u+2B}) &= \frac{-b|\nabla\nabla u|^2 - (b-1)|\nabla\bar{\nabla}u|^2}{u+2B} + \frac{(-\Delta u + b|\nabla u|^2)(2B-a)}{(u+2B)^2} \\ &+ \frac{2\bar{\nabla}u\nabla(\frac{-\Delta u + b|\nabla u|^2}{u+2B})}{u+2B}.\end{aligned}$$

Let $G = \frac{-\Delta u + b|\nabla u|^2}{u+2B}$ and by maximum principle,

$$\frac{d}{dt}G_{\max} \leq -(b-1)\frac{|\nabla\bar{\nabla}u|^2}{u+2B} + \frac{(-\Delta u + b|\nabla u|^2)(2B-a)}{(u+2B)^2}.$$

In local coordinates,

$$(\Delta u)^2 = (\sum_i u_{i\bar{i}})^2 \leq n\sum_i u_{i\bar{i}}^2 = n|\nabla\bar{\nabla}u|^2,$$

and therefore,

$$\begin{aligned}\frac{d}{dt}G_{\max} &\leq -(b-1)\frac{(\Delta u)^2}{n(u+2B)} + \frac{(-\Delta u + b|\nabla u|^2)(2B-a)}{(u+2B)^2} \\ &\leq \frac{(-\Delta u)}{u+2B}\{\frac{2B-a}{u+2B} - \frac{1}{n}\frac{(-\Delta u)}{u+2B}\} + \frac{b|\nabla u|^2(2B-a)}{(u+2B)^2}. \end{aligned} \tag{18}$$

By Lemmas 2 and 5 we may assume that $\frac{2B-a}{u+2B}$ is bounded from above by a uniform constant. We have also proved the estimate (10) on $|\nabla u|$. If

$$-\Delta u >> u + 2B, \tag{19}$$

by (18) we would have that $\frac{d}{dt}G_{\max} < 0$ for big values of t. This would imply $-\Delta u(t) \le C(u+2B)$, for some uniform constant C and all big values of t, which contradicts (19). Therefore, there exists a uniform constant C such that (11) holds. □

Proposition 7. *There is a $C = C(A)$, such that* $\mathrm{Vol}(B(x,1)) \ge C$*, for any metric g satisfying $|R| \le 1$ on $B(x,1)$, where $\partial B(x,1) \ne \emptyset$.*

Proof. Let $g(t)$ be as before, a solution to a normalized Kähler Ricci flow equation, and let $\tilde{g}(s)$ be a solution to the equation $\frac{d}{ds}\tilde{g}(s) = -2\mathrm{Ric}(\tilde{g}(s))$. Reparametrization between these two flows is given by $\tilde{g}(s) = (1-2s)g(t(s))$, where $t(s) = -\ln(1-2s)$. The first flow has a solution for $t \in [0,\infty)$ and the second one has a maximal solution for $s \in [0,1/2)$. The scalar curvature rescales as $R(\tilde{g}(s)) = \frac{R(g(t(s)))}{1-2s} \le \frac{1}{1-2s}$. The following improvement of Perelman's noncollapsing result (noticed by Perelman himself) that requires only a scalar curvature bound can be found in [9]. The result was communicated to Kleiner and Lott by Tian. It says that there is a universal constant $\kappa = \kappa(\tilde{g}(0))$, so that for an unnormalized Ricci flow $\frac{d}{ds}\tilde{g}(s) = -2\mathrm{Ric}(\tilde{g}(s))$, if $R(\tilde{g}(s) \ge -\frac{1}{r^2}$ in a ball $B_{\tilde{g}(s)}(p,r)$, then $\mathrm{Vol}_{\tilde{g}(s)}B_{\tilde{g}(s)}(p,r) \ge \kappa r^{2n}$. The detailed arguments of the proof can be found in [9] and [14], but for the convenience of a reader we will include it here as well. We argue by contradiction, that is, assume there are sequences $p_k \in M$ and $t_k \to \infty$ so that $|R| \le \frac{C}{r_k^2}$, but $\mathrm{Vol}(B_k)r_k^{-2n} \to 0$ as $k \to \infty$, where $B_k = B_{t_k}(p_k, r_k)$. Let $\tau = r_k^2$. Define

$$u_k = e^{C_k}\phi(r_k^{-1}\mathrm{dist}(x,p_k)) \tag{20}$$

at t_k, where ϕ is a smooth function on R, equal 1 on $[0,1/2]$, decreasing on $[1/2,1]$ and equal 0 on $[1,\infty)$. C_k is a constant to make u satisfy the constraint

$$\begin{aligned}(4\pi)^n &= e^{2C_k}r_k^{-2n}\int_{B(p_k,r_k)}\phi(r_k^{-1}\mathrm{dist}(x,p_k))^2 dV \\ &\le e^{2C_k}r_k^{-2n}\mathrm{Vol}(B_k).\end{aligned}$$

Since $r_k^{-n}\text{Vol}B_k \to 0$, this shows that $C_k \to +\infty$. We compute

$$
\begin{aligned}
\mathcal{W}(u_k) &= (4\pi)^{-n} r_k^{-2n} e^{2C_k} \int_{B(p_k,r_k)} (4|\phi'(r_k^{-1}\text{dist}(x,p_k))|^2 - 2\phi^2 \ln\phi) dV \\
&+ r_k^2 \int_{B(p_k,r_k)} Ru^2 (4\pi)^{-n} r_k^{-n} dV - n - 2C_k \\
&\le (4\pi)^{-n} r_k^{-2n} e^{2C_k} \int_{B(p_k,r_k)} (4|\phi'|^2 - 2\phi^2 \ln\phi) dV \\
&+ r_k^2 \max_{B_k} R - n - 2C_k.
\end{aligned}
$$

Let $V(r) = \text{Vol}(B(p_k, r))$. The necessary ingredients of the argument are that

(a) $r_k^{-2n}\text{Vol}(B(p_k, r_k)) \to 0$.

(b) $r_k^2 R$ is uniformly bounded above.

(c) $\frac{\text{Vol}(B(p_k,r_k))}{\text{Vol}(B(p_k,r_k/2))}$ is uniformly bounded above.

Suppose that $r_k^{-2n}\text{Vol}(B(p_k, r_k)) \to 0$ and $r_k^2 R \le n(n-1)$ on B_k. If $\frac{\text{Vol}(B(p_k,r_k))}{\text{Vol}(B(p_k,r_k/2))} < 3^n$ for all k, we are done. If not, for a given k we have that $\frac{\text{Vol}(B(p_k,r_k))}{\text{Vol}(B(p_k,r_k/2))} \ge 3^n$. Let $r_k' = r_k/2$. We have $(r_k')^{-2n}\text{Vol}(B(p_k, r_k')) \le r_k^{-2n}\text{Vol}(B(p_k, r_k))$ and $(r_k')^2 R \le n(n-1)$ on $B(p_k, r_k')$. Replace r_k by r_k'. If $\frac{\text{Vol}(B(p_k,r_k))}{\text{Vol}(B(p_k,r_k/2))} < 3^n$ we stop. If not, then we repeat the process and replace r_k by $r_k/2$. At some point we will achieve that $\frac{\text{Vol}(B(p_k,r_k))}{\text{Vol}(B(p_k,r_k/2))} < 3^n$. We then take this new subsequence $\{p_k, r_k)\}_{k=1}^{\infty}$ into further discussion. Hence $V(r_k) - V(r_k/2) \le C'V(r_k/2)$. Therefore

$$
\begin{aligned}
\int_{B(p_k,r_k)} (4|\phi'|^2 - 2\phi^2 \ln\phi) dV \le\ & C \quad (V(r_k) - V(r_k/2)) \\
\le\ & CV(r_k/2) \\
\le\ & C \int_{B_k} \phi^2 dV.
\end{aligned}
$$

Plugging this into the previous estimate for $\mathcal{W}$ and using the constraint $(4\pi\tau_k)^{-n}\int_M e^{-u_k} dV_{t_k} = 1$, we get

$$
\mathcal{W}(u_k) \le C'' - 2C_k. \tag{21}
$$

Since $C_k \to +\infty$ and $\mu(g(t_k), r_k^2) \le \mathcal{W}(g(t_k), u_k, r_k^2)$, we conclude that $\mu(g(t_k), r_k^2) \to -\infty$. By the condition (a) we have $A \le \mu(g(t_k), r_k^2) \to -\infty$ which is impossible.

The previous argument, since $R(\tilde{g}(s)) \le \frac{1}{1-2s}$ implies $\mathrm{Vol}_{\tilde{g}(s)} B_{\tilde{g}(s)}(x, \sqrt{1-2s}) \ge \kappa(1-2s)^n$, which by rescaling implies $\mathrm{Vol}B(x,1) \ge \kappa$ at metric $g(t)$, where κ is a constant depending only on an initial metric $g(0)$. □

Claim 8. *There is a uniform constant C such that*

$$u(y,t) \le C\mathrm{dist}_t^2(x,y) + C,$$

$$R(y,t) \le C\mathrm{dist}_t^2(x,y) + C,$$

$$|\nabla u| \le C\mathrm{dist}_t(x,y) + C,$$

where $u(x,t) = \min_{y\in M} u(y,t)$.

Proof. By Lemma 5 we can assume $u \ge \delta > 0$, since otherwise we can consider $u + 2B + \delta$ instead of u. From (10) it follows that $\sqrt{u}$ is a Lipshitz function since $|\nabla(\sqrt{u})| \le C = C(\delta)$ and therefore,

$$\begin{aligned} |\sqrt{u}(y,t) - \sqrt{u}(z,t)| &\le \frac{|\nabla u|(p,t)}{2\sqrt{u}}\mathrm{dist}_t(y,z) \\ &\le \bar{C}\mathrm{dist}_t(y,z), \end{aligned}$$

and therefore,

$$\begin{aligned} u(y,t) &\le (\tilde{C}\mathrm{dist}_t(y,z) + \sqrt{u}(x,t))^2 \\ &\le C_1\mathrm{dist}_t^2(x,y) + C_1 u(x,t). \end{aligned}$$

Assume $u(x,t) \ge K(t)$. Then $u(y,t) \ge K(t)$ for all $y \in M$ and we would have,

$$(2\pi)^n = \int_M e^{-u} dV_t \le e^{-K(t)}\mathrm{Vol}(M) \to 0,$$

if $K(t) \to \infty$, which is not possible. Therefore, $u(x,t) \le K$, for a constant that does not depend on t and finally

$$u(y,t) \le C\mathrm{dist}_t^2(y,x) + \tilde{C}, \tag{22}$$

for some uniform constants C and $\tilde{C}$. Other two estimates in the claim follow from (22) and Proposition 6. □

By Claim 8 it follows that if we manage to estimate the diameter, we will get uniform bounds on the scalar curvature and the C^1 norm of u.

3 A uniform upper bound on diameters

In this section we want to prove the following proposition which will finish the proof of Theorem 1.

Proposition 9. *There is a uniform constant C such that* $\mathrm{diam}(M, g(t)) \le C$*.*

The proof goes by contradiction argument. Assume that the diameters are unbounded in time. Denote by $d_t(z) = \mathrm{dist}_t(x, z)$ where $u(x,t) = \min_{y\in M} u(y,t)$.

Let $B(k_1, k_2) = \{z : 2^{k_1} \le d_t(z) \le 2^{k_2}\}$. Consider an annulus $B(k, k+1)$. By Claim 8 we have that $R \le C2^{2k}$ on $B(k, k+1)$. Interval $[2^k, 2^{k+1}]$ fits 2^{2k} balls of radii $\frac{1}{2^k}$. By Claim 8 and Proposition 7 we have that at time t

$$\mathrm{Vol}(B(k, k+1)) \ge \sum_i \mathrm{Vol}(B(x_i, 2^{-k})) \ge 2^{2k}2^{-2kn}C. \tag{23}$$

Claim 10. *For every $\epsilon > 0$ we can find $B(k_1, k_2)$, such that if* $\mathrm{diam}(M, g(t))$ *is large enough, then*
(a) $\mathrm{Vol}(B(k_1, k_2)) < \epsilon$ *and*
(b) $\mathrm{Vol}(B(k_1, k_2)) \le 2^{10n}\mathrm{Vol}(B(k_1 + 2, k_2 - 2))$.

Proof. Since $\mathrm{Vol}_t(M)$ is constant along the flow and therefore uniformly bounded, if diameter is sufficiently big, there is k_0 such that for all $k_2 \ge k_1 \ge k_0$, we have that $\mathrm{Vol}(B(k_1, k_2)) < \epsilon$. If our estimate (b) did not hold, that is, if

$$\mathrm{Vol}(B(k_1, k_2)) \ge 2^{10n}\mathrm{Vol}(B(k_1 + 2, k_2 - 2)),$$

we would consider $B(k_1 + 2, k_2 - 2)$ instead and ask whether (b) holds for that ball. Assume that for every p, at the p-th step we are still not able to find our radii so that (a) and (b) are satisfied. In that case, at the p-th step we would have

$$\mathrm{Vol}(B(k_1, k_2)) \ge 2^{10np}\mathrm{Vol}(B(k_1 + 2p, k_2 - 2p)).$$

In particular, assume we have the above estimate at the p-th step so that $k_1 + 2p + 1 \sim k_2 - 2p$, which is for $2p \sim \frac{k_2 - k_2 - 1}{2}$ (*). Take $k_1 = k/2$ and $k_2 = 3k/2$ for $k >> 1$. In that case (*) becomes $p \sim k/4$, $k_1 + 2p \sim k$ and $k_2 - 2p \sim k + 1$. Combining this with (23) yields

$$\epsilon > \mathrm{Vol}(B(k_1, k_2)) \ge 2^{10nk/4}\mathrm{Vol}(B(k, k+1)) \ge 2^{10nk/4}C2^{2k}2^{-2nk}.$$

This leads to contradiction if we let $k \to \infty$. This finishes the proof of our claim. $\square$

For every t for which diameter of $(M, g(t))$ becomes very big, find k_1 and k_2 as in Claim 10. Then we have the following lemma.

Lemma 11. *There exist r_1, r_2 and a uniform constant C such that $2^{k_1} \le r_1 \le 2^{k_1+1}$, $2^{k_2-1} \le r_2 \le 2^{k_2}$ and*

$$\int_{B(r_1,r_2)} R \le CV,$$

where $\epsilon > 0$ is the same as in Claim 10, $B(r_1,r_2) = \{z \in M : r_1 \le \mathrm{dist}_t(z) \le r_2\}$ and $V = \mathrm{Vol}(B(k_1,k_2))$.

Proof. We will first prove the existence of r_1, such that $2^{k_1} \le r_1 \le 2^{k_1+1}$ and

$$\mathrm{Vol}S(r_1) \le 2\frac{V}{2^{k_1}}, \tag{24}$$

where $S(r)$ is a metric sphere of radius r. We have that,

$$\frac{d}{dr}\mathrm{Vol}(B(r)) = \mathrm{Vol}S(r). \tag{25}$$

Assume that for all $r \in [2^{k_1}, 2^{k_1+1}]$ we have $\mathrm{Vol}(S(r)) \ge 2\frac{V}{2^{k_1}}$. Integrate (25) in r. Then,

$$\begin{aligned}\mathrm{Vol}(B(k_1,k_1+1)) &= \int_{2^{k_1}}^{2^{k_2}} \mathrm{Vol}(S(r))dr \\ &> 2\frac{V}{2^{k_1}}2^{k_1} = 2V = 2\mathrm{Vol}(B(k_1,k_2)),\end{aligned}$$

which is not possible, since $k_2 >> k_1$ by the proof of Claim 10. If for all $r \in [2^{k_2-1}, 2^{k_2}]$ we have that $\mathrm{Vol}(S(r)) \ge 2\frac{V}{2^{k_2}}$, similarly as above we would get $\mathrm{Vol}(B(k_2-1,k_2)) > V = B(k_1,k_2)$, which is not possible. Therefore, there exists $r_2 \in [2^{k_2-1}, 2^{k_2}]$ such that

$$\mathrm{Vol}S(r_2) \le 2\frac{V}{2^{k_2}}. \tag{26}$$

Estimates (24), (26), together with bounds on ∇u obtained in Claim 8 imply

$$\begin{aligned}\int_{B(r_1,r_2)} R &= \int_{B(r_1,r_2)} (R-n) + n\mathrm{Vol}(B(r_1,r_2)) \\ &= -\int_{B(r_1,r_2)} \Delta u + n\mathrm{Vol}(B(r_1,r_2)) \\ &\le \int_{S(r_1)} |\nabla u| + \int_{S(r_2)} |\nabla u| \le \frac{V}{2^{k_1}}C2^{k_1+1} + \frac{V}{2^{k_2}}C2^{k_2+1} \\ &= \tilde{C}V < \tilde{C}\epsilon.\end{aligned}$$

$\square$

We can now finish the proof of Proposition 9.

Proof of Proposition 9. The proof of the proposition is similar to the proof of Perelman's noncollapsing theorem from [12]. Assume $\mathrm{diam}(M, g(t))$ is not uniformly bounded in t, that is, there exists a sequence $t_i \to \infty$ such that $\mathrm{diam}(M, g(t_i)) \to \infty$. Let $\epsilon_i \to 0$ be a sequence of positive numbers. By Claim 10 we can find sequences k_1^i and k_2^i such that

$$\mathrm{Vol}_{t_i} B_{t_i}(k_1^i, k_2^i) < \epsilon_i, \tag{27}$$

$$\mathrm{Vol}(B_{t_i}(k_1^i, k_2^i)) \le 2^{10n} \mathrm{Vol}(B(k_1^i + 2, k_2^i - 2)). \tag{28}$$

For each i, find r_1^i and r_2^i as in Lemma 11. Let ϕ_i be a sequence of cut off functions such that $\phi(z) = 1$ for $z \in [2^{k_1^i+2}, 2^{k_2^i-2}]$ and equal zero for $z \in (-\infty, r_1^i] \cup [r_2^i, \infty)$. Let $u_i(x) = e^{C_i}\phi_i(\mathrm{dist}_{t_i}(x, p_i))$ such that $(2\pi)^{-n}\int_M u_i^2 = 1$. This implies

$$\begin{aligned}
(2\pi)^n &= e^{2C_i}\int_M \phi_i^2 \\
&\le e^{2C_i}\mathrm{Vol}_{t_i} B_{t_i}(k_1^i, k_2^i + 1) \\
&\le e^{2C_i}\epsilon_i.
\end{aligned}$$

Since $\epsilon_i \to 0$, this is possible only if $\lim_{i\to\infty} C_i = -\infty$. By Perelman's monotonicity formula,

$$\begin{aligned}
A &\le \mathcal{W}(g(t_i), u_i, 1/2) \\
&= (2\pi)^{-n}e^{2C_i}\int_{B_{t_i}(r_1^i, r_2^i)} (4|\phi_i'(\mathrm{dist}_{t_i}(y))|^2 - 2\phi_i^2 \ln \phi_i) dV_{t_i} \\
&+ (2\pi)^{-n}\int_{B_{t_i}(r_1^i, r_2^i)} R u_i^2 dV_{t_i} - 2n - 2C_i.
\end{aligned} \tag{29}$$

First of all by Lemma 11 and (28) we have

$$\begin{aligned}
\int_{B_{t_i}(r_1^i, r_2^i)} R u_i^2 &\le e^{2C_i}\int_{B_{t_i}(r_1^i, r_2^i)} R \\
&\le \tilde{C} e^{2C_i}\mathrm{Vol}_{t_i} B_{t_i}(k_1^i, k_2^i) \\
&\le \tilde{C} e^{2C_i} 2^{10n}\mathrm{Vol}_{t_i} B_{t_i}(k_1^i + 2, k_2^i - 2) \\
&\le \tilde{C} 2^{10n}\int_M u_i^2 dV_{t_i} = \tilde{C} 2^{10n}(2\pi)^n.
\end{aligned}$$

By (28) we also have

$$
\begin{aligned}
& e^{2C_i}\int_{B_{t_i}(r_1^i,r_2^i)}(4|\phi_i'(\mathrm{dist}_{t_i}(y))|^2-2\phi_i^2\ln\phi_i)dV_{t_i}\le \\
\le\ & Ce^{2C_i}\mathrm{Vol}_{t_i}B_{t_i}(k_1^i,k_2^i)\le e^{2C_i}C2^{10n}\mathrm{Vol}_{t_i}B_{t_i}(k_1^i+2,k_2^i-2) \\
\le\ & C2^{10n}\int_M u_i^2 = C2^{10n}(2\pi)^n.
\end{aligned}
$$

By (29) we get

$$A\le \bar{C}-2C_i\to-\infty,$$

as $i\to\infty$ and we get a contradiction. Therefore, there is a uniform bound on $(M,g(t))$, which gives us uniform bounds on scalar curvatures and $|u(y,t)|_{C^1}$.

□

4 Convergence of the Kähler Ricci flow

In this section we will apply Perelman's boundness results on the scalar curvature and the diameter of the Kähler Ricci flow to show its sequential convergence towards singular metrics that are smooth and satisfy the Kähler Ricci soliton equation outside a singular set. We would like to understand more closely a singular metric that we get as a limit in Theorem 12. Our final goal as a long term project is to prove that a singular set S is a variety, that is, that it has some structure than only being a closed set.

Theorem 12. *Let $g(t)$ be a Kähler Ricci flow on a compact, Kähler manifold M, given by*

$$\frac{d}{dt}g_{i\bar{j}}=g_{i\bar{j}}-R_{i\bar{j}}=\partial_i\partial_{\bar{j}}u.$$

Assume the Ricci curvature is uniformly bounded along the flow. For every sequence $t_i\to\infty$ there is a subsequence so that $(M,g(t_i+t))$ converges to $(M_\infty,g_\infty(t))$ in the following sense:

(i) M_∞ is smooth outside a singular set S which is of codimension at least 4 and the convergence is smooth off S.

(ii) A singular metric $g_\infty(t)$ satisfies a Kähler Ricci soliton equation $g_\infty-\mathrm{Ric}(g_\infty)=\partial\bar{\partial}f_\infty$, with $(f_\infty)_{ij}=(f_\infty)_{\bar{i}\bar{j}}=0$ outside S.

(iii) A potential function f_∞ is smooth off S and there is a uniform constant C so that $|f_\infty(t)|_{C^1(M_\infty\setminus S)}\le C$.

In [15] we have proved that if the Ricci curvatures are uniformly bounded, i.e. $|\mathrm{Ric}| \le C$ for all times t, then for every sequence $t_i \to \infty$ there exists a subsequence such that $(M, g(t_i + t)) \to (M_\infty, g_\infty(t))$ and the convergence is smooth outside a singular set S, which is at least of codimension four and M_∞ is a smooth manifold off S. We also showed that a singular set S does not depend on time t. Moreover, $g_\infty(t)$ solves the Kähler-Ricci flow equation on $M_\infty \backslash S$. We want to show that $g_\infty(t)$ is actually a Kähler Ricci soliton off the singular set.

Due to Perelman we have the following uniform estimates for the Kähler Ricci flow: there are uniform constants C and κ such that for all t,

1. $|u(t)|_{C^1} \le C$,
2. $\mathrm{diam}(M, g(t)) \le C$,
3. $|R(g(t))| \le C$,
4. $(M, g(t))$ is κ-noncollapsed.

This together with the uniform lower bound on the Ricci curvatures along the flow gives a uniform Sobolev constant, that is, there is a uniform constant C_S so that for any $v \in C_0^1(M)$ we have that

$$(\int_M v^{\frac{4n}{2n-2}} dV_{g(t)})^{\frac{2n}{2n-2}} \le C_S \int_M |\nabla v|^2 dV_{g(t)}, \tag{30}$$

for all times $t \ge 0$. This enables us to work with integral estimates. The proof of Theorem 12 will be completed after we prove Proposition 13 and Proposition 14.

Proposition 13. *A singular metric g_∞ that arises in Theorem 12 satisfies the Kähler Ricci soliton equation on $M_\infty \backslash S$.*

Proof. Notice that $\mu(g(t), 1/2) \le \mathcal{W}(g(t), u(t), 1/2) \le C$, for a uniform constant C, due to Perelman's estimates mentioned above (recall that $u(t)$ is the Ricci potential of metric $g(t)$). This yields a finite $\lim_{t\to\infty} \mu(g(t), 1/2)$. Let f_t be a minimizer of $\mathcal{W}$ with respect to metric $g(t)$ and let $f_t(s)$ be a solution of the equation

$$\frac{d}{ds} f_t(s) = -\Delta f_t(s) + |\nabla f_t(s)|^2 - R(s) + 2n, \tag{31}$$

for $s \in [0,t]$. Fix $A > 0$. By monotonicity of Perelman's functional W we also have

$$
\begin{aligned}
0 &\leq (2\pi)^{-n} \int_0^A \int_M |R_{j\bar{k}} + \nabla_j \bar{\nabla}_k f_{t_i+A}(t_i+s) - g_{j\bar{k}}|^2 e^{-f_{t_i+A}(t_i+s)} dV_{t_i+s} ds + \\
&+ (2\pi)^n \int_0^A \int_M (|\nabla_j \nabla_k f_{t_i+A}(t_i+s)|^2 + |\bar{\nabla}_j \bar{\nabla}_k f_{t_i+A}(t_i+s)|^2) e^{-f_{t_i+A}(t_i+s)} dV_{t_i+s} ds \\
&= \mathcal{W}(g(t_i+A), f_{t_i+A}, 1/2) - \mathcal{W}(g(t_i), f_{t_i+A}(t_i), 1/2) \\
&\leq \mu(g(t_i+A), 1/2) - \mu(g(t_i), 1/2),
\end{aligned}
$$

where the right hand side of the previous estimate tends to zero as $i \to \infty$, since there is a finite $\lim_{t\to\infty} \mu(g(t), 1/2)$. Moreover, for any compact set $K \subset M_\infty \backslash S$ we have

$$
\begin{aligned}
&\lim_{i\to\infty} \int_0^A \int_K |R_{j\bar{k}}(t_i+s) + \nabla_j \bar{\nabla}_k f_{t_i+A}(t_i+s) - g_{j\bar{k}}(t_i+s)|^2 e^{-f_{t_i+A}(t_i+s)} dV_{t_i+s} ds + \\
+ &\int_0^A \int_K (|\nabla_j \nabla_k f_{t_i+A}(t_i+s)|^2 + |\bar{\nabla}_j \bar{\nabla}_k f_{t_i+A}(t_i+s)|^2) e^{-f_{t_i+A}(t_i+s)} dV_{t_i+s} ds = 0,
\end{aligned}
$$

which implies for almost all $s \in [0,A]$ and almost all $x \in K$,

- $\lim_{i\to\infty} |R_{j\bar{k}}(t_i+s) + \nabla_j \bar{\nabla}_k f_{t_i+A}(t_i+s) - g_{j\bar{k}}(t_i+s)|^2 e^{-f_{t_i+A}(t_i+s)} = 0.$
- $\lim_{i\to\infty} |\nabla_j \nabla_k f_{t_i+A}(t_i+s)|^2 e^{-f_{t_i+A}(t_i+s)} = 0.$

Let $D_l \subset M_\infty$ be a sequence of open sets where $S \subset D_l$ and $\bar{D}_l \to S$ as $l \to \infty$. We know that $g(t_i+t) \to g_\infty(t)$ smoothly on $M_\infty \backslash D_l$. Function $f_{t_i+A}(t_i+s)$ satisfies an evolution equation (31) and we have the geometries $g(t_i+s)$ are uniformly bounded on $M_\infty \backslash D_l$ for all i and all $s \in [0,A]$ (those bounds depend on l, that is, on the closeness to the singular set S). Standard parabolic estimates as in [15] and [16] give there are uniform constants $C(p,q,l)$ so that $|\frac{\partial^p}{\partial s^p} \nabla^q f_{t_i+A}(t_i+s)| \leq C(p,q,l)$. Using the uniform derivative bounds from above, by Arzela Ascoli theorem and diagonalizing the sequence (by letting l bigger and bigger) we can find a subsequence so that $f_{t_i+A}(t_i+s) \stackrel{C^k(M_\infty\backslash S)}{\to} f_\infty(s)$. Moreover, this limit $f_\infty(s)$ satisfies

$$
R_{j\bar{k}}(g_\infty(s)) + \nabla_j \bar{\nabla}_k f_\infty(s) - (g_\infty)_{j\bar{k}}(s) = 0. \tag{32}
$$

$$
\nabla_i \nabla_j f_\infty = \bar{\nabla}_i \bar{\nabla}_j f_\infty = 0. \tag{33}
$$

□

Relations (32) and (33) give,

$$\sup_{M_\infty \setminus S} |D^2 f_\infty| \leq C < \infty. \tag{34}$$

The previous estimate helps us prove the following proposition.

Proposition 14. *There is a constant C so that $|f_\infty|_{C^1(M_\infty\setminus S)} \leq C$.*

Proof. We will first mention few results and some notation from [3] and [4], which our proof will rely upon.

A point $y \in M_\infty$ is called regular, if for some k, every tangent cone at y is siometric to R^{2n}. Let $\mathcal{R}_k$ denote the set of k-regular points and put $\mathcal{R} = \cup_k \mathcal{R}_k$, the *regular set*. A point $y \in M_\infty$ is acalled *singular*, if it is not regular. Denote by S the set of singular points. In [3] it has been shown that under the assumption $|\mathrm{Ric}(g(t))| \leq C$, we have $\mathcal{R} = \mathcal{R}_n$. Moreover, one of the results in [5] is that $\dim S \leq n-4$. The ϵ-regular set, $\mathcal{R}_\epsilon$, consists of those points y, such that every tangent cone, (Y_y, y_∞), satisfies $d_{GH}(B_1(y_\infty, B_1(0)) < \epsilon$. In [3] it was shown that for the uniform bound on the Ricci curvatures, there is an ϵ_0, so that for every $\epsilon < \epsilon_0$, $\mathcal{R}_\epsilon = \mathcal{R}$ and $\mathcal{R}_\epsilon \cap S = \emptyset$. That means we can write

$$M_\infty = \mathcal{R}_\epsilon \cup S,$$

for $\epsilon < \epsilon_0$. Fix $x_0 \in \mathcal{R}$. By Theorem 3.9 in [4], there exists $\mathcal{C}(x_0) \subset \mathcal{R}$, with $\nu(M_\infty \setminus \mathcal{C}(x_0)) = 0$ (ν is the unique limit measure, which in our noncollapsed case is exactly Hausdorff measure), such that for all $y \in \mathcal{C}(x_0)$ and $\epsilon > 0$, there exists a minimal geodesic from x_0 to y, which is contained in $\dot{\mathcal{R}}_\epsilon$. If we choose ϵ small enough, $\mathcal{R}_\epsilon = \mathcal{R}$ and $\mathcal{R}$ is an open set. This means for almost all $y \in \mathcal{R}$ there is a minimal geodesic, call it γ, connecting x_0 and y, all contained in $\mathcal{R}$. For such y, we have

$$|Df_\infty(x_0) - Df_\infty(y)| \leq C \sup_{M_\infty \setminus S} |D^2 f_\infty| \mathrm{length}(\gamma) \leq \tilde{C},$$

since we have an estimate (34) and since $\mathrm{length}(\gamma) \leq D$, where D is a uniform bound on the diameters of $(M, g(t))$. Since $|Df_\infty(x_0)|$ is a finite number, we get $|Df_\infty|(x) \leq \tilde{C}_1$ for almost all $x \in \mathcal{R}$. On the other hand, since f_∞ is a smooth function on $\mathcal{R} = M_\infty \setminus S$, we get

$$\sup_{M_\infty \setminus S} |Df_\infty| \leq \tilde{C}_1.$$

By similar arguments, we also have

$$\sup_{M_\infty \setminus S} |f_\infty| \leq \tilde{C}_2.$$

□

References

[1] H.D.Cao: Deformation of Kähler metrics to Käher-Einstein metrics on compact Kähler manifolds; Invent. math. 81 (1985) 359–372.

[2] H.D.Cao, B.L.Chen, X.P.Zhu: Ricci flow on compact Kähler manifold of positive bisectional curvature; math.DG/0302087.

[3] J.Cheeger, T.Colding: On the structure of spaces with Ricci curvature bounded below. I; J.Differential Geometry 45, (1997) 406–480.

[4] J.Cheeger, T.Colding: On the structure of spaces with Ricci curvature bounded below. II; J.Differential Geometry 52, (1999) 13-35.

[5] J.Cheeger, T.H.Colding, G.Tian: On the singularities of spaces with bounded Ricci curvatures; GAFA, Geom. funct. anal. 12, (2002) 873–914.

[6] Chen, X.X., G.Tian: Ricci flow on Kähler Einstein surfaces; Invent.Math., 147 (2002) 487–544.

[7] Chen, X.X., G.Tian: Ricci flow on Kähler Einstein surfaces; Inventiones Math. 147 (2002) 487–544.

[8] Chen X.X., G.Tian: Ricci flow on Kähler manifolds; C.R.Acad.Sci.Paris Sr. I Math. 332 (2001) 245–248.

[9] B.Kleiner, J.Lott: Notes on Perelman's papers; http://www.math.lsa.umich.edu/research/ricciflow/posting123004.pdf.

[10] D.Gilbarg, N.S.Trudinger: Elliptic partial differential equations of second order; Springer-Verlag, 1983.

[11] Q.Han, F.Lin: Elliptic partial differential equations; Courant Institute of Mathematical Sciences, 1997, ISBN 0-9658703-0-8.

[12] G. Perelman: The entropy formula for the Ricci flow and its geometric applications; arXiv:math.DG/0211159.

[13] G.Perelman: private correspondence.

[14] N.Sesum, G.Tian, X.Wang: Notes on Perelman's paper on the entropy fomula for the Ricci flow andits geometric applications.

[15] N.Sesum: Convergence of a Kähler Ricci flow; preprint, arXiv:math.DG/0402238 (submitted to the Mathematical Research Letters).

[16] N. Sesum: Convergence of a Kähler-Ricci flow, arXiv:math.DG/0402238, to appear in Mathematical Research Letters.

[17] G.Tian, X.H.Zhu: Uniqueness of Kähler Ricci solitons; Acta Math. 184 (2000), 271–305.

[18] G.Tian, X.H.Zhu: A new holomorphic invariant and uniqueness of Kähler Ricci solitons; Comm.Math.Helv. 77 (2002) 297-325.

[19] Yau, S.T.: Harmonic functions on complete Riemannian manifolds; Comm. Pure Appl. Math. 28 (1975) 201-228.