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Renormalization Group Flow for General σ-Models

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Abstract. The renormalization group flow for σ-models with base space of dimension 1 or 2 is investigated. In two dimensions it is shown that the flow is singular towards the UV for a generic target space. In one dimension it is shown that there are IR fixed points coming from negatively curved symmetric spaces.

The quantum σ-model based on a Riemannian manifold M is the quantum field theory of harmonic maps from $\mathbb{R}^d$ to M. This is a perturbatively renormalizable theory (in dim. reg.) when $d \leqq 2$. For $d=2$ the classical theory has a scale invariance which is generally broken on the quantum level, the breaking being given by the β-function. This function was first computed to the two-loop level in [1], which also studied the flow toward the infra-red. Such is the region of interest in statistical mechanics, in which IR fixed points give the long distance limits of scalar theories for $d<4$ [2]. In other areas, the tree level of (super) string theories is given by (super) quantum σ-models. There is evidence that for various degrees of supersymmetry the β-function is given exactly by its one-loop approximation, which would imply that in order to have the scale invariance desired for string theory, it suffices that M be a ((hyper)-Kähler) Ricci-flat manifold [3]. Finally, the Hamiltonian of the quantum σ-model is a renormalized version of the formal Laplacian + potential term acting on functions on the loop space ΩM. This may be of mathematical interest.

It is an open question as to when the quantum σ-model exists as a continuum field theory. So far it has been constructed in the $d=2$ case when $M=S^N$ and a hierarchical propagator is used [4]. It is generally believed that for a continuum QFT based on perturbation theory to exist, one must have asymptotic freedom at short distance. This is needed so that one can solve the renormalization group (RG) flow without singularities toward the UV. For the $d=2$ σ-model, the asymptotic freedom condition means that M must be Ricci-nonnegative. We wish to look more carefully at the flow toward the UV. We find that asymptotic freedom is not enough. One also needs an exceptional smoothness of the metric which roughly indicates that the only interesting manifolds for which the σ-model exists are homogeneous spaces and nonnegatively-curved Einstein manifolds. This may affect approaches to the construction of these models.

Another question is whether the perturbation predictions for the renormalization flow are accurate. An interesting case is that of $d=1$. There are then no UV divergences, but the bad IR divergences require renormalization. The one-loop perturbation theory predicts a nontrivial IR fixed point when M is a negatively-curved Einstein manifold. As the $d=1$ QFT is just quantum mechanics, one can solve this theory exactly in the Hamiltonian approach and check the perturbation results. (I thank J. Zinn-Justin for this observation.) Using harmonic analysis on symmetric spaces, we show that for negatively curved symmetric spaces there is indeed a nontrivial IR fixed point and compute its location in coupling-constant space for the case when M is a simply-connected hyperbolic manifold. We also show that there are some nonsymmetric IR fixed points.

I. The UV Flow for $d=2$

There are two reasons that one wishes to have a singularity-free RG flow [5].

1. The flow gives the effective coupling constants at different energy scales in the renormalized field theory, and one needs finite coupling constants in order to have finite Green's functions.

2. In order to construct a continuum field theory, the RG equation tells how the bare coupling constants must depend on the UV cutoff. Recent work shows that for a number of asymptotically free field theories, in order to obtain a continuum limit it is enough to compute the bare couplings at a cutoff Λ using the two-loop β-function. Then if one integrates down to a fixed energy scale and takes the $\Lambda \to \infty$ limit, one obtains some continuum limit [6].

Reason 2 indicates that when flowing toward a Gaussian fixed point, it is enough to consider the two-loop RG flow. In the case of the σ-model we will only consider the one-loop flow for technical simplicity; the results should extend to the two-loop case. As mentioned above, the one-loop result should be exact given enough supersymmetry.

Using dimensional regularization, the one-loop RG equation is

$$\frac{d}{dT} g_{ij} = R_{ij}(g), \tag{1}$$

with $T \to \infty$ being the UV limit [1]. In the other direction (the $T \to \infty$ IR limit) this same equation has coincidentally been studied in order to find which topological types of manifolds can support a metric of positive Ricci curvature [7]. In order to see the qualitative nature of the flow of (1), consider a C^2 metric g on S^2. By the Riemann mapping theorem, we can choose isothermal coordinates so that $g = e^{2\phi} g_0$, with g_0 being the standard S^2 metric. Then (1) becomes $\frac{d}{dT} e^{2\phi} = 1/2(R(g_0) - \nabla^2 \phi)$. This is a nonlinear heat equation going backwards in time, and one would not expect that there is a solution for arbitrary smooth initial data.

A trivial solution is given by homotheities of g_0:

$$\phi(T)=1/2\ln(c+(1/2)R(g_0)T).$$

Linearizing (1) around this solution we have that $\delta\phi$ satisfies

$$\frac{d}{dT}(e^{2\phi}\delta\phi)=-1/4\nabla^2\delta\phi.$$

Defining $S=\frac{2}{R_0}\ln\left(\frac{R_0}{2c}T+1\right)$, this becomes $\frac{d}{ds}\delta\phi=-1/4\nabla^2\delta\phi$. If this is to have a smooth solution for all $s\geqq 0$ one needs that $e^{-s/4\nabla^2}\delta\phi(0)\in L^2$ for all $s\geqq 0$. That is, if $\delta\phi(0)=\Sigma c_i\chi_i$ is the eigenfunction decomposition of $\delta\phi(0)$, then one needs that $|c_i|^2=o(e^{-K\lambda_i})$ for all $K\geqq 0$. Clearly the generic smooth variation $\delta\phi(0)$ will not satisfy this.

In order to do such a linearization analysis in higher dimensions, note that there is a trivial class of solutions to (1) given by nonnegatively-curved Einstein manifolds. If $R_{ij}(g_0)=\alpha(g_0)_{ij}$, $\alpha\geqq 0$, then a solution is $g(T)=(1+\alpha T)g_0$. As there are no longer isothermal coordinates, one wishes to formulate a smoothness criterion for the metric which is diffeomorphism invariant. There is a natural operator on symmetric 2-tensors $h\in\Gamma(S^2TM)$ given by the Lichnerowicz Laplacian: $(\Delta h)_{ij}=-h_{ij;k}{}^{k}+R_{ik}h^k{}_j+R_{jk}h^k{}_i-2R_{ikjl}h^k{}_l$. We show that to linearized order, any smooth solution of (1) must be such that the coefficients in the eigenfunction decomposition of $R_{ij}(g(0))$ (with respect to Δ) fall off faster than any exponential in the eigenvalue.

Proposition. *Let $\mathcal{M}$ denote the affine space of C^∞ metrics on a compact manifold M, in the Fréchet topology. Suppose that $g:[0,1]\times[0,\infty]\to\mathcal{M}$ is a smooth map such that $g(0,T)=(1+\alpha T)g(0,0)$ for some $\alpha\geqq 0$, and $\forall\mathscr{E}$, $\frac{d}{dT}g(\mathscr{E},T)=\mathrm{Ric}\,g(\mathscr{E},T)$. Then*

$$\forall v\geqq 0,\quad \lim_{\mathscr{E}\to 0}|e^{v\Delta}\mathrm{Ric}\,g(\mathscr{E},0)|^2_{L^2},\quad \left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}|e^{v\Delta}\mathrm{Ric}\,g(\mathscr{E},0)|^2_{L^2},$$

and $\left.\frac{d^2}{d\mathscr{E}^2}\right|_{\mathscr{E}=0}|e^{v\Delta}\mathrm{Ric}\,g(\mathscr{E},0)|^2_{L^2}$ *are all finite.*

Proof. First, $|e^{v\Delta}\mathrm{Ric}\,g(0,0)|^2=|e^{v\Delta}\alpha g(0,0)|^2=\alpha^2|g(0,0)|^2<\infty$. To proceed, we need the linearization formula for the Ricci tensor. For $h\in\Gamma(S^2TM)$, one has $\left.\frac{d}{d\beta}\right|_{\beta=0}\mathrm{Ric}(g+\beta h)=1/2\Delta h-\mathrm{div}^*\,\mathrm{div}\,Gh$, where $(Gh)_{ij}=h_{ij}-1/2g_{ij}h^k{}_k$, $(\mathrm{div}\,h)_i=-h_{ij;}{}^{j}$ and $(\mathrm{div}^*V)_{ij}=1/2(V_{i;j}+V_{j;i})$ [8]. If $h=\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}g(\mathscr{E},T)$, then $\frac{d}{dT}h=(1/2\Delta-\mathrm{div}^*\,\mathrm{div}\,G)h$, where the operators are with respect to $g(0,T)$. Equivalently, $\frac{d}{dT}h=(1+\alpha T)^{-1}(1/2\Delta-\mathrm{div}^*\,\mathrm{div}\,G)h$, where the operators are now with respect to $g(0,0)$, or $\frac{d}{ds}h=(1/2\Delta-\mathrm{div}^*\,\mathrm{div}\,G)h$ with $S=1/\alpha\ln(1+\alpha T)$. By hypothesis, $h(s)\in L^2(S^2TM)$ for all $s\geqq 0$. Given $z>0$, we have a solution of

$$\frac{dk}{dr}=-(1/2\Delta-\mathrm{div}^*\,\mathrm{div}\,G)k\quad\text{for}\quad r\in[0,z]\quad\text{given by}\quad k(r)=h(z-r).\qquad(2)$$

Let us solve the equations $\frac{d}{dr}l=-1/2\Delta l$ and $\frac{d}{dr}X=-(\alpha X-1/2\operatorname{div}Gl)$ for $r\in[0,z]$ with $l\in\Gamma(S^2TM)$, $X\in\Gamma(TM)$, $l(0)=k(0)$ and $X(0)=0$. Then

$$\frac{d}{dr}(l+\mathscr{L}_Xg)=-1/2\Delta l-\alpha\mathscr{L}_Xg+1/2\mathscr{L}_{\operatorname{div}Gl}g=-(1/2\Delta-\operatorname{div}^*\operatorname{div}G)l-\mathscr{L}_X\operatorname{Ric}g$$
$$=-(1/2\Delta-\operatorname{div}^*\operatorname{div}G)(l+\mathscr{L}_Xg).$$

By the uniqueness of solutions to (2), [7], we have $k=l+\mathscr{L}_Xg$. Thus over any interval we can assume that h has the form $m+\mathscr{L}_Yg$ with $\frac{d}{ds}m=1/2\Delta m$.

Lemma. $\left(\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}\Delta\right)g=-\Delta h.$

Proof. Because $\Delta_{g+\mathscr{E}h}(g+\mathscr{E}h)=0$,

$$\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}\Delta_{g+\mathscr{E}h}(g+\mathscr{E}h)=\left(\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}\Delta_q\right)g+\Delta_gh=0. \quad\square$$

We need that at $T=0$,

$$\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}(e^{v\Delta}\operatorname{Ric})=e^{v\Delta}\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}\operatorname{Ric}+\int_0^v e^{w\Delta}\left.\frac{d\Delta}{d\mathscr{E}}\right|_{\mathscr{E}=0}e^{(v-w)\Delta}dw\operatorname{Ric}$$
$$=e^{v\Delta}\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}\operatorname{Ric}+1/\Delta(e^{v\Delta}-I)\left.\frac{d\Delta}{d\mathscr{E}}\right|_{\mathscr{E}=0}\operatorname{Ric}$$
$$=e^{v\Delta}\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}\operatorname{Ric}-\alpha(e^{v\Delta}-I)h.$$

Also,

$$\left.\frac{d^2}{d\mathscr{E}^2}\right|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric}=e^{v\Delta}\left.\frac{d^2}{d\mathscr{E}^2}\right|_{\mathscr{E}=0}\operatorname{Ric}+2\int_0^v e^{w\Delta}\left.\frac{d\Delta}{d\mathscr{E}}\right|_{\mathscr{E}=0}e^{(v-w)\Delta}\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}\operatorname{Ric}dw$$
$$+\int_0^v e^{w\Delta}\left.\frac{d^2\Delta}{d\mathscr{E}^2}\right|_{\mathscr{E}=0}e^{(v-w)\Delta}\operatorname{Ric}dw+2\int_0^v\int_0^w e^{z\Delta}\left.\frac{d\Delta}{d\mathscr{E}}\right|_{\mathscr{E}=0}$$
$$\times e^{(w-z)\Delta}\left.\frac{d\Delta}{d\mathscr{E}}\right|_{\mathscr{E}=0}e^{(v-w)\Delta}\operatorname{Ric}dz\,dw.$$

Let $Q(\eta,\eta')$ denote the inner product on $\Gamma(S^2TM)$. Then

$$\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}|e^{v\Delta}\operatorname{Ric}|^2$$
$$=\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}Q(e^{v\Delta}\operatorname{Ric},e^{v\Delta}\operatorname{Ric})$$
$$=\left.\frac{dQ}{d\mathscr{E}}\right|_{\mathscr{E}=0}(e^{v\Delta}\operatorname{Ric},e^{v\Delta}\operatorname{Ric})+2Q\left(e^{v\Delta}\operatorname{Ric},\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric}\right)$$
$$=\left.\frac{dQ}{d\mathscr{E}}\right|_{\mathscr{E}=0}(\operatorname{Ric},\operatorname{Ric})+2Q\left(\operatorname{Ric},e^{v\Delta}\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}\operatorname{Ric}+\int_0^v e^{w\Delta}\left.\frac{d\Delta}{d\mathscr{E}}\right|_{\mathscr{E}=0}e^{(v-w)\Delta}\operatorname{Ric}dw\right)$$
$$=\left.\frac{dQ}{d\mathscr{E}}\right|_{\mathscr{E}=0}(\operatorname{Ric},\operatorname{Ric})+2Q\left(\operatorname{Ric},\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}\operatorname{Ric}\right)+2vQ\left(\operatorname{Ric},\left.\frac{d}{d\mathscr{E}}\right|_{\mathscr{E}=0}\operatorname{Ric}\right).$$

By the smoothness assumption on the map g, this is finite.

Similarly,

$$\begin{aligned}\frac{d^2}{d\mathscr{E}^2}\bigg|_{\mathscr{E}=0}|e^{v\Delta}\operatorname{Ric}|^2 &= \frac{d^2}{d\mathscr{E}^2}Q(e^{v\Delta}\operatorname{Ric},e^{v\Delta}\operatorname{Ric})\\
&=\frac{d^2Q}{d\mathscr{E}^2}\bigg|_{\mathscr{E}=0}(e^{v\Delta}\operatorname{Ric},e^{v\Delta}\operatorname{Ric})\\
&\quad+4\frac{dQ}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}\left(e^{v\Delta}\operatorname{Ric},\frac{d}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric}\right)\\
&\quad+2Q\left(\frac{d}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric},\frac{d}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric}\right)\\
&\quad+2Q\left(e^{v\Delta}\operatorname{Ric},\frac{d^2}{d\mathscr{E}^2}\bigg|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric}\right)\\
&=\frac{d^2Q}{d\mathscr{E}^2}\bigg|_{\mathscr{E}=0}(\operatorname{Ric},\operatorname{Ric})+4\frac{dQ}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}\left(\operatorname{Ric},\frac{d}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric}\right)\\
&\quad+2Q\left(\frac{d}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric},\frac{d}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric}\right)\\
&\quad+2Q\left(\operatorname{Ric},\frac{d^2}{d\mathscr{E}^2}\bigg|_{\mathscr{E}=0}\operatorname{Ric}+2\frac{d\Delta}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}\frac{e^{v\Delta}-I}{\Delta}\frac{d\Delta}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}\operatorname{Ric}\right.\\
&\quad\left.+v\frac{d^2\Delta}{d\mathscr{E}^2}\bigg|_{\mathscr{E}=0}\operatorname{Ric}+2\frac{d\Delta}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}\frac{1}{\Delta^2}(e^{v\Delta}-I-v\Delta)\frac{d\Delta}{d\mathscr{E}}\bigg|_{\mathscr{E}=0}\operatorname{Ric}\right).\end{aligned}$$

The only part involving $\frac{d^2}{d\mathscr{E}^2}\Big|_{\mathscr{E}=0}g(\mathscr{E},0)$ is

$$\frac{d^2Q}{d\mathscr{E}^2}\bigg|_{\mathscr{E}=0}(\operatorname{Ric},\operatorname{Ric})+2Q\left(\operatorname{Ric},\frac{d^2}{d\mathscr{E}^2}\bigg|_{\mathscr{E}=0}\operatorname{Ric}\right)+2vQ\left(\operatorname{Ric},\frac{d^2\Delta}{d\mathscr{E}^2}\bigg|_{\mathscr{E}=0}\operatorname{Ric}\right),$$

which is finite by the smoothness assumption. For the terms only involving $h_0=\frac{d}{d\mathscr{E}}\Big|_{\mathscr{E}=0}g(\mathscr{E},0)$, under the decomposition $h=m+\mathscr{L}_Y g$, the $\mathscr{L}_Y g$ term is an infinitesimal diffeomorphism which preserves $Q(e^{v\Delta}\operatorname{Ric},e^{v\Delta}\operatorname{Ric})$, and so can be dropped. Thus WLOG we can assume $\frac{dh}{ds}=1/2\Delta h$ with $h(s)\in L^2(S^2TM)$ for all $s\geqq 0$. That is, $h_0\in e^{-1/2\Delta s}(L^2)$ for all $s\geqq 0$. To finish the proposition, it suffices to prove that $\frac{d}{d\mathscr{E}}\Big|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric}$ and the terms of $\frac{d^2}{d\mathscr{E}^2}\Big|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric}$ only involving first derivatives are all finite.

We have $\frac{d}{d\mathscr{E}}\Big|_{\mathscr{E}=0}e^{v\Delta}\operatorname{Ric}=e^{v\Delta}(1/2\Delta-\operatorname{div}^*\operatorname{div}G)h_0-\alpha(e^{v\Delta}-I)h_0$. Now

$$\begin{aligned}(1/2\Delta-\operatorname{div}^*\operatorname{div}G)\operatorname{div}^*\operatorname{div}Gh_0&=(1/2\Delta-\operatorname{div}^*\operatorname{div}G)\mathscr{L}_{1/2\operatorname{div}Gh_0}g\\
&=\mathscr{L}_{1/2\operatorname{div}Gh_0}\operatorname{Ric}=\alpha\operatorname{div}^*\operatorname{div}Gh_0\end{aligned}$$

and so $\Delta \operatorname{div}^* \operatorname{div} G = 2((\operatorname{div}^* \operatorname{div} G)^2 + \alpha \operatorname{div}^* G)$, from which $e^{v\Delta} \operatorname{div}^* \operatorname{div} G = \operatorname{div}^* \operatorname{div} G e^{2v(\operatorname{div}^* \operatorname{div} G + \alpha)}$. Choose constants w and β such that $2v|\operatorname{div}^* \operatorname{div} G + \alpha| \leqq w\Delta + \beta$. Then the formal expression $e^{2v(\operatorname{div}^* \operatorname{div} G + \alpha)} e^{-(w\Delta + \beta)}$ defines a bounded quadratic form on $e^{-2v(\operatorname{div}^* \operatorname{div} G + \alpha)^*}(L^2)$, and so extends to a bounded operator on L^2. Write h_0 as $e^{-z\Delta}\tilde{h}$ for $z > \max(w, v)$ and $\tilde{h} \in L^2(S^2TM)$. Then

$$\frac{d}{d\mathscr{E}}\bigg|_{\mathscr{E}=0} e^{v\Delta} \operatorname{Ric} = (1/2 e^{v\Delta}\Delta - \alpha e^{v\Delta} + \alpha) e^{-z\Delta}\tilde{h} - e^{\beta} \times \operatorname{div}^* \operatorname{div} G (e^{2v(\operatorname{div}^* \operatorname{div} G + \alpha)} e^{-(w\Delta + \beta)}) e^{-(z-w)\Delta}\tilde{h}$$

is in $L^2(S^2TM)$.

The terms remaining of $\frac{d^2}{d\mathscr{E}^2}\Big|_{\mathscr{E}=0} e^{v\Delta} \operatorname{Ric}$ are

$$2\frac{d\Delta}{d\mathscr{E}}\bigg|_{\mathscr{E}=0} \frac{e^{v\Delta} - I}{\Delta} \frac{d}{d\mathscr{E}}\bigg|_{\mathscr{E}=0} \operatorname{Ric}$$

and

$$2\frac{d\Delta}{d\mathscr{E}}\bigg|_{\mathscr{E}=0} \frac{1}{\Delta^2}(e^{v\Delta} - v\Delta - I)\frac{d\Delta}{d\mathscr{E}}\bigg|_{\mathscr{E}=0} \operatorname{Ric}.$$

The first is

$$2\frac{d\Delta}{d\mathscr{E}}\bigg|_{\mathscr{E}=0} \frac{e^{v\Delta} - I}{\Delta}(1/2\Delta - \operatorname{div}^* \operatorname{div} G)h_0$$
$$= \frac{d\Delta}{d\mathscr{E}}\bigg|_{\mathscr{E}=0} (e^{v\Delta} - I)h_0 - \frac{d\Delta}{d\mathscr{E}}\bigg|_{\mathscr{E}=0} \operatorname{div}^* \operatorname{div} G \frac{1}{\operatorname{div}^* \operatorname{div} G + \alpha}(e^{2v(\operatorname{div}^* \operatorname{div} G + \alpha)} - I)h_0,$$

which is in $L^2(S^2TM)$ by the above argument. The second is $-2\alpha \frac{d\Delta}{d\mathscr{E}}\Big|_{\mathscr{E}=0} \frac{1}{\Delta^2}(e^{v\Delta} - v\Delta - I)\Delta h_0$, which is also in $L^2(S^2TM)$. □

The above smoothness condition on the Ricci tensor is clearly not satisfied for a generic smooth metric. Some classes of metrics for which it is satisfied are those of nonnegatively curved homogeneous and Einstein spaces, for which we know that the RG equation is solvable. Thus it seems to be a reasonable conjecture that the above condition is necessary to have a smooth RG flow. Otherwise, the flowed metric may become bumpy and blow up, possibly in infinitesimal time. Conversely, the metric will become smoother when flowing toward the IR, although the average curvature may increase.

We have shown in a linearized approximation that the condition of the proposition is necessary for a smooth flow. One may ask whether this is a good approximation, because if the metric blows up then nonlinear effects could be important. It is easy to see that some smoothness condition on the initial metric is necessary. A priori, one could conceive of a solution of (1) for which $g(0)$ is in C^α, $\alpha < \infty$. This gives a solution of $\frac{d}{dr}\tilde{g} = -\operatorname{Ric}(\tilde{g})$ for $r \in [0, \mathscr{E}]$ with $\tilde{g}(\mathscr{E})$ in C^α. Using the trick of [9], fix a metric $\varrho \in C^\infty(S^2TM)$ and consider the equation $\frac{d}{dr}\gamma$

$= -\mathrm{Ric}(\gamma) + \mathrm{div}^* \varrho^{-1} \mathrm{div}\, G\varrho$, $\gamma(0) = \tilde{g}(0)$. By parabolic theory [10], $\gamma(r)$ is in $C^\infty(S^2TM)$ for all $r > 0$. Solve $\frac{d}{dr}\phi = -1/2\varrho^{-1}\mathrm{div}\, G\varrho$ for $\phi(r) \in \mathrm{Diff}\, M$, with $\phi(0) = I$. Then ϕ is C^∞ for all $r \geqq 0$ and $\frac{d}{dr}(\phi^*\gamma) = -\mathrm{Ric}(\phi^*\gamma)$, $(\phi^*\gamma)(0) = \tilde{g}(0)$, from which $\tilde{g}(v) = (\phi^*\gamma)(r)$ is C^∞ for $r > 0$. Thus $g(0) = \tilde{g}(\mathscr{E})$ must be C^∞.

As for the physical interpretation of a flow singularity, it is conceivable that it is due to a perturbative expansion around a false vacuum. For example, in the large N Gross-Neveu model [5] the perturbative β-function obtained by summing bubble diagrams predicts tachyons. However, if one allows a vacuum expectation of $\bar{\psi}\psi$, then there is a dynamical mass generation whose bare coupling constant dependence is given by the β-function. The supposed singularity occurs when flowing toward the IR and the interpretation is that the IR behaviour of the true vacuum (with mass generation) is different from that of the perturbative vacuum, while the UV behavior is the same. For the σ-model one would need a new vacuum with different UV behaviour, which would seem to be difficult to have in a UV asymptotic free theory.

One could also ask whether there is a nontrivial UV fixed point. For constant curvature spaces the β-function has been computed to 3-loop order and also to $1/N$ order in the $1/N$ expansion [11], but the evidence is inconclusive.

II. The IR Flow for $d = 1$

Given a complete Riemannian manifold (M, g) and a C^∞ map $\phi : \mathbb{R} \to M$, the energy of ϕ is $E(\phi) = \frac{1}{2\mu} \int_{-\infty}^{\infty} g_{\mu\nu}(\phi(T)) \frac{d\phi^\mu}{dT} \frac{d\phi^\nu}{dT} dT$, where μ is a constant with units of energy. Let us approximate this by a map $\phi : \mathbb{Z} \to M$ and a functional $\mathscr{L}(\phi) = \frac{1}{2\mu} \sum_{i \in \mathbb{Z}} d^2(\phi(i), \phi(i+1))$. We want to look at the scaling behaviour of the measure $\mu_0 \in \mathscr{M}(M^{\mathbb{Z}})$ given by $\mu_0 = e^{-L} \Pi \sqrt{g(\phi_i)} d\phi_i$. Let $T : M^{\mathbb{Z}} \to M^{\mathbb{Z}}$ be defined by $(T\phi)(i) = \phi(2i)$, and let $D : \mathscr{M}(M^{\mathbb{Z}}) \to \mathscr{M}(M^{\mathbb{Z}})$, the decimation operator, be the push-forward T_*. A scaling limit is given by a sequence $\{c_i\}_{i=0}^\infty$ such that $c_i D^i \mu_0$ has a limit; the c_i's give a rescaling of the ground state energy. First, the structure

$$\mu_0 = \exp - \sum_i f(\phi(i), \phi(i+1)) \Pi \sqrt{g(\phi_i)} d\phi_i$$

is preserved under D; one has

$$D\mu_0 = \exp - \sum_i f'(\phi(i), \phi(i+1)) \Pi \sqrt{g(\phi_i)} d\phi_i$$

with

$$e^{-f'}(m_1, m_2) = \int_M \exp - (f(m_1, m) + f(m, m_2)) \sqrt{g(m)} dm\,.$$

For example, if $M = \mathbb{R}^n$ and $f(m_1, m_2) = \frac{g}{2\mu}|m_1 - m_2|^2$, then $f'(m_1, m_2) = \frac{g}{4\mu}|m_1 - m_2|^2 - \frac{n}{2}\ln\frac{\pi\mu}{g}$, in accord with the result $\beta(g) = -g$.

We will consider M's which are simply-connected irreducible noncompact symmetric spaces G/K. The decimation operator becomes

$$Dh(m_1, m_2) = \int_M h(m_1, m)h(m, m_2)\sqrt{g(m)}dm.$$

Lemma. *Suppose that h is G-invariant; that is, $\forall \gamma \in G$, $h(\gamma m_1, \gamma m_2) = h(m_1, m_2)$. Then Dh is G-invariant.*

Proof.

$$\begin{aligned} Dh(\gamma m_1, \gamma m_2) &= \int h(\gamma m_1, m)h(m, \gamma m_2)\sqrt{g(m)}dm \\ &= \int h(m_1, \gamma^{-1}m)h(\gamma^{-1}m, m_2)\sqrt{g(m)}dm \\ &= \int h(m_1, m')h(m', m_2)\sqrt{g}(\gamma m')d(\gamma m') \\ &= \int h(m_1, m')h(m', m_2)\sqrt{g}(m')dm' = Dh(m_1, m_2). \quad \square \end{aligned}$$

Thus the property of G-invariance is preserved by D.

Lemma. *There is an isomorphism between the G-invariant elements of $C^\infty(M \times M)$ and the C^∞ radial functions on M (i.e. the K-bivariant functions on G). Under this isomorphism, D becomes convolution on radial functions.*

Proof. Let us write $m \in M$ as γK for some $\gamma \in G$. Given a G-invariant h, define $l \in C^\infty(M)$ by $l(\gamma K) = h(K, \gamma K)$. Because $h(K, \gamma K) = h(K, \gamma kK)$, this is well-defined. Then $l(k\gamma K) = h(K, k\gamma K) = h(k^{-1}K, \gamma K) = h(K, \gamma K) = l(\gamma K)$, showing that l is radial. Conversely, given a radial l, define h by $h(\gamma_1 K, \gamma_2 K) = l(\gamma_1^{-1}\gamma_2 K)$. Because $l((\gamma_1 k_1)^{-1}\gamma_2 k_2 K) = l(k_1^{-1}\gamma_1^{-1}\gamma_2 k_2 K) = l(\gamma_1^{-1}\gamma_2 K)$, this is well-defined. Then $h(\gamma\gamma_1 K, \gamma\gamma_2 K) = l((\gamma\gamma_1)^{-1}\gamma\gamma_2 K) = l(\gamma_1^{-1}\gamma_2 K) = h(\gamma_1 K, \gamma_2 K)$, showing that h is G-invariant. Putting $\mathrm{vol}(K) = 1$, one has

$$\begin{aligned} (Dl)(\gamma K) = (Dh)(K, \gamma K) &= \int_G h(K, \gamma' K)h(\gamma' K, \gamma K)d\gamma' \\ &= \int_G l(\gamma')l(\gamma'^{-1}\gamma)d\gamma'. \quad \square \end{aligned}$$

For all reference on harmonic analysis on G/K, we refer to [12]. If $G = NAK$ is the Iwasawa decomposition of G, the radial functions ϕ_λ are defined by $\phi_\lambda(\gamma K) = \int_K \exp((i\lambda + \varrho)A(k\gamma))dk$ for $\lambda \in a^*$, where $A(\gamma) \in a$ is given uniquely by $\gamma \in NA(\gamma)K$ and $\varrho = 1/2 \sum_{\substack{\text{pos.}\\ \text{roots}}} \alpha$. The Fourier transform of a C_0^∞ radial function f is given by $\tilde{f}(\lambda) = \int_G f(\gamma K)\phi_{-\lambda}(\gamma K)d\gamma$, and the inverse transform is $f(\gamma K) = \tilde{c}\int_{a^*} \tilde{f}(\lambda)\phi_\lambda(\lambda K)|c(\lambda)|^{-2}d\lambda$, where $\tilde{c}$ is some constant and $c(\lambda)$ is the Harish-Chandra c-function. Furthermore, $\widetilde{Df} = \tilde{f}^2$. First, consider the case that $-I$ is in the Weyl group W.

Proposition. *Suppose that $f \in C_0^\infty(M)$ is nonnegative and $|\tilde{f}|^2$ has a unique maximum in a^* modulo the action of W. Then for some sequence $\{c_i\}_{i=1}^\infty$, $c_i D^i f \to \phi_0$ pointwise.*

Proof. We have that $\widetilde{D^i f} = \tilde{f}^{2^i}$ and $\tilde{f}^{2^i} \in L^2(a^*, |c(\lambda)|^{-2})$. Because $-I \in W$, $\phi_\lambda^* = \phi_{-\lambda} = \phi_\lambda$, and so $\tilde{f}$ is real. By the Paley-Wiener theorem, $\tilde{f}^{2^i}$ falls off faster than any power on a^*. We also need that $c(\lambda)$ is meromorphic on $a^*_{\mathbb{C}}$ and polynomially bounded on a^*, and $c(0) \neq 0$.

Because $\tilde{f}^{2^i} = (\tilde{f}^2)^{2^{i-1}}$, we can assume that $\tilde{f}$ is positive. For all $N > 0$, $\tilde{f}^N |c|^{-2} \in L^1(a^*)$ and $\mu_N = \tilde{f}^N |c|^{-2} d\lambda / \int \tilde{f}^N |c|^{-2} d\lambda$ is a probability measure on a^* which is W invariant. First, suppose that the maximum of $\tilde{f}$ is away from the origin. If C is a Weyl chamber, suppose that the maximum is at $z \in C$. Then it suffices to show that the sequence of prob. measures $v_N = (\# W)\mu_N|_C$ on C converges to δ_z. WLOG, assume that $\tilde{f}(z) = 1$. Choose $\delta > 0$ such that $|\lambda - z| < \delta \Rightarrow |c(\lambda)|^{-2} \leqq \alpha |\lambda - z|^j$ for some $j \geqq 0$. $\forall 0 < \mathscr{E} < 2\delta$, choose open balls $D(\mathscr{E})$ and $E(\mathscr{E})$ in C around z such that $D(\mathscr{E}) \subset \tilde{f}^{-1}[1-\mathscr{E}, 1] \subset E(\mathscr{E})$. As $E \to 0$, $\operatorname{rad} E(\mathscr{E}) \to 0$. Now

$$\int_{C \setminus E(\mathscr{E})} \tilde{f}^N |c|^{-2} d^n\lambda = \int_{C \setminus E(\mathscr{E})} \tilde{f}^{N-1} (\tilde{f} |c|^{-2}) d^n\lambda \leqq (1-\mathscr{E})^{N-1} \int_C \tilde{f} |c|^{-2} d^n\lambda ,$$

and

$$\int_{D(\mathscr{E}/2)} \tilde{f}^N |c|^{-2} d^n\lambda \geqq \left(1 - \frac{\mathscr{E}}{2}\right)^N \alpha \int_{D(\mathscr{E}/2)} |\lambda|^j d^n\lambda$$
$$= \left(1 - \frac{\mathscr{E}}{2}\right)^N \alpha \Omega_{n-1} \frac{1}{n+j+1} \left(\frac{\mathscr{E}}{2}\right)^{n+j+1} .$$

Thus

$$v_N(C \setminus E(\mathscr{E})) \leqq \frac{(\# W)(1-\mathscr{E})^{N-1} \int_C \tilde{f} |c|^{-2} d^n\lambda}{\left(1 - \frac{\mathscr{E}}{2}\right)^N \alpha \Omega_{n-1} \frac{1}{n+j+1} \left(\frac{\mathscr{E}}{2}\right)^{n+j+1}} \to 0$$

as $N \to \infty$ for all $\mathscr{E}$. Thus $v_N \to \delta_z$ and $(D^N f)(\gamma K) / \int \tilde{f}^N |c|^{-2} d^n\lambda \to \phi_z(\gamma K)$ for all γ. However, ϕ_z is only nonnegative when $z = 0$ [16], which is a contradiction. Thus the maximum of $\tilde{f}$ must be at the origin. The same argument shows that $\mu_N \to \phi_0$, and so $(D^N f)(\gamma K) / \int \tilde{f}^N |c|^{-2} d^n\lambda \to \phi_0(\gamma K)$ for all γ. □

The Weyl group contains $-I$ for $M = H^n$, the simply connected n-dim hyperbolic space, and so the above proposition applies. It is clear in the proof that the only property of $\tilde{f}$ needed is that it falls off faster than some power, which will be the case for $f(\gamma K) = \exp(-(1/2d^2(K, \gamma K))$. Thus the scaling limit for μ_0 is given by the 2-site energy $E(\gamma_1 K, \gamma_2 K) = -\ln \phi_0(\gamma_1^{-1} \gamma_2 K)$. To write this more explicitly, take an H^n of some constant curvature and represent it as conformally equivalent to the flat metric on the unit ball B^n in $\mathbb{R}^n$. We will write the scaling limit as a function of coordinates in B^n; the scaling limit cannot depend on the original sectional curvature, and so we can assume that it is -1. The metric on B^n is then $ds^2 = 4\Sigma dX_i^2 / (1 - \Sigma X_i^2)$ and the ϕ_λ are radial eigenfunctions of Δ with ϕ_0 corresponding to the lowest eigenvalue $(1/4)(n-1)^2$. In terms of the hyperbolic distance r from the origin, the eigenvalue equation is $f_{,rr} + (n+1)\coth r f_{,r} = -\alpha f$. For $\alpha = (1/4)(n-1)^2$ the solution is $\phi_0(r) = {}_2F_1\left(\frac{n-1}{2}, \frac{n-1}{2}; \frac{n}{2}; -\sinh^2 \frac{r}{2}\right)$. Thus the scaling limit is given by the energy $E(m_1, m_2) = -\ln \phi_0(d(m_1, m_2))$, where d is the (curvature $= -1$) hyperbolic distance from m_1 to m_2; this is independent of the original sectional curvature. For $n=3$, ϕ_0 has the simple form $\frac{r}{\sinh r}$. Note that ϕ_0 is not in $L^2(M)$ and $D\phi_0$ does not exist; this is due to the infinite scaling given by $\{c_i\}$ and is why a sequence must used to approach the fixed point ϕ_0.

All of the information about the scaling limit is given by ϕ_0. However, one could ask where the fixed point is in terms of the renormalized metric. For example, let Δ_1 denote the Laplacian with respect to the (sect. curv. $=-1$) metric. Define the renormalized sect. curv. to be $\alpha = \left(-\frac{1}{n}\Delta_1 \ln\phi_0(0,m)|_{m=0}\right)^{-1}$. This is designed so that if $\phi_0(0,m)$ were equal to $\exp -\frac{1}{2}d^2(0,m)$ with d the geodesic distance coming from the (sect. curv. $=-K$) metric, then $\alpha=-K$. One finds that this gives $\alpha = -\frac{4}{(n-1)^2}$.

For general symmetric spaces, one can also see the fixed point as a scaling limit of expectation values. Because of the IR divergences in one dimension, the usual expectation of products of field values does not exist even for the free field. In order to come up with well-defined expectations, consider the lattice σ-model on $\mathbb{Z}$ and require the fields to satisfy $\phi(0)=m_0$, $\phi(T)=m_1$. Upon taking the continuum limit, the mass of this set of fields becomes $e^{-T\Delta}(m_0,m_1)$. That is, we are looking at the expectation $G(m_0,m_1;T)=\langle\delta_{m_0}\phi(0)\delta_{m_1}\phi(T)\rangle$. This is well-defined for the free field with action $L=-\frac{1}{2}g\int_{\infty}^{\infty}|\phi(s)|^2ds$, and one has

$$G(m_0,m_1;T)=\left(\frac{2\pi T}{g}\right)^{-n/2}\exp(-g/2T|m_0-m_1|^2).$$

The factor $T^{-n/2}$ gives the canonical scaling dimension of δ_m to be $n/4$.

In the case of H^2 one can write $e^{-T\Delta}(m_1,m_2)$ explicitly as

$$(4\pi T)^{-3/2}e^{-T/4}\int_{d(m_1,m_2)}^{\infty}\beta e^{-\beta^2/4T}/(\cosh\beta-\cosh d(m_1,m_2))^{-1/2}d\beta,$$

where d is the geod. distance in the (sect. curv. $=-1$) metric. The $e^{-T/4}$ factor comes from the fact that the L^2-spectrum starts at $1/4$, and can be removed by a change in the energy scale. The $(4\pi T)^{-3/2}$ factor gives the anomalous scaling dimension for the operator δ_m to be $3/4$. Upon removing these factors, there is a $T\to\infty$ limit given by $\int_{d(m_1,m_2)}^{\infty}\beta(\cosh\beta-\cosh d(m_1,m_2))^{-1/2}d\beta$, which equals $\phi_0(d(m_1,m_2))$ up to a constant.

For general irreducible noncompact symmetric spaces, one has

$$e^{-T\Delta}(K,\gamma K)=(\text{const})\int_{a^*}e^{-TE(\lambda)}\phi_\lambda(\gamma K)|c(\lambda)|^{-2}d\lambda,$$

where $E(\lambda)$ satisfies $\Delta\phi_\lambda=E(\lambda)\phi_\lambda$ and has a minimum at $0\in a^*$. As in the above proposition, there is a function $s(T)$ such that $\forall\gamma\in G$ $\lim_{T\to\infty}s(T)e^{-T\Delta}(K,\gamma K)=\phi_0(\gamma K)$. The large T behaviour of s is given as follows: let H be the Hessian of

$E(\lambda)$ around 0 and suppose that $|c(\lambda)|^{-2}=O(|\lambda|^j)$ along each ray going to 0. Then

$$\begin{aligned} e^{-T\Delta}(K,\gamma K) &\sim (\mathrm{const})\int e^{-1/2T\langle\lambda,H\lambda\rangle}\phi_0(\gamma K)|c(\lambda)|^{-2}d^n\lambda \\ &= (\mathrm{const})\int e^{-1/2\langle\lambda,H\lambda\rangle}\phi_0(\gamma K)\left|c\left(\frac{\lambda}{\sqrt{T}}\right)\right|^{-2} T^{-n/2}d^n\lambda \sim T^{-\frac{n+j}{2}}. \end{aligned}$$

Thus the scaling dimension of δ_m is $\frac{n+j}{4}$.

We have shown that all constant curvature metrics on B^n flow to $-\ln\phi_0$. One could ask whether this is true of all negatively curved metrics. Such is not the case, as is shown by the following construction. Inside H^n, patch a large region with sect. curv. $-\alpha$, $0<\alpha<1$, to make a manifold (M,g). By making the patched region sufficiently large, one can make an C_0^∞ function on M whose energy is arbitrarily close to $\alpha\frac{(n-1)^2}{4}$. Thus the spectrum extends below $\left(\frac{n-1}{4}\right)^2$.

Proposition. *Let Δ_0 denote the self-adjoint extension of the Laplacian on C_0^∞ functions on H^n and let Δ_g denote the closure of the Laplacian on C_0^∞ functions on (M,g). Then the spectrum of Δ_g below $\frac{(n-1)^2}{4}$ is discrete.*

Proof. We will show, using Weyl's theorem [13], that the essential spectrum of Δ_g is that of Δ_0, namely $\left[\frac{(n-1)^2}{4}, \infty\right]$. For this it suffices to show that $T\equiv(\Delta_0+I)^{-1}-(\Delta_g+I)^{-1}$ is compact. Let σ be a C_0^∞ function on M which is 1 on the patching region, and let M_σ be the corresponding multiplication operator. Then

$$T=(\Delta_g+I)^{-1}(\Delta_g-\Delta_0)(\Delta_0+I)^{-1}=(\Delta_g+I)^{-1}M_\sigma(\Delta_g-\Delta_0)M_\sigma(\Delta_0+I)^{-1}$$

and T^* (with respect to g) is $(\Delta_0^*+I)^{-1}M_\sigma(\Delta_g-\Delta_0^*)M_\sigma(\Delta_g+I)^{-1}$. The Hilbert-Schmidt norm of T^n, if it exists, is $\mathrm{Tr}\,T^{*n}T^n=\mathrm{Tr}\,M_\sigma A M_\sigma B$, where

$$A=(\Delta_g-\Delta_0^*)M_\sigma(\Delta_g+I)^{-1}T^{*N-1}(\Delta_g+I)^{-1}$$

and

$$B=(\Delta_g-\Delta_0)M_\sigma(\Delta_0+I)^{-1}T^{N-1}(\Delta_0^*+I)^{-1}.$$

In terms of operator kernels,

$$\mathrm{Tr}\,T^{*N}T^N=\int\sigma(x)A(x,y)\sigma(y)B(y,x)\sqrt{g(x)}\sqrt{g(y)}\,dxdy.$$

Because A and B are pseudo-differential operators of order $-2N$, $A(x,y)$ and $B(y,x)$ are continuous when $N>n/2$, and so $\mathrm{Tr}\,T^{*N}T^N$ is finite. Thus T^N is Hilbert-Schmidt and T is compact. □

Thus there is a minimal L^2 eigenvalue $\beta<\frac{(n-1)^2}{4}$. By the results of [14], the β-eigenspace is one-dimensional and has a positive L^2-eigenfunction χ. Then $\lim_{T\to\infty} e^{-T\Delta}(x,y)=e^{-T\beta}\chi(x)\chi(y)$, and after rescaling the energy the scaling limit is given by χ, which could be almost anything. Thus there is no unique scaling limit for arbitrary negatively curved metric. On the other hand, one might suspect that there is a neighborhood of the constant curvature metrics in some norm which flows to ϕ_0. This seems plausible because almost all Brownian paths on a negatively curved manifold go to the boundary at ∞ [15]. The Brownian motion is nonrecurrent, as on flat $\mathbb{R}^n$ for $n>3$ as opposed to $\mathbb{R}^n$ for $n \leqq 2$. In the flat case this is responsible for the fact that $\Delta + V$ has a bound state for arbitrary negative V when $n \leqq 2$, but not when $n \geqq 3$ (by the Birman-Schwinger bound [13]).

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References

1. Friedan, D.: Nonlinear models in $2+\mathscr{E}$ dimensions. Phys. Rev. Lett. **45**, 13, 1057 (1980); Ann. Phys. **163**, 2, 318 (1985)
2. Brézin, E., Le Guillou, J.C., Zinn-Justin, J.: Discussion of critical phenomena for general n-vector models. Phys. Rev. B **10**, 3, 892 (1974)
3. Alvarez-Gaumé, L., Ginsparg, P.: Finiteness of Ricci-flat nonlinear σ models. Commun. Math. Phys. **102**, 2, 311 (1985)
4. Gawędzki, K., Kupiainen, A.: In preparation
5. Gross, D.: Applications of the renormalization group. In: Methods in field theory. Balian, R., Zinn-Justin, J. (eds.). Amsterdam: North-Holland 1975
6. Gawędzki, K., Kupiainen, A.: Gross-Neveu model through convergent perturbation expansions. Commun. Math. Phys. **102**, 1 (1985)
7. Hamilton, R.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. **17**, 255 (1982)
8. Berger, M., Ebin, D.: Some decompositions of the space of symmetric tensors on a Riemannian manifold. J. Differ. Geom. **3**, 379 (1969)
9. De Turck, D.: Correction to: Deforming metrics in the direction of the Ricci tensors (to appear)
10. Friedman, A.: Partial differential equations of parabolic type. New York: Prentice-Hall 1964
11. Brézin, E., Hikama, S.: Three-loop calculations in the two-dimensional non-linear σ model. J. Phys. A **11**, 6, 1141 (1978)
12. Helgason, S.: Groups and geometric analysis. New York: Academic Press 1984
13. Reed, M., Simon, B.: Analysis of operators. New York: Academic Press 1978
14. Sullivan, D.: Related aspects of positivity. IHES preprint 1983
15. Prat, J.: Etude asymptotique du mouvement Brownien sur une variété Riemannienne à courbure négative. C.R. Acad. Sci. Paris Sér. A–B **272**, A 1586 (1971)
16. Furstenberg, H.: Translation-invariant cones of functions on semi-simple Lie groups. Bull. Am. Math. Soc. **71**, 271 (1965)

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