Real anomalies

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The relationship between global anomalies of quantum theory and the topology of spaces of real Fredholm operators is shown. The spectral properties of such operators and how they are seen in examples of global anomalies on both compact and noncompact space-times are discussed.

I. INTRODUCTION

It has become clear that many of the anomalies of quantum field theory are due to the nontrivial topology of various moduli spaces, such as the space of connections on a fixed vector bundle modulo the group of gauge transformations.[1] More abstractly, one can view the moduli space as parametrizing a family of Dirac-type operators, and so one is using the particular family of operators in order to view the space $\mathscr{F}$ of all complex Fredholm operators.

If one captures a nontrivial cohomology class of $\mathscr{F}$ by means of a family of operators then this may prevent one from defining the renormalized determinant of the operators in a nice way. To be more precise, in free fermionic path integrals there are two types of determinants which arise. In the Lagrangian the relevant differential operator (the inverse of the covariance) may either map one function space to itself, or to another.

In the second case the determinant can be complex and its anomalous symmetry properties reflect the topology of $\mathscr{F}$. In the first case the determinant is always real. One can view the underlying function space as a real vector space and because fermion fields anticommute, the differential operator must be real and skew adjoint. It turns out that the space $\mathscr{F}_1 R$ of real skew-adjoint Fredholm operators has a very rich topology[2] and we wish to show that much of this topology can be seen in quantum field theories (QFT's). This is manifested both in the existence of zero eigenvalues for Dirac-type operators and in the occurrence of global anomalies, the original example of which is Witten's SU(2) anomaly.[3]

When one rotates fermions from Minkowski space to Euclidean space, one may seem to lose special properties, such as the existence of Majorana or Weyl representations. In Euclidean space, these special Minkowski properties are seen in the existence of operators which anticommute with the Euclidean Dirac operator. In general, one can consider the spaces $\mathscr{F}_k R$ which consist of the elements of $\mathscr{F}_1 R$ which anticommute with a Clifford algebra of operators. These spaces have a topology which is different but related to that of $\mathscr{F}_1 R$. We also give examples of how this refined structure is seen in QFT's.

The structure of this paper is as follows. In Sec. I, we review the topology of some spaces of Fredholm operators. In Sec. III, we discuss how this topology is seen in terms of the spectra of such operators. In Sec. IV, we give examples of QFT's on compact space-times which see the topology of the $\mathscr{F}_k R$'s. These examples are more-or-less known, but we hope that it may help to see them in a unified way, and that the derivations of the indices may be new. In Sec. V, we give some new examples of global anomalies on noncompact space-times. These examples are analogs of the Gell-Mann–Lévy σ model[4] and show that the existence of a global anomaly does not necessarily ruin consistency of a QFT. In Sec. VI, we sketch how the anomalies involving real operators can be understood in terms of analogs of the determinant line bundle of Quillen.[5]

Notation: $\{\sigma^j\}_{j=1}^3$ will denote the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\{\gamma^j_M\}_{j=0}^3$ will denote the $(-+++)$ real Dirac matrices: $\gamma^0_M = I \otimes i\tau^2$, $\gamma^1_M = I \otimes \tau^3$, $\gamma^2_M = \sigma^1 \otimes \tau^1$, and $\gamma^3_M = \sigma^3 \otimes \tau^1$, and γ^5_M will denote $\gamma^0_M\gamma^1_M\gamma^2_M\gamma^3_M$, satisfying $(\gamma^5_M)^2 = -I$, $(\gamma^5_M)^T = -\gamma^5_M$. $\{\gamma^j_E\}_{j=0}^3$ will denote $(++++)$ complex Dirac matrices satisfying $\gamma^i_E\gamma^j_E + \gamma^j_E\gamma^i_E = 2\delta^{ij}$, and γ^5_E will denote $\gamma^0_E\gamma^1_E\gamma^2_E\gamma^3_E$, satisfying $(\gamma^5_E)^2 = I$, $(\gamma^5_E)^\dagger = \gamma^5_E$. A handy reference for Clifford algebra structures is Ref. 6.

II. REVIEW OF TOPOLOGY OF OPERATOR SPACES

Let H be a complex Hilbert space and consider the space of Fredholm operators

$$\mathscr{F} = \{T \in B(\mathscr{H}) \colon \dim \ker T < \infty \ \text{ and } \ \dim \ker T^* < \infty\}.$$

(If one wishes to consider unbounded Fredholm operators one can generally modify the function spaces to reduce the bounded case.) Put $\mathscr{F}_1 = \{T \in \mathscr{F} : T^* = -T$ and the essential spectrum of iT intersects both components of $\mathbf{R} - \{0\}\}$. One has that $\mathscr{F}$ is a classifying space for complex K theory, i.e., for all compact topological spaces X, the Grothendieck group $K(X)$ of virtual vector bundles over X satisfies $K(X) \cong [X, \mathscr{F}]$, where $[X, \mathscr{F}]$ denotes the homotopy classes of maps from X to $\mathscr{F}$.[7] The relationship is as follows: over $\mathscr{F}$ one has the virtual vector bundle Index with fiber [Ker T]−[Coker T] over an element $T \in \mathscr{F}$. Then any element of $K(X)$ can be written as ϕ^* Index for some $\phi \in [X, \mathscr{F}]$. As a consequence, $\mathscr{F}$ has the homotopy type of $\mathbf{Z} \times BU(\infty)$ where the $\mathbf{Z}$ factor refers to the ordinary index of an operator and $BU(\infty)$ is the classifying space for the group $U(\infty)$. By Bott periodicity, $\pi_{k+2}(\mathscr{F}) = \pi_k(\mathscr{F})$ and these homotopy groups $\pi_l(\mathscr{F}) \cong K(S^l)$ are listed in Table I.

Similarly, $\mathscr{F}_1$ is a classifying space for K^{-1}, i.e., $K^{-1}(X) \cong [X, \mathscr{F}_1]$. Then $\pi_{k+2}(\mathscr{F}_1) = \pi_k(\mathscr{F}_1)$ and the

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TABLE I. Homotopy groups of complex operator spaces.

	F	F_1
π_0	Z	0
π_1	0	Z

TABLE II. Homotopy groups of real operator spaces.

	F_0R	F_1R	F_2	F_3R	F_4R	F_5R	F_6R	F_7R
π_0	Z	Z_2	Z_2	0	Z	0	0	0
π_1	Z_2	Z_2	0	Z	0	0	0	Z
π_2	Z_2	0	Z	0	0	0	Z	Z_2
π_3	0	Z	0	0	0	Z	Z_2	Z_2
π_4	Z	0	0	0	Z	Z_2	Z_2	0
π_5	0	0	0	Z	Z_2	Z_2	0	Z
π_6	0	0	Z	Z_2	Z_2	0	Z	0
π_7	0	Z	Z_2	Z_2	0	Z	0	0

homotopy group are listed in Table I. The relationship between $\mathscr{F}$ and $\mathscr{F}_1$ can be seen as a suspension.[8] Let $\Omega\mathscr{F}$ denote the paths in $\mathscr{F}$ from I to $-I$. Then there is a map ϕ: $\mathscr{F}_1\rightarrow\Omega\mathscr{F}$ given by $\phi(T)=\{\cos\pi t+T\sin\pi t\text{: } 0\leqslant t\leqslant 1\}$, which can be shown to a homotopy equivalence. Similarly, let J be an operator unitarily equivalent to $\left(\begin{smallmatrix} i & 0\\ 0 & -i\end{smallmatrix}\right)$ and put

$$\mathscr{F}_2=\{T\in\mathscr{F}_1\text{: } TJ+JT=0\}.$$

Let $\Omega\mathscr{F}_1$ denote the paths in $\mathscr{F}_1$ from J to $-J$. Then there is a homotopy equivalence ϕ_1: $\mathscr{F}_2\rightarrow\Omega\mathscr{F}$, given by $\phi_1(T)=\{J\cos\pi t+T\sin\pi t,\ 0\leqslant t\leqslant 1\}$. Because $\mathscr{F}_2$ is isomorphic to $\mathscr{F}$, one has $\mathscr{F}\sim\mathscr{F}_2\sim\Omega\mathscr{F}_1\sim\Omega^2\mathscr{F}$, which shows the Bott periodicity.

It is now easy to state the relationship between the axial anomaly of QFT and the topology of $\mathscr{F}$. Consider, for example, the space $\mathscr{A}$ of connections on $S^4\times SU(N)$, $N>2$, and the group $\mathscr{G}$ of gauge transformations which are the identity at a point ∞ on S^4. Then the determinant line bundle $\Lambda^{\max}$ Index has first Chern class which is a nontrivial element of $H^2(\mathscr{F},\mathbf{R})=\pi_2(\mathscr{F})\otimes\mathbf{R}=\mathbf{R}$, and which is pulled back via the Dirac operator to give a nontrivial element of $H^2(\mathscr{A}/\mathscr{G},\mathbf{R})=H^2(\Omega^3(SU(N)),\mathbf{R})=\mathbf{R}$.[1] (More precisely, under $\not\partial$: $\mathscr{A}\rightarrow\mathscr{F}$, the pullback $\not\partial^*\ \Lambda^{\max}$ Index is a $\mathscr{G}$-equivariant line bundle over $\mathscr{A}$ which pushes forward to a line bundle on $\mathscr{A}/\mathscr{G}$.) To see this another way, fix $A_0\in\mathscr{A}$ such that $\not\partial_{A_0}$ is invertible. Put

$$\mathscr{K}=\{T\in B(H)\text{: } T-I \text{ is compact and } T \text{ is invertible}\}.$$

Then there is a map ρ: $\mathscr{G}\rightarrow\mathscr{K}$ given by $\rho(g)=\not\partial_{A_0}^{-1}\not\partial_{g\cdot A_0}$. Now $\mathscr{K}$ is homotopically equivalent to $U(\infty)$, or $\mathscr{F}_1$, and $\rho^*\text{: }H^1(\mathscr{K},\mathbf{R})\rightarrow H^1(\mathscr{G},\mathbf{R})$ is nontrivial from $\mathbf{R}$ to $\mathbf{R}$. This is a precise form of the intuitive idea that the phase of the chiral determinant changes by a nontrivial multiple of 2π when going around a nontrivial loop in $\mathscr{G}$. Finally, from the Hamiltonian viewpoint consider the analogous spaces for $S^3\times SU(N)$. The Dirac Hamiltonian $\not\partial_A$: $\Gamma(S)\rightarrow\Gamma(S)$ is skew adjoint and gives a map σ: $\mathscr{A}/\mathscr{G}\rightarrow\mathscr{F}_1$. The nontriviality of $\sigma^*H^3(\mathscr{F}_1,\mathbf{R})\in H^3(\mathscr{A}/\mathscr{G},\mathbf{R})$ leads to a Hamiltonian interpretation of the axial anomaly.[9]

Let us now consider the space $\mathscr{F}_0\mathbf{R}$ of real Fredholm operators on a real Hilbert space $\mathscr{H}_{\mathbf{R}}$. For a compact topological space X with involution τ, let $KR(X)$ denote the Grothendieck group of virtual complex vector bundles over X with an antilinear involution covering τ.[10] [If τ is the identity then $KR(X)=KO(X)$]. One has $KR(X)\cong[X,\mathscr{F}_0R]$. It follows that $\mathscr{F}_0R$ is homotopically equivalent to $\mathbf{Z}\times BO(\infty)$ and $\pi_{k+8}(\mathscr{F}_0R)=\pi_k(\mathscr{F}_0R)$. The homotopy groups are listed in Table II.

In order to get the higher KR functors, let C_k denote the real Clifford algebra generated by $\{e_i\}_{i=1}^k$ with relations $e_ie_j+e_je_i=-2\delta_{ij}, e_i^*=-e_i$. Let ρ: $C_k\rightarrow B(H_{\mathbf{R}})$ be a faithful * representation of C_k. Put

$$\mathscr{G}_k=\{T\in\mathscr{F}_0R\text{: } T=-T^*,\ T\rho(e_i)+\rho(e_i)T=0 \text{ for } 1\leqslant i\leqslant k-1\}.$$

For $k\equiv-1 \pmod 4$ put $\mathscr{F}_kR=\mathscr{G}_k$. For $k\equiv-1 \pmod 4$ there is a slight subtlety: put $\mathscr{F}_kR=\{T\in\mathscr{G}_k$: the essential spectrum of $\rho(e_1)\cdots\rho(e_{k-1})T$ intersects both components of $\mathbf{R}-0\}$. Then $KR^{-k}(X)\cong[X,\mathscr{F}_kR]$.[2] As a consequence, the homotopy groups are those listed in Table II. The various $\mathscr{F}_kR$'s are again linked by suspension maps: put $\Omega\mathscr{F}_{k-1}R=\{$paths in $\mathscr{F}_{k-1}R$ from $\rho(e_{k-1})$ to $-\rho(e_{k-1})\}$. Then ϕ: $\mathscr{F}_kR\rightarrow\Omega\mathscr{F}_{k-1}R$ is given by

$$\phi(T)=\{\rho(e_{k-1})\cos\pi t+T\sin\pi t,\ 0\leqslant t\leqslant 1\}.$$

The various spaces $\mathscr{F}_kR$ have simple interpretations.

k=1: $\mathscr{F}_1R$ is the space of real skew-adjoint Fredholm operators on $H_{\mathbf{R}}$.

k=2: Because $\mathscr{F}_2R$ consists of the elements of $\mathscr{F}_1R$ which anticommute with $\left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)$, they all have the form $\left(\begin{smallmatrix} A & B\\ B & -A\end{smallmatrix}\right)$ with A and B real and skew adjoint. Then $(v,w)\rightarrow(Av+Bw,Bv-Aw)$ and $v+iw\rightarrow(A+iB)(v-iw)$, showing that $\mathscr{F}_2R$ is the space of skew-adjoint antilinear Fredholm operators on a complex Hilbert space. Note that

$$\begin{pmatrix} V\\ W\end{pmatrix}^T\begin{pmatrix} A & B\\ B & -A\end{pmatrix}\begin{pmatrix} V\\ W\end{pmatrix}=\mathrm{Re}(v-iw)^\dagger(A+iB)(v+iw),$$

showing that $\mathscr{F}_2R$ can also be thought of as skew-symmetric Fredholm operators from a complex Hilbert space to its complex conjugate. Finally, because

$$\begin{pmatrix} V\\ W\end{pmatrix}^T\begin{pmatrix} A & B\\ B & -A\end{pmatrix}\begin{pmatrix} V\\ W\end{pmatrix}=\mathrm{Re}(v+iw)^T(A+iB)(v+iw),$$

these operators arise when writing complex Berezin integrals (i.e., Berezin integrals whose total integral is a complex Pfaffian).

k=3: $\mathscr{F}_3R$ consists of the underlying real Fredholm operators coming from skew-adjoint operators on $H_{\mathbf{R}}\otimes\mathbf{C}^2$ of the form $i\alpha_0+\sigma^1\alpha_1+\sigma^2\alpha_2+\sigma^3\alpha_3$, with $\alpha_0,\alpha_1,\alpha_2,\alpha_3\in B(H_{\mathbf{R}})$, which satisfy the essential spectrum property. This anticommutes with the operators $\rho(e_1)$ and $\rho(e_2)$ given by $\rho(e_1)x=\sigma_2\bar{x}$, $\rho(e_2)x=i\sigma_2\bar{x}$: the complex structure comes from $\rho(e_1)\rho(e_2)$.

k=4: $\mathscr{F}_4R$ consists of Fredholm operators of the form $\left(\begin{smallmatrix} 0 & B\\ -B^T & 0\end{smallmatrix}\right)$ acting on $(H_{\mathbf{R}}\otimes\mathscr{H})\oplus(H_{\mathbf{R}}\otimes\mathscr{H})$, where

$B \in B(H_{\mathbf{R}} \otimes \mathscr{H})$ commutes with the quaternions $\mathscr{H}$. The operators $\rho(e_1)$, $\rho(e_2)$, and $\rho(e_3)$ are $i \otimes \sigma_3$, $j \otimes \sigma_3$, and $k \otimes \sigma_3$.

k=5: $\mathscr{F}_5 R$ consists of Fredholm operators of the form $\left(\begin{smallmatrix} 0 & B \\ B & 0 \end{smallmatrix}\right)$ acting on $H_{\mathbf{R}} \otimes (\mathscr{H} \oplus \mathscr{H})$ where $B \in B(H_{\mathbf{R}} \otimes \mathscr{H})$ is skew and commutes with $\mathscr{H}$. The operators $\rho(e_1)$, $\rho(e_2)$, $\rho(e_3)$, and $\rho(e_4)$ are $1 \otimes i\sigma_2$, $i \otimes \sigma_3$, $j \otimes \sigma_3$, and $k \otimes \sigma_3$.

k=6: For $M \in M(2,\mathscr{H})$, let $L(M)$ denote left multiplication by M on $\mathscr{H} \oplus \mathscr{H}$ and for $q \in \mathscr{H}$, let $R(q)$ denote right multiplication by q on $\mathscr{H} \oplus \mathscr{H}$. Then $\mathscr{F}_6 R$ consists of Fredholm operators of the form $B_1 R(j) + B_2 R(k)$ acting on $(H_{\mathbf{R}} \otimes \mathscr{H}) \oplus (H_{\mathbf{R}} \otimes \mathscr{H})$, with B_1 and B_2 being self-adjoint operators in $B(\mathscr{H}_{\mathbf{R}})$. The operators $\rho(e_1),...,\rho(e_5)$ are $L\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right) R(i)$, $L\left(\begin{smallmatrix} 0 & j \\ -j & 0 \end{smallmatrix}\right) R(i)$, $L\left(\begin{smallmatrix} 0 & k \\ -k & 0 \end{smallmatrix}\right) R(i)$, $L\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) R(i)$, and $L\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) R(i)$.

k=7: Because $C_6 = M(8,\mathbf{R})$, $\mathscr{F}_1 R$ consists of $\{\rho(e_1)\cdots\rho(e_6)T: T \in B(\mathscr{H}_{\mathbf{R}}), T^* = T, T$ is Fredholm and the essential spectrum of T lies on both sides of $\mathbf{R} - \{0\}\}$.

k=8: Let $J_1,...,J_6$ denote a representation of the generators of C_6 on $\mathbf{R}^8$ and put $\epsilon = J_1,...,J_6$. Then $\mathscr{F}_8 R$ consists of Fredholm operators of the form $\left(\begin{smallmatrix} A & \epsilon B \\ \epsilon B & -A \end{smallmatrix}\right)$ acting on $H_{\mathbf{R}} \otimes (\mathbf{R}^8 \oplus \mathbf{R}^8)$ with $A \in B(H_{\mathbf{R}})$ skew and $B \in B(H_{\mathbf{R}})$ self-adjoint. The operators $\rho(e_1),...,\rho(e_7)$ are $J_1 \otimes \sigma_1,...,J_6 \otimes \sigma_1$ and $I \otimes i\sigma_2$. The Bott periodicity is seen in the fact that $\mathscr{F}_8 R$ is isomorphic to $\mathscr{F}_0 R$.

III. SPECTRAL PROPERTIES OF REAL INDEX THEORY

We will be interested in the π_0 and π_1 homotopy groups of operator spaces. First, for the complex Fredholm operators $\pi_0(\mathscr{F}) = \mathbf{Z}$ shows that $\mathscr{F}$ breaks into connected components labeled by the index of an operator. That $\pi_1(\mathscr{F}_1)$ equals $\mathbf{Z}$ can be seen using spectral flow. Given a smooth map: $S^1 \to \mathscr{F}_1$, we have that the spectrum of $i\Phi(e^{2\pi i\epsilon})$ is uniformly bounded away from zero as ϵ varies in $[0,1]$, with the possible exception of a finite number of eigenvalues. Because the spectrum for $\epsilon = 1$ is the same as that for $\epsilon = 0$, the generic circle of operators will have a finite number of eigenvalues which flow from negative to positive when going from $\epsilon = 0$ to $\epsilon = 1$; this number defines the spectral flow F: $[S^1, \mathscr{F}^1] \to \mathbf{Z}$. If the operators $i\Phi(e^{2\pi i\epsilon})$ are actually self-adjoint first-order elliptic differential operators acting on cross sections of a vector bundle E over a compact manifold M, one can compute $F(\Phi)$ by means of the eta invariant.[11] Given such an operator H, define

$$\eta(H) = \lim_{s \to 0} \sum_{\lambda_i \neq 0} \lambda_i |\lambda_i|^{-s-1}.$$

If $H(\epsilon)$ is a one-parameter family of such operators then $\eta(H(\epsilon))$ can have integer jumps as eigenvalues cross the origin, but $(d/d\epsilon)\eta(\mathscr{H}(\epsilon))$ has a smooth extension which can be computed in terms of local quantities. Then

$$0 = 2F(\Phi) + \int_0^1 \frac{d}{d\epsilon} \eta(i\Phi(e^{2\pi i\epsilon}))d\epsilon \tag{1}$$

gives an effective way to compute $F(\Phi)$. One can also compute F as an index by means of a "desuspension." Consider the operator $D = \partial/\partial\epsilon + i\Phi(e^{2\pi i\epsilon})$ acting on cross sections of the pullback of E to $S^1 \times M$. Then $F(\Phi) = \text{Index } D$.[11] Similarly, for some Fredholm operators on a noncompact complete manifold, one can define a generalized eta invariant[12] and the spectral flow is again given by (1); however, the expression $(d/d\epsilon)\eta(H(\epsilon))$ then depends both on local quantities and on the behavior of $H(\epsilon)$ at infinity.

For an operator $T \in \mathscr{F}_k R$, one has that ker T is a C_{k-1} module.[2] If it is not a C_k module then T represents a nontrivial element of $\pi_0(\mathscr{F}_k R)$. Thus the connected components of $\mathscr{F}_k R$ are labeled as follows:

$k = 0$: Index T, $\quad k = 1$: dim ker T (mod 2),

$k = 2$: $\frac{1}{2}$ dim ker T (mod 2),

$k = 4$: $\frac{1}{4}$ dim ker $T = \text{Index}_{\mathscr{H}} B$.

In order to see $\pi_1(\mathscr{F}_k R)$ spectrally, consider first the case $k = 1$. Then one has a one-parameter family $iT(\epsilon)$ of self-adjoint operators, each of which has spectrum symmetric around the origin. As ϵ ranges from 0 to 1 the spectral flow of $iT(\epsilon)$ is zero because of the symmetry, but a finite number of pairs of eigenvalues can be switched. This number (mod 2) then labels the class of $\pi_1(\mathscr{F}_1 R) = \mathbf{Z}_2$ in which $T(\epsilon)$ lies. [Because the switching can be seen by watching what happens near the origin of the spectrum, the definition makes sense even if the operators $iT(\epsilon)$ have continuous spectrum.]

For the case $k = 3$, let $T(\epsilon)$ be a one-parameter family in $\mathscr{F}_3 R$. Viewing $T(\epsilon)$ as a complex operator as in Sec. II, one sees that if x is an eigenvector of $iT(\epsilon)$ with real eigenvalue λ then $\sigma_2\bar{x}$ is also an eigenvector with an eigenvalue λ. Thus there is an action of the complex Clifford algebra $C_1^{\mathbf{C}}$ on the discrete eigenspaces of $iT(\epsilon)$ given by $x \to \sigma_2\bar{x}$ and so the eigenspaces have even complex dimension (one cannot solve $\sigma_2\bar{x} = \alpha x$ with $\alpha \in \mathbf{C}$). The class of $T(\epsilon)$ in $\pi_1(\mathscr{F}_3 R) = \mathbf{Z}$ is labeled by $\frac{1}{2}$ of the spectral flow of $T(\epsilon)$.

Finally, for $k = 7$ the operators are self-adjoint and $\pi_1(\mathscr{F}_7 R)$ is labeled by the spectral flow.

For real first-order differential operators there is a desuspension which maps $\pi_1(\mathscr{F}_k R)$ to $\pi_0(\mathscr{F}_{k+1} R)$. If $T(\epsilon)$ is a one-parameter family of operators in $\mathscr{F}_k R$ then formally $\left(\begin{smallmatrix} \partial_\epsilon & T(\epsilon) \\ T(\epsilon) & -\partial_\epsilon \end{smallmatrix}\right)$ is in $\mathscr{F}_{k+1} R$, as a differential operator on $S^1 \times M$. However, there is a slight subtlety, since to obtain the isomorphism between $\pi_1(\mathscr{F}_k R)$ and $\pi_0(\mathscr{F}_{k+1} R)$ one must also twist the bundles over S^1 by the Hopf bundle $\mathbf{H}$, the flat $\mathbf{R}$ bundle over S^1 with holonomy -1. To be more precise, we state the following.

Proposition 1: Let $T(\epsilon)$ be a circle of elliptic first-order real differential operators acting on $\Gamma(E)$, with $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$ being a real vector bundle over a compact manifold M. Suppose that each $T(\epsilon)$ is in $\mathscr{F}_k R$. Let Φ_1: $S^1 \times M \to S^1$ and Φ_2: $S^1 \times M \to M$ be the projection maps. Consider the first-order operator D acting on $\Gamma(\Phi_1^* \mathbf{H} \otimes (\Phi_2^* E \oplus \Phi_2^* E))$ given locally by $D = \left(\begin{smallmatrix} \partial_\epsilon & T(\epsilon) \\ T(\epsilon) & -\partial_\epsilon \end{smallmatrix}\right)$. Then under the isomorphism $KR^{-k}(S^1) \to KR^{-(k+1)}(pt)$, the topological index of the family $T(\epsilon)$ is mapped to the topological index of D.

Proof: Let T^{vert} denote the vertical directions in $T(S^1 \times M)$, i.e., $T^{\text{vert}} = S^1 \times TM$. Let η be a fixed element of $KR^{-1}(TS^1)$. Consider the diagram

$$\begin{tikzcd}
KR^{-k}(T^{\mathrm{vert}}) \arrow[r, "\cdot\otimes\eta"] \arrow[d, "t\text{-ind}"'] & KR^{-k}(T^{\mathrm{vert}})\otimes KR^{-1}(TS^1) \arrow[r, "\alpha"] & KR^{-(k+1)}(T(S^1\times M)) \arrow[dr, "t\text{-ind}"] & \\
 & & & KR^{-(k+1)}(\mathrm{pt.}) \\
KR^{-k}(S^1) \arrow[r, "\pi^*\otimes\eta"] & KR^{-k}(TS^1)\otimes KR^{-1}(TS^1) \arrow[r, "\beta"] & KR^{-(k+1)}(TS^1) \arrow[ur, "t\text{-ind}"'] &
\end{tikzcd}$$

where the maps are as follows: t-ind is the topological index[13] which generally maps $KR^*(P\times TX)$ to $KR^*(P)$, where P and X are compact manifolds. $\pi^*: KR^*(S^1)\to KR^*(TS^1)$ is the map induced from the projection π: $TS^1\to S^1$. α is multiplication in $KR^*(T(S^1\times M))$ and β is multiplication in $KR^*(TS^1)$. The multiplicative property of t-ind ensures that that diagram commutes. Thus the only problem is to find η such that $t\text{-ind}\circ\beta\circ(\pi^*\otimes\eta)$ is the identity map from $KR^{-k}(S^1)$ to $KR^{-(k+1)}(\text{pt.})$. It is easily checked that this η is the symbol of the operator ∂_ϵ acting on $\Gamma(\mathbf{H})$, which proves the proposition. ∎

One can easily generalize the Proposition to the case of a fibration over S^1. In a special case, the element of $\pi_1(\mathscr{F}_1R)$ represented by a circle of real skew-adjoint operators can be computed by means of spectral flow. Suppose that for all $e^{2\pi i\epsilon}\in S^1$, $T(\epsilon)$ commutes with a fixed $J\in B(H_{\mathbf{R}})$ satisfying $J^2=-I$, $J^*=-J$. Then J provides a complex structure on $H_{\mathbf{R}}$ and we can write $T(\epsilon)$ as $\begin{pmatrix} A(\epsilon) & B(\epsilon)\\ -B(\epsilon) & A(\epsilon)\end{pmatrix}$. Over the complexes this is equivalent to $\begin{pmatrix} A+iB(\epsilon) & 0\\ 0 & (A-iB)(\epsilon)\end{pmatrix}$ and for each eigenvector $x\in H_{\mathbf{R}}\otimes C$ of $(A+iB)(\epsilon)$ with eigenvalue $i\lambda$, there is a corresponding eigenvector $\bar{x}$ of $(A-iB)(\epsilon)$ with eigenvalue $-i\lambda$. It follows that each eigenvalue $i\lambda$ of $A+iB$ gives a pair $(i\lambda,-i\lambda)$ of eigenvalues of T, and the spectral flow of $i(A+iB)$ equals the number of eigenvalue rearrangements of $T (\mathrm{mod}\, 2)$. Thus the class in $\pi_1(\mathscr{F}_1R)=\mathbf{Z}_2$ represented by $T(\epsilon)$ is labeled by the spectral flow of $i(A+iB)(\epsilon)\ (\mathrm{mod}\, 2)$.

IV. QFT's ON COMPACT SPACES

The topology of real operator spaces arises in two distinct ways in QFT. The first way uses the π_0 invariant to ensure zero eigenvalues for some differential operator T. The physical interpretation of such a zero eigenvalue depends on whether the operator arises from a Lagrangian or a Hamiltonian. If T enters in a Euclidean fermionic Lagrangian in the form $\langle\Psi,T\Psi\rangle$ then a zero eigenvalue can prevent tunnelling between different "θ vacua."[14,15] On the other hand, if T gives the spatial Hamiltonian for a fermionic theory then there are degenerate ground states arising from ker T.[16]

The second way uses the π_1 invariant to label global anomalies. This means that there may be an obstruction to defining a renormalized determinant function for a family of operators. If one is dealing with a circle of operators then it is possible that when one attempts to define the the determinant smoothly along the loop, the spectral properties of the operator force the putative determinant to change sign when going around the circle. (For a more precise interpretation, see Sec. VI.)

Our examples all involve Dirac-type operators. Because the Clifford algebra structure depends strongly on the dimension of the manifold, we will list examples by dimension and restrict to the case of perturbatively renormalizable field theories, i.e., dim$\leqslant 4$. Of course, there are mathematically interested examples in all dimensions. In general, one has that on a k-dimensional manifold, the real Dirac operator (involving only the metric) lies in $\mathscr{F}_kR$.[17]

A. One dimension

Let M be an oriented Riemannian manifold with $\pi_1(M)=0$, let γ: $S^1\to M$ be a smooth path in M, and let (γ,Ψ): $S^1\to TM$ cover γ. The Lagrangian for $N=\frac{1}{2}$ supersymmetric geodesic motion is

$$L(\gamma,\Psi)=\frac{1}{2}\int_{S^1}[|\dot{\gamma}|^2-\langle\Psi,\nabla_{\dot{\gamma}}\Psi\rangle]\,dT,$$

where the Ψ fields are formally anticommuting. Upon doing a formal integration over Ψ in the functional integral $\int e^{-L}\,\mathscr{D}\gamma\,\mathscr{D}\Psi$, one is left with $\int e^{-(1/2)\int_{S^1}|\dot{\gamma}|^2\,dT}\times(\det{}^{1/2}\nabla_{\dot{\gamma}})\mathscr{D}\gamma$. If one tries to define $\det^{1/2}\nabla_{\dot{\gamma}}$ by a regularized product of the positive eigenvalues of $i\nabla_{\dot{\gamma}}$ then the obstruction to a smooth definition is the possibility of an odd number of eigenvalue rearrangements of $i\nabla_{\dot{\gamma}}$ when going around a loop of γ's, i.e., the possibility of a map $S^1\to[S^1,M]$ giving a nontrivial element of $\pi_1(\mathscr{F}_1R)$. In this example one can compute $|\det^{1/2}\nabla_{\dot{\gamma}}|$ explicitly and see whether there is a smooth definition of $\det^{1/2}\nabla_{\dot{\gamma}}$,[18] but one can also see this via Proposition 1.

Proposition 2: There is a loop in Map (S^1,M) whose image is nontrivial in $\pi_1(\mathscr{F}_1R)$ iff M is not spin.

Proof: Let γ: $T^2\to M$ be a loop in Map(S^1,M). Because M is oriented, γ^*TM is an SO(N) bundle over T^2. Let A be the pullback of the Riemannian connection on TM to γ^*TM. Let S be the flat $\mathbf{R}$ bundle on $T^2=S^1\times S^1$ with the holonomy -1 on the first S^1 factor. By Proposition 1, the element of $\pi_1(\mathscr{F}_1R)$ given by γ is the same as the element of $\pi_0(\mathscr{F}_2R)$ given by $\begin{pmatrix}\partial_0 & \partial_1+A_1\\ \partial_1+A_1 & -\partial_0\end{pmatrix}$ acting on $\Gamma(E)$, $E=(\gamma^*TM\oplus\gamma^*TM)\otimes S$. Because the index of D in $\pi_0(\mathscr{F}_2R)$ is a homotopy invariant, it only depends on the topological class of the real vector bundle E. For $n>2$ the SO(n) bundles on T^2 are classified by $H^2(T^2,\mathbf{Z}_2)=\mathbf{Z}_2$, which can be considered to be the element of $\pi_1(\mathrm{SO}(n))$ used in gluing the ends of $S^1\times I$ to construct a bundle over T^2.

Let V denote a nontrivial $\mathbf{R}^3$ bundle over T_2. Now γ^*TM is classified as a real bundle by $\gamma^*w_2(M)$, where $w_2(M)\in\mathscr{H}^2(M,\mathbf{Z}_2)$ is the second Stiefel–Whitney class, and so we can instead compute the index of $\widetilde{D}=\begin{pmatrix}\partial_0 & D_1\\ D_1 & -\partial_0\end{pmatrix}$ acting on either $\Gamma((T^2\times\mathbf{R}^{\dim M})\otimes S)$ if $\gamma^*w_2(M)$ is trivial, or $\Gamma(((T^2\times\mathbf{R}^{\dim M-3})\oplus V)\otimes S)$ if $\gamma^*w_2(M)$ is nontrivial. However, this is computed to be $\gamma^*w_2(M)[T^2]\in\mathbf{Z}_2$, the evaluation of $\gamma^*w_2(M)$ on T^2. As one can pick up a nontrivial

$w_2(M)$ by some mapping of T^2, it follows that the index of $\nabla_{\dot\gamma}$ in $\pi_1(\mathscr{F}_1 R)$ is zero for all γ: $T^2 \to M$ iff $w_2(M) = 0$, i.e., M is spin. One has the same story for $n = 2$. ■

Under canonical quantization one sees that the Hamiltonian corresponding to $L(\gamma,\Psi)$ is $\frac{1}{2} \not{D}_M^2$,[18] which makes sense iff M is spin. Thus in this case a global anomaly causes nonexistence of the quantum theory.

B. Two dimensions

As one has Majorana–Weyl spinors in two-dimensional Minkowski spaces, one can consider the fermionic Lagrangian $L(\Psi) = \int \overline{\Psi} \partial_+ \Psi$, where ∂_+ maps S_+ to S_-. The total integral $\int e^{iL(\Psi)} \mathscr{D}\Psi$ is formally $\det^{1/2} \partial_+$, which Wick rotates to $\det^{1/2} \partial_{\bar z}$. Now on a compact two-dimensional Riemannian spin manifold one only has Majorana spinors, and the real Dirac operator can be written as $\not{D} = \sigma^1 D_0 + \sigma^3 D_1$. The Minkowski–Weyl property can be seen in the fact that $\not{D}$ lies in $\mathscr{F}_2 R$, as it anticommutes with $i\sigma_2$. Then $D_0 + iD_i$ is skew symmetric and one can form the complex Berezin integral $\int \mathscr{D}\Psi \exp - \int \Psi^T (D_0 + iD_1)\Psi$, with total formal integral $\det^{1/2} \partial_{\bar z}$.

The class of $\not{D}$ in $\pi_0(\mathscr{F}_2 R)$ is labeled by $\frac{1}{2} \dim_{\mathbf{R}} \ker \not{D}$ (mod 2). On a Riemann surface of genus g there are $2^{g-1}(2^g + 1)$ spin structures for which it is nontrivial and $2^{g-1}(2^g - 1)$ for which it is trivial.[19]

C. Three dimensions

Let A be a real gauge field and consider the Minkowski–Majorana action

$$L = \int \overline{\Psi} (i\sigma^2 D_0 + \sigma^1 D_1 + \sigma^3 D_2) \Psi \, d^3 x.$$

After integrating out the fermions in $\int e^{iL(\Psi)} \mathscr{D}\Psi$ one is left with $\det^{1/2} \not{D}_A$.

As there are no Euclidean Majorana spinors in three dimensions, let S be the complex spinor bundle over S^3, let E be an $\mathbf{R}^N$ vector bundle over S^3 with connection A, and consider the Euclidean Lagrangian

$$L = \int_{S^3} \Psi^\dagger (\sigma^1 D_1 + \sigma^2 D_2 + \sigma^3 D_3) \Psi, \text{ for } \Psi \in \Gamma(S \otimes E).$$

The Minkowski–Majorana property is seen in the fact that $\not{D}_A = \sigma^1 D_1 + \sigma^2 D_2 + \sigma^3 D_3$ lies in $\mathscr{F}_3 R$, which implies that all eigenspaces of $i\not{D}_A$ are even dimensional. We may try to define the formal integral $\int e^{-L(\Psi)} \mathscr{D}\Psi = \det^{1/2} \not{D}_A$ by multiplying the eigenvalues of $i\not{D}_A$ with half-multiplicity. This will only be well-defined when going around a circle of operators if one-half of the spectral flow around the circle is even, i.e., if the circle is trivial in $\pi_1(\mathscr{F}_3 R)$ (mod 2).

Proposition 3: Let $A(\epsilon)$ be a one-parameter family of connections on E, $0 \leqslant \epsilon \leqslant 1$, with $A(1) = g \cdot A(0)$ for a gauge transformation g: $S^3 \to \mathrm{SO}(N)$. Then the class of $\not{D}_{A(\epsilon)}$ in $\pi_1(\mathscr{F}_3 R) = \mathbf{Z}$ is $g^*\omega[S^3]$, where $\omega \in H^3(\mathrm{SO}(N),\mathbf{Z})$ is given by the three-form $\omega = (-1/48\pi^2) \mathrm{Tr}(g^{-1} dg)^3$. This can be odd for some choice of $A(\epsilon)$ iff $N > 3$.

Proof: Let L be the $\mathbf{R}^N$ bundle over $S^1 \times S^3$ formed from the trivial bundle over $I \times S^3$ by identifying the fiber over $\{1\} \times S^3$ with the g-twisted fiber over $\{0\} \times S^3$, and by then tensoring with the pullback of $\mathbf{H}$ to $S^1 \times S^3$. Let T denote the real Dirac operator on $S^1 \times S^3$ twisted by L and let $\widetilde{T}$ denote the complex Dirac operator on $S^1 \times S^3$ from $S^+ \otimes L$ to $S^- \otimes L$. By Proposition 1, the class of $\not{D}_{A(\epsilon)}$ in $\pi_1(\mathscr{F}_3 R)$ equals the class of T in $\pi_0(\mathscr{F}_4 R)$, which is one-half of the index of $\widetilde{T}$. Now the family of connections $A(\epsilon)$ give a connection B on the trivial bundle over $I \times S^3$ by $B_0 = 0$, $B_i(\epsilon, X) = A_i(\epsilon)(X)$, which extends to a connection on L. We can homotopy B to $B_0 = 0$, $B_i(\epsilon, X) = \epsilon(g^{-1} dg)_i$ without changing the index. Then the index of $\widetilde{T}$ is given by

$$\begin{aligned}
&\int_{S^1 \times S^3} \mathrm{Tr}\, e^{F(B)/2\pi i} \\
&= -\frac{1}{8\pi^2} \int_{S^1 \times S^3} \mathrm{Tr}(d\epsilon \wedge g^{-1} dg \\
&\qquad + (\epsilon - \epsilon^2)(g^{-1} dg)^2)^2 = -\frac{1}{24\pi^2} \int_{S^3} \mathrm{Tr}(g^{-1} dg)^3,
\end{aligned}$$

so the class in $\pi_0(\mathscr{F}_4 R)$ is

$$-\frac{1}{48\pi^2} \int_{S^3} \mathrm{Tr}(g^{-1} dg)^3.$$

One can check that for $N = 3$, the pullback of ω from SO(3) to SU(2) is twice the generator of $\mathscr{H}(\mathrm{SU}(2),\mathbf{Z})$. Since every map from S^3 to SO(3) factors through SU(2), it follows the evaluation of $g^*\omega$ on S^3 on S^3 is always even. On the other hand, for $N = 4$ the pullback of ω from SO(4) to SU(2) via $\mathrm{SU}(2) \xrightarrow{\cdot,1} \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$ gives the generator of $\mathscr{H}^3(\mathrm{SU}(2),\mathbf{Z})$. As one can embed SO(4) in SO(N) for $N \geqslant 4$, it follows that $g^*\omega[S^3]$ can always be odd for some g if $N \geqslant 4$. ■

One could also compute this invariant by computing one-half of the spectral flow directly. This is perhaps more physical, as for fixed ϵ there will be a term in the Euclidean effective action equal to $\pm \frac{1}{4} i\pi\eta(i\not{D}_{A(\epsilon)})$.[20,21]

D. Four dimensions

In four-dimensional Minkowski space we have massive or massless Majorana spinors, or massless Weyl spinors, but not both simultaneously. To see how this is reflected in the Euclidean action, consider the real Euclidean Dirac operator $\not{D} = \Sigma_{\mu=0}^3 \gamma^\mu D_\mu$ with $\gamma^0 = \gamma_M^0 \otimes i\tau^2$ and $\gamma^j = \gamma_M^j \otimes I$. As D anticommutes with the operators $\rho(e_1) = \gamma_M^0 \otimes \tau^1$, $\rho(e_2) = \gamma_M^0 \otimes \tau^3$, and $\rho(e_3) = \gamma_M^5 \otimes I$, it lies in $\mathscr{F}_4 R$ and gives a quaternionic operator. The natural way to form a massive Dirac operator is by $\not{D}_m = \not{D}\, 1 + m\rho(e_3)$, which lies in $\mathscr{F}_3 R$, as it anticommutes with $\rho(e_1)$ and $\rho(e_2)$. Using the complex structure provided by $\rho(e_1)\rho(e_2)$, one can see that $\not{D}_m$ is the underlying real operator for the complex skew-adjoint operator $T_m = i\gamma_E^5(\Sigma_{\mu=0}^3 \gamma_E^\mu D_\mu + m)$. One can then use the action $L(\Psi) = \int \Psi^\dagger T_m \Psi$ to form a complex Berezin integral $\int \mathscr{D}\Psi\, e^{-L(\Psi)}$ to describe massive Euclidean Dirac spinors. This Berezin integral satisfies reflection positivity and the reconstructed Hilbert space is the Fock space of the massive Minkowski Dirac spinor, with the standard second-quantized Dirac Hamiltonian. Although this way of handling Euclidean Dirac spinors may be unconventional, one can see, for example, that the total Berezin integral is formally $\det i\gamma_E^5(\Sigma_{\mu=0}^3 \gamma_E^\mu D_\mu + m)$, which formally equals

$\det(\Sigma_{\mu=0}^3 \gamma_E^\mu D_\mu + m)$, the result obtained from the usual field-doubling method.[22]

In order to deal with Euclidean Majorana spinors, one must use the symmetries of $\not{D}_m$. Because $\not{D}_m$ is $\mathscr{F}_3 R$, there is an operator A satisfying $A^T = -A$ and $AT_m^* + T_m A = 0$. Then $(A^{-1}T_m)^T = -A^{-1}T_m$ and one can use $A^{-1}T_m$ to form the action $\widetilde{L}(\Psi) = \int \Psi^T A^{-1} T_m \Psi$ for complex four-component Ψ. In terms of the charge conjugation operator, $\widetilde{L}$ can be written in the following way. In a given representation $\{\gamma_\mu^E\}_{\mu=0}^3$ of the Dirac algebra, let C satisfy $C\gamma^\mu C^{-1} = -\gamma^{\mu T}$, $C^T = -C$, $C^\dagger = C^{-1} = -C^*$. Then the charge conjugation operator $\Psi \to \Psi^c = C^{-1}\Psi^*$ is intrinsically defined and one can write $\widetilde{L}(\Psi)$ as $\int (\Psi^c)^\dagger (\Sigma_{\mu=0}^3 \gamma^\mu D_\mu + m)\Psi$, with $C(\Sigma_{\mu=0}^3 \gamma^\mu D_\mu + m)$ a skew-symmetric operator. One can form the complex Berezin integral $\int e^{-\widetilde{L}}(\Psi)\mathscr{D}\Psi$ whose total integral is the complex Pfaffian $\det^{1/2} C(\Sigma_{\mu=0}^3 \gamma^\mu \mathscr{D}_\mu + m)$, the $\frac{1}{2}$ reflecting that one is dealing with Majorana spinors. This gives the same way to handle Euclidean Majorana spinors as was probably by Nicolai.[23]

The Weyl property is seen in the fact that $\not{D}$ anticommutes with the self-adjoint operator $\rho(e_1)\rho(e_2)\rho(e_3)$. Writing $\not{D}$ as $\begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$, the quaternionic operator B is the Euclidean equivalent of the chiral Minkowski Dirac operator ∂ and $\det^{1/2} \not{D} \cong \det B$ is the Wick rotation of $\det \not{\partial}$. One can couple an $O(N)$ gauge field A to $\not{D}$ to obtain an operator $\not{D}_A$ in $\mathscr{F}_4 R$, but one can go further and use the quaternionic nature of $\not{D}$ to naturally couple an $\mathrm{Sp}(N)$ gauge field V. Let us write V as $V^{(1)} + V^{(i)}i + V^{(j)}j + V^{(k)}k$ with $V^{(1)}$ skew symmetric and $V^{(i)}$, $V^{(j)}$, and $V^{(k)}$ symmetric. Then

$$\not{D}_V = \sum_{\mu=0}^{3} \gamma^\mu (\partial_\mu + V_\mu^{(1)} + V_\mu^{(i)} i + V_\mu^{(j)} j + V_\mu^{(k)} k)$$

lies in $\mathscr{F}_1 R$ and anticommutes with $\rho(e_1)\rho(e_2)\rho(e_3)$.

The class of $\not{D}$ in $\pi_0(\mathscr{F}_4 R) = \mathbf{Z}$ is labeled by $\frac{1}{4}$ Index B, which is $\frac{1}{2}\hat{A}(M)$ for a pure Dirac operator acting on the real spinors $\Gamma(S)$. As the other homotopy groups of $\mathscr{F}_4 R$ vanish up to π_3, a more interesting example is given by $\not{D}$ coupled to an $\mathrm{Sp}(N)$ gauge field, the original global anomaly of Witten.[3] If $V(\epsilon)$ is a one-parameter family of $\mathrm{Sp}(N)$ connections on an $\mathscr{H}^N$ vector bundle E over M, with $V(1)$ differing from $V(0)$ by a gauge transformation g, then by Proposition 1 the class of $\not{D}_V$ in $\pi_1(\mathscr{F}_1 R) = \mathbf{Z}_2$ is given by $\frac{1}{2}\dim_{\mathbf{R}} \ker T$ (mod 2), with $T = \begin{pmatrix} \partial_\epsilon & \not{D}_{V(\epsilon)} \\ \not{D}_{V(\epsilon)} & -\partial_\epsilon \end{pmatrix}$ acting on cross sections of the $\mathscr{H}^{2N}$ vector bundle over $S^1 \times M$ created by twisting the ends of $I \times ((E \otimes S) \oplus (E \otimes S))$ together by $-g$. Now

$$\begin{aligned}
\tfrac{1}{2}\dim_{\mathbf{R}} \ker T &= \tfrac{1}{2}\dim_{\mathbf{C}} \ker T \\
&= \tfrac{1}{2}\dim_{\mathbf{C}} \ker \rho(e_1)\rho(e_2)\rho(e_3) T \\
&= \tfrac{1}{2}\dim_{\mathbf{C}} \ker \rho(e_1)\rho(e_2)\rho(e_3) \\
&\quad \times \begin{pmatrix} 0 & -i\partial_\epsilon + \not{D}_{V(\epsilon)} \\ i\partial_\epsilon + \not{D}_{V(\epsilon)} & 0 \end{pmatrix} \\
&= \tfrac{1}{2}\dim_{\mathbf{C}} \ker \rho(e_1)\rho(e_2)\rho(e_3)(\partial_\epsilon + i\not{D}_{V(\epsilon)}) \\
&\quad + \tfrac{1}{2}\dim_{\mathbf{C}} \ker \rho(e_1)\rho(e_2)\rho(e_3) \\
&\quad \times (\partial_\epsilon - i\not{D}_{V(\epsilon)}).
\end{aligned}$$

However, over the complexes both $\rho(e_1)\rho(e_2)\rho(e_3) \times (\partial_\epsilon + i\not{D}_{V(\epsilon)})$ and $\rho(e_1)\rho(e_2)\rho(e_3)(\partial_\epsilon - i\not{D}_{V(\epsilon)})$ are equivalent to the real operator $\rho(e_1)\rho(e_2)\rho(e_3)\partial_\epsilon + \not{D}_{V(\epsilon)}$. Thus

$$\tfrac{1}{2}\dim_{\mathbf{R}} \ker T = \dim_{\mathbf{R}} \ker(\rho(e_1)\rho(e_2)\rho(e_3)\partial_{(\epsilon)} + \not{D}_{V(\epsilon)}).$$

In the case of $M = S^4$, the $\mathrm{Sp}(N)$ bundles over $S^1 \times S^4$ are classified by $\mathscr{H}^5(S^1 \times S^4, \mathbf{Z}_2) = \mathbf{Z}_2$, which can be thought of as the element of $\pi_4(\mathrm{Sp}(N)) = \mathbf{Z}_2$, used to join the ends of $I \times S^4$. Upon twisting by the Hopf bundle over S^1, one can check that $\dim_{\mathbf{R}} \ker(\rho(e_1)\rho(e_2)\rho(e_3)\partial_\epsilon + \not{D}_{V(\epsilon)})$ (mod 2), equals the class of g in $\pi_4(\mathrm{Sp}(N))$, which is Witten's orignal calculation. [In this case, because one is dealing with chiral spinors, one can also see that there is a global anomaly using the results of Witten–Bismut–Freed.[24] They showed that the holonomy of the Quillen connection on the determinant line bundle is, when going around the loop $V(\epsilon)$, $\exp - \pi i(\eta(\not{\mathscr{D}}) + \dim_{\mathbf{C}} \ker \not{\mathscr{D}})$ where, in our case, $\not{\mathscr{D}} = i(\rho(e_1)\rho(e_2)\rho(e_3)\partial_{(\epsilon)} + \not{\mathscr{D}}_{V(\epsilon)})$. Because $-i\not{\mathscr{D}}$ lies in $\mathscr{F}_1 R$, the spectral symmetry ensures that $\eta(\not{\mathscr{D}})$ is zero. Thus the holonomy is 1 if $\dim_{\mathbf{C}} \ker \not{\mathscr{D}}$ is even and -1 if $\dim_{\mathbf{C}} \ker \not{\mathscr{D}}$ is odd, showing that in the latter case there is a global anomaly in the sense that the Quillen connection has nontrivial holonomy.]

V. QFT's ON NONCOMPACT SPACES

In general the index of a family of Fredholm operators on a noncompact space is harder to compute than in the compact case. We will only consider complex skew-adjoint operators T whose underlying real operator $T_{\mathbf{R}}$ lies in $\mathscr{F}_1 R$. In general $\det^{1/2} T_{\mathbf{R}} = |\det T|$, and so an odd spectral flow in a family $T(\epsilon)$ prevents the smooth definition of $\det^{1/2} T_{\mathbf{R}}(\epsilon)$. This is seen in the fact that a class $[T_{\mathbf{R}}(\epsilon)]$ in $\pi_1(\mathscr{F}_1 R) = \mathbf{Z}_2$ can be computed using the spectral flow (mod 2) of a circle of operators $T(\epsilon)$, which in turn can be computed using the generalized eta invariant. If $H(\epsilon)$ is a skew-adjoint operator which arises in the Lagrangian of a d-dimensional Euclidean QFT, the most practical way that we know to compute $\eta(H(\epsilon))$ is to regard $H(\epsilon)$ as the Hamiltonian for a $(d+1)$-dimensional Minkowski QFT and compute the vacuum expectation of the change operator $Q = \int \langle j^0(X) \rangle d^d X$, which then gives $\eta(H(\epsilon))$ via the equation $Q = -\frac{1}{2}\eta$.[12] One can calculate Q (or more precisely, $dQ/d\epsilon$) for the $(d+1)$-dimensional theory via a gradient expansion[25] and then the spectral flow is simply the change in Q when going around the circle.

One way of producing the d-dimensional Lagrangian is as follows: let $\{M_i\}_{i=1}^{2d+1}$ be mutually anticommuting self-adjoint matrices and for a map ϕ: $\mathbf{R}^d \to \mathbf{R}^{d+1}$, consider the operator $T = \Sigma_{j=1}^d M^j \partial_j + \Sigma_{j=d+1}^{2d+1} iM^j \phi^j$ with ϕ approaching constants radially. This will be Fredholm iff $|\phi(x)|^2$ is bounded away from zero for large x and then the large x behavior of $\phi/|\phi|$ gives a map $\tilde{\phi}$: $S^{d-1} \to S^d$. We will show that under the one-parameter family of $\tilde{\phi}$'s that starts and ends with a point map, and covers S^d, there is an odd spectral flow and so a global anomaly.

A d-dimensional Euclidean Lagrangian incorporating T is

$$L(\phi,\psi) = \int_{\mathbf{R}^d} \frac{1}{2} \sum_{j=1}^{d+1} |\nabla \phi^j|^2 + V\left(\sum_{j=1}^{d+1} \phi_j^2\right) + \frac{1}{2} \psi^\dagger T \psi.$$

The ϕ-ψ couplings are Yukawa type, and these Lagrangians can be thought to be analogs of the Gell-Mann–Lévy σ model[4] (as opposed to the more recent definitions of a σ model[26]!). L has an $SO(d+1)$ global symmetry, which we will argue to be broken by the global anomaly.

A. One dimension

Consider the Euclidean action

$$L^{(1)}(\phi,\psi) = \int_{\mathbf{R}} \frac{1}{2} (\partial_1 \phi_1)^2 + \frac{1}{2} (\partial_1 \phi_2)^2 + V(\phi_1^2 + \phi_2^2) + \tfrac{1}{2} \Psi^\dagger (\sigma^3 \partial_1 + i\sigma^1 \phi_1 + i\sigma^2 \phi_2) \Psi,$$

for $\phi_1, \phi_2 \in C^\infty(\mathbf{R}^1)$ and $\Psi \in C^\infty(\mathbf{R}^1) \otimes \mathbf{C}^2$, where V goes to ∞ as its argument goes to ∞. Here $L^{(1)}$ has an $SO(2)$ symmetry given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \to \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

$\Psi \to e^{(1/2) i\alpha\sigma_3} \Psi$. First consider the case that V has a minimum away from the origin. Then there will be finite action bosonic paths which (as x_1 ranges from $-\infty$ to ∞) go from one point in the minimum well to another. Formally, the integration over these paths gives the $SO(2)$ symmetry in the quantum theory.

Let T denote the skew operator $\sigma^3 \partial_1 + i\sigma^1 \phi_1 + i\sigma^2 \phi_2$. Consider a family $\phi(\epsilon)$ of background bosonic configurations with $\phi_{1\epsilon}(-\infty) = \phi_{2\epsilon}(-\infty) = 0$ and $\phi_{1\epsilon}(\infty) = \cos\epsilon$, $\phi_{2\epsilon}(\infty) = \sin\epsilon$, as sketched in Fig. 1. As the fermionic integration in $\int e^{-L^{(1)}} \mathscr{D}\phi\, \mathscr{D}\Psi$ leaves a factor of $|\det T|$, if there is an odd spectral flow in $T(\epsilon)$ then one might expect that the instanton sum is ill-defined and the $SO(2)$ symmetry is broken.

For the operator $iT(\epsilon)$ one can show that the derivative of the generalized η invariant is

$$\frac{d}{d\epsilon} \eta(iT(\epsilon)) = \frac{1}{\pi} \frac{1}{\phi_1^2 + \phi_2^2} \left(\frac{d\phi_1}{d\epsilon} \phi_2 - \frac{d\phi_2}{d\epsilon} \phi_1 \right) \Bigg|_{-\infty}^{\infty} = \frac{1}{\pi} \frac{d}{d\epsilon} TAN^{-1} \frac{\phi_1}{\phi_2} \Bigg|_{-\infty}^{\infty}$$

(Refs. 12 and 25). This can be seen by computing the vacuum charge for the two-dimensional Minkowski Lagrangian

$$L^{(2)}(\Psi) = i \int_{\mathbf{R}^2} \frac{1}{2} \overline{\Psi} (\partial_0 + \sigma^3 \partial_1 + i\sigma^1 \phi_1 + i\sigma^2 \phi_2) \Psi,$$

as was done in Ref. 25; the relevant Feynman diagram is that of Fig. 2. Thus there is odd spectral flow as ϵ goes from 0 to 2π.

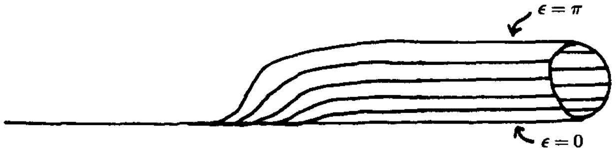

FIG. 1. A one-parameter family of background ϕ's in one dimension.

As our model is quantum mechanical, one can also analyze it in a Hamiltonian approach. The Hamiltonian $\mathscr{H}$, acting on $C^\infty(\mathbf{R}^2) \otimes \Lambda^*(\mathbf{C}^2)$, is

$$\mathscr{H} = \tfrac{1}{2}(-\partial_1^2 - \partial_2^2) + V(X_1^2 + X_2^2) + \tfrac{1}{2} \Psi^\dagger (\sigma^2 X_1 - \sigma^1 X_2) \Psi$$

where the operators Ψ_i satisfy $\{\Psi_i^*, \Psi_j\} = 2\delta_{ij}$. The U(1) charge is

$$Q = -i(X^1 \partial_2 - X^2 \partial_1) + \tfrac{1}{4} i \Psi^\dagger \sigma_3 \Psi,$$

and commutes with $\mathscr{H}$. Representing the complex Clifford algebra generated by the Ψ's on $\Lambda^*(\mathbf{C}^2)$ via $\Psi_i \to \sqrt{2} I(e_i)$, $\Psi_i^* \to \sqrt{2} E(e_i)$, we can split H as $H_1 \oplus H_2$ where H_1 and H_2 act on $C^\infty(\mathbf{R}^2) \otimes \Lambda^{\text{even}}(\mathbf{C}^2)$ and $C^\infty(\mathbf{R}^2) \otimes \Lambda^{\text{odd}}(\mathbf{C}^2)$, respectively, and are given by

$$H_1 = \tfrac{1}{2}(-\partial_1^2 - \partial_2^2) + V(X_1^2 + X_2^2)$$

and

$$H_2 = \tfrac{1}{2}(-\partial_1^2 - \partial_2^2) + V(X_1^2 + X_2^2) + \begin{pmatrix} 0 & -iX_1 - X_2 \\ iX_1 - X_2 & 0 \end{pmatrix}.$$

Because Q has integer spectrum on $C^\infty(\mathbf{R}^2) \otimes \Lambda^{\text{even}}(\mathbf{C}^2)$ and half-integer spectrum on $C^\infty(\mathbf{R}^2) \otimes \Lambda^{\text{odd}}(\mathbf{C}^2)$, the $SO(2)$ symmetry of the ground state will be broken iff the ground state is in $C^\infty(\mathbf{R}^2) \otimes \Lambda^{\text{odd}}(\mathbf{C}^2)$. However, for all $\Psi \in C_0^\infty(\mathbf{R}^2)$, $\langle \Psi | H_1 | \Psi \rangle = \langle \Psi \otimes \binom{1}{0} | H_2 | \Psi \otimes \binom{1}{0} \rangle$ and so

$$(\inf_{\Psi \in C_0^\infty(\mathbf{R}^2) \otimes \Lambda^{\text{odd}}(\mathbf{C}^2)} \langle \Psi | H_2 | \Psi \rangle) \leqslant (\inf_{\Psi \in C_0^\infty(\mathbf{R}^2) \otimes \Lambda^{\text{even}}(\mathbf{C}^2)} \langle \Psi | H_1 | \Psi \rangle),$$

implying that the ground state is indeed in $C^\infty(\mathbf{R}^2) \otimes \Lambda^{\text{odd}}(\mathbf{C}^2)$. In this example it is clear that the existence of a global anomaly does not make a theory inconsistent but merely breaks a global symmetry; this appears to be related to the fact that the anomaly occurs in a global rather than local symmetry.

The functional integral argument for global symmetry breaking required that V have a nontrivial minimum in order that the fermionic operator in the background field be Fredholm. However, from the Hamiltonian argument one sees that symmetry is broken no matter what V is. This can be seen in the functional integral approach by compactifying the space-time from $\mathbf{R}$ to $[-\beta,\beta]$. If there is a symmetry breaking for each β then one would expect the same as β goes to ∞. A convenient choice of fermionic boundary conditions which preserves the $SO(2)$ symmetry is the Atiyah–Patodi–Singer (APS) boundary condition.[11] This requires that $\Psi(\beta)$ lie in the positive eigenspace of $-\sigma^2\phi_1(\beta) + \sigma^1\phi_2(\beta)$ and that $\Psi(-\beta)$ lie in the negative eigenspace of $-\sigma^2\phi_1(-\beta) + \sigma^1\phi_2(-\beta)$.

Proposition 4: Let $T(\epsilon)$ be a family of operators on $C^\infty([-\beta,\beta]) \otimes \mathbf{C}^2$ given by $\sigma^3\partial_1 + i\sigma^1\phi_1 + i\sigma^2\phi_2$ with the

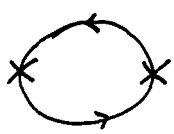

FIG. 2. One-dimensional spectral flow.

APS boundary condition, where $\binom{\phi_{1\epsilon}(-\beta)}{\phi_{2\epsilon}(-\beta)}$ is a nonzero vector independent of ϵ and $\binom{\phi_{1\epsilon}(\beta)}{\phi_{2\epsilon}(\beta)} = \binom{\cos\epsilon}{\sin\epsilon}$ .Then as ϵ goes from 0 to 2π, there is an odd spectral flow of $iT(\epsilon)$.

Proof: Let $V(\epsilon)$ denote the vector $\binom{\sqrt{\phi_1^2+\phi_2^2}\ (-\beta)}{-(\phi_2 - i\phi_1)\ (-\beta)}$ and let $W(\epsilon)$ denote the vector $\binom{\sqrt{\phi_1^2+\phi_2^2}\ (\beta)}{(\phi_2 - i\phi_1)\ (\beta)}$; the APS condition is that $\Psi_\epsilon(-\beta)$ is proportionate to $V(\epsilon)$ and $\Psi_\epsilon(\beta)$ is proportionate to $W(\epsilon)$. Because the spectral flow is a homotopy invariant, we can compute it for any loop in $\mathscr{F}_1$ homotopic to $T(\epsilon)$. In particular, for $0 \leqslant \alpha \leqslant 1$, consider the loop of operators on $C^\infty([-\beta,\beta]) \otimes \mathbf{C}^2$ given by $\sigma^3\partial_1 + \alpha i\sigma^1\phi_1 + \alpha i\sigma^2\phi_2$ with the boundary condition that $\Psi_\epsilon(-\beta)$ is proportionate to $V(\epsilon)$ and $\Psi_\epsilon(\beta)$ is proportionate to $W(\epsilon)$. One can check that this gives a smooth homotopy within the class of elliptic self-adjoint boundary value problems[27] and so it suffices to compute the spectral flow at $\alpha = 0$. Then the spectrum is

$$\left\langle \frac{1}{4\beta}\left[(2n+1)i\pi + \ln\frac{\phi_1 + i\phi_2}{\sqrt{\phi_1^2+\phi_2^2}}(-\beta) - \ln\frac{\phi_1 + i\phi_2}{\sqrt{\phi_1^2+\phi_2^2}}(\beta)\right]\right\rangle n\in\mathbf{Z},$$

which has an odd spectral flow as ϵ goes from 0 to 2π. ■

In higher dimensions, we will only consider the case when V has a minimum away from the origin, so that the instantonlike background fields give Fredholm fermionic operators. Presumably one could put the theory in a finite volume, as we have done in one dimension, and conclude that there is a global anomaly with no restriction on V.

B. Two dimensions

Consider the two-dimensional Euclidean Lagrangian

$$L^{(2)} = \int_{\mathbf{R}^2} \sum_{j=1}^{3} \frac{1}{2}(\partial_\mu\phi_j)^2 + V\left(\sum_{j=1}^{3}\phi_j^2\right) + \frac{1}{2}\Psi^\dagger\left(\sum_{j=1}^{2}\gamma_E^j\partial_j + i\gamma_E^0\phi_1 + i\gamma_E^3\phi_2 + i\gamma_E^5\phi_3\right)\Psi$$

for $\phi_1,\phi_2,\phi_3 \in C^\infty(\mathbf{R}^2)$ and $\Psi \in C^\infty(\mathbf{R}^2)\otimes\mathbf{C}^4$. Here $L^{(2)}$ has an SO(3) symmetry which rotates the ϕ's. If $T(\epsilon)$ is a one-parameter family of skew Fredholm operators of the form

$$T = \sum_{j=1}^{2}\gamma^j{}_E\partial_j + i\gamma_E^0\phi_1 + i\gamma_E^3\phi_2 + i\gamma_E^5\phi_3$$

then we will compute the generalized η invariant of $T(\epsilon)$ by considering iT to be equivalent to the Hamiltonian of the three-dimensional Minkowski Lagrangian

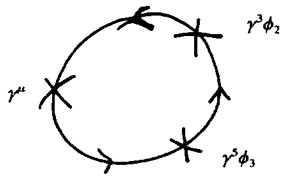

FIG. 3. Three-dimensional spectral flow.

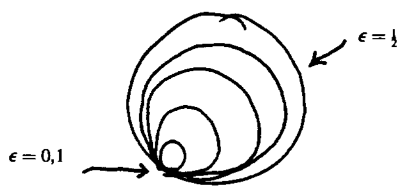

FIG. 4. A one-parameter family of background ϕ's (at ∞) in two dimensions.

$$L^{(3)} = \int_{\mathbf{R}^3}\overline{\Psi}\times\left(-i\gamma_E^0\partial_0 + \sum_{j=1}^{2}\gamma_E^j\partial_j + \phi_1 - i\gamma_E^3\phi_2 - i\gamma_E^5\phi_3\right)\Psi.$$

The Feynman diagram to compute the vacuum charge is that of Fig. 3 and letting n^a denote $\phi^a/|\phi|$, one finds

$$\frac{dQ}{d\epsilon} = \frac{d}{d\epsilon}\int_{\mathbf{R}^2}\frac{1}{8\pi}\epsilon_{abc}n^a dn^b\wedge dn^c.$$

This is simply the infinitesimal change in the volume on S^2 swept out by the curve $\phi/|\phi|: S^1 \to S^2$, where the S^1 is a large circle in $\mathbf{R}^2$, and we have normalized the volume form on S^2 to have mass 1. Consider a one-parameter family of loops on S^2 as in Fig. 4. If each loop represents the behavior of $\phi/|\phi|$ for large radius in $\mathbf{R}^2$, for some ϵ, then as ϵ goes from 0 to 1 it follows that there is an odd change in the vacuum charge, and so an odd spectral flow in $iT(\epsilon)$. Presumably this spectral flow breaks the global SO(3) symmetry.

C. Three dimensions

Consider the three-dimensional Euclidian Lagrangian

$$L^{(3)} = \int_{\mathbf{R}^3}\frac{1}{2}\sum_{j=0}^{3}(\partial_\mu\phi_j)^2 + V\left(\sum_{j=0}^{3}\phi_j^2\right) + \frac{1}{2}\Psi^\dagger\left(\sum_{j=1}^{3}\gamma_E^j\partial_j + i\gamma_E^0\phi_0 + i\gamma_E^5\otimes\vec{\phi}\cdot\vec{\tau}\right)\Psi,$$

with $\phi_0,\ldots,\phi_3 \in C^\infty(\mathbf{R}^3)$ and $\Psi \in C^\infty(\mathbf{R}^3)\otimes\mathbf{C}^8$. There is a naive SO(4) symmetry which rotates the ϕ's, and as before we will compute the spectral flow for the fermionic differential operator by computing the vacuum charge of the four-dimensional Minkowski Langrangian

$$L^{(4)} = \int_{\mathbf{R}^4}\frac{1}{2}\overline{\Psi}\left(-i\gamma_E^0\partial_0 + \sum_{j=1}^{3}\gamma_E^j\partial_j + \phi_0 + i\gamma^5\vec{\phi}\cdot\vec{\tau}\right)\Psi.$$

This calculation was done in Ref. 25 and the relevant Feynman diagram is that of Fig. 5. Letting n^a denote $\phi^a/|\phi|$, the result was

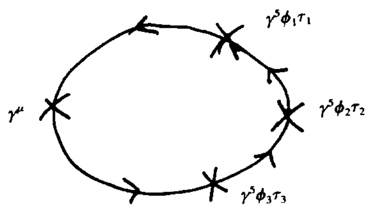

FIG. 5. Three-dimensional spectral flow.

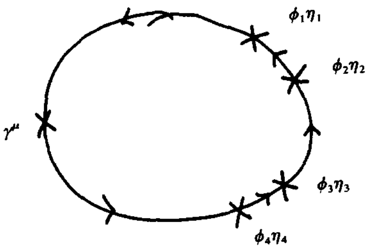

FIG. 6. Four-dimensional spectral flow.

$$\frac{dQ}{d\epsilon} = \frac{d}{d\epsilon}\int_{\mathbf{R}^3} \frac{1}{12\pi^2} \epsilon_{abcd} n^a \, dn^b \wedge dn^c \wedge dn^d.$$

As before, this is the infinitesimal change in the normalized volume on S^3 swept out by a family of immersed S^2's, and by choosing the family to cover S^3, we ensure that there is an odd spectral flow in the fermionic differential operator.

D. Four dimensions

Let $\{\eta^i\}_{i=1}^4$ be another copy of the Dirac matrices $\{\gamma_E^i\}_{i=0}^3$ and put $\eta^5 = \eta^1\eta^2\eta^3\eta^4$. Consider the four-dimensional Euclidean action

$$L^{(4)} = \int_{\mathbf{R}^4} \sum_{j=0}^{4} (\partial_\mu \phi_j)^2 + V\left(\sum_{j=0}^{4} \phi_j^2\right) + \frac{1}{2}\Psi^\dagger\left(\sum_{\mu=1}^{4} \gamma_E^\mu \partial_\mu + i\gamma_E^5(\phi_0\eta^5 + \vec{\phi}\cdot\vec{\eta})\right)\Psi,$$

where $\phi_0,\ldots,\phi_4 \in C^\infty(\mathbf{R}^4)$ and $\Psi \in C^\infty(\mathbf{R}^4) \otimes \mathbf{C}^{16}$. This is an analog of the linear σ model[4] with the target space being $\mathbf{R}^5$ instead of $\mathbf{R}^4$. The naive SO(5) global symmetry rotates the ϕ's. The corresponding five-dimensional Minkowski Lagrangian is

$$L^{(5)} = \int_{\mathbf{R}^5} \frac{1}{2}\bar{\Psi} \times \left(-i\gamma_E^5 \partial_0 + \sum_{\mu=1}^{4} \gamma_E^\mu \partial_\mu + \phi_0 \eta^5 + \vec{\phi}\cdot\vec{\eta}\right)\Psi.$$

The Feynman diagram to compute the vacuum charge is that of Fig. 6, and the result is (letting $n^a = \phi^a/|\phi|$)

$$\frac{dQ}{d\epsilon} = \frac{d}{d\epsilon}\frac{1}{64\pi^2}\int_{\mathbf{R}^4} \epsilon_{abcde} n^a \, dn^b \wedge dn^c \wedge dn^d \wedge dn^e.$$

Once again, this is the infinitesimal volume on S^4 swept out by the ϕ field at ∞, and by a suitable choice of $\phi(\epsilon)$ there will be an odd spectral flow in the fermionic differential operator.

VI. DETERMINANT BUNDLES

Over the space of Fredholm operators $\mathscr{F}$ one has the virtual index bundle Index and its highest exterior power, the line bundle Det. For Dirac-type operators Quillen showed how to define a natural metric on Det.[5] We wish to show how to extend these constructions to the other classes of Fredholm operators.

First, consider the space $\mathscr{F}_1$ of skew-adjoint complex Fredholm operators. The heuristic obstruction to defining a determinant function on $\mathscr{F}_1$ is the possible change of sign in going around a loop, that is, the mod 2 reduction of $\pi_1(\mathscr{F}_1) = \mathbf{Z}$. Abstractly one can form a flat $\mathbf{R}$ bundle over $\mathscr{F}_1$ via the homomorphism ρ: $\pi_1(\mathscr{F}_1) \to \mathrm{End}(\mathbf{R})$ which takes $1 \in \pi_1(\mathscr{F}_1)$ to the operator of multiplication by -1. To be more concrete, let us consider a space $\mathscr{S}$ of skew-adjoint Dirac type operators $\not{D}_s$ on a compact spin manifold, possibly coupled to an external vector bundle. As in Ref. 5, $\mathscr{S}$ can be covered by open sets $\{\cup_\alpha\}_{\alpha\in\mathbf{R}^+}$, so that for $s \in \cup_\alpha$, $i\not{D}_s$ has no eigenvalue of $\pm\alpha$. Then the transition functions (for $\alpha < \beta$) $g_{\alpha\beta}(S) = \Pi_{\alpha<|\lambda_i|<\beta}\lambda_i(i\not{D}_s)$ define an $\mathbf{R}$ bundle DET over $\mathscr{S}$. That is, $v_\alpha \in \mathbf{R}$ in a trivialization over $\cup_\beta$ corresponds to $v_\beta = g_{\alpha\beta} v_\alpha$ in trivialization over $\cup_\beta$, and so there is a well-defined Quillen metric on DET given by

$$\|v_\alpha\|^2 = v_\alpha^2\left(\prod_{|\lambda_i|>\alpha} \lambda_i^2(i\not{D}_s)\right),$$

where the product is understood to be defined using zeta-function regularization. The unique connection on $\Gamma(\mathrm{DET})$ which preserves $\|\cdot\|$ is given in trivialization α by

$$A_\alpha = \lim_{s\to 0} \sum_{|\lambda_i|>\alpha} \lambda_i^{-s-1}\, d\lambda_i$$

and is flat. Thus under parallel transport in patch α, the quantity $(\Pi_{|\lambda_i|>\alpha}|\lambda_i|)v_\alpha$ is constant. One can convince oneself that the holonomy around a loop is the spectral flow (mod 2).

Now consider the space $\mathscr{F}_1 R$ of real skew-adjoint Fredholm operators. We can abstractly define a flat $\mathbf{R}$ bundle via the homomorphism ρ: $\pi_1(\mathscr{F}_1 R) = \mathbf{Z}_2 \to \mathrm{End}(\mathbf{R})$ which takes 1 to -1. For a space $\mathscr{S}$ of real skew-adjoint Dirac-type operators, define the covering $\{\cup_\alpha\}_{\alpha\in\mathbf{R}^+}$, as above. Over $\cup_\alpha$ we have the $\mathbf{R}$ bundle $\Lambda^{\max}(V_\alpha)$, where $V_\alpha = \oplus_i\{\text{eigenspaces of eigenvalue } \lambda_i, |\lambda_i| < \alpha\}$. If $\alpha < \beta$ then over $\cup_\alpha \cap \cup_\beta$, T defines a two-form on $V_\beta - V_\alpha$ (by $\Sigma_{\alpha<\lambda_i<\beta}\lambda_i e_i \wedge \bar{e}_i$, e_i orthonormal) and an isomorphism from $\Lambda^{\max}(V_\alpha)$ to $\Lambda^{\max}(V_\beta)$ via exterior multiplication by $T^{(1/2)(\dim_{\mathbf{R}} V_\beta - \dim_{\mathbf{R}} V_\alpha)}$; then the bundles $\Lambda^{\max}(V_\alpha)$ patch together to give an $\mathbf{R}$ bundle Pfaff over $\mathscr{S}$. There is a metric on Pfaff given by

$$\|\Lambda(e_i)_\alpha\|^2 = |\Lambda(e_i)|_H^2 \prod_{\lambda_i>\alpha} \lambda_i^2(i\not{D}_s),$$

where $|\cdot|_{\mathscr{H}}$ denotes the metric induced from the Hilbert space $\mathscr{H}$ and there is a compatible flat connection. One can see that the holonomy of the connection around a loop is the number of eigenspace rearrangements (mod 2).

Because the elements of $\mathscr{F}_2 R$, can be written as $A + iB$ with A and B skew symmetric, the natural function to consider is the complex Pfaffian. Freed has shown that for Dirac-type operators in $\mathscr{F}_2 R$, the determinant line bundle of Quillen has a natural square root, the complex Pfaffian line bundle, with induced metric and connection.[24]

Finally, the elements of $\mathscr{F}_3 R$ can be considered to be skew-adjoint complex operators which anticommute with a complex antilinear map. Then the even dimensionality of the eigenspaces allows us to canonically take the square root of the transition functions used to define DET for the $\mathscr{F}_1$ case. In this way one obtains a flat line bundle $\mathrm{DET}^{1/2}$ which has holonomy around a loop given by $\frac{1}{2}$ (spectral flow) (mod 2).

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