– Monday, June 20: Overview lecture by John Lott –

There is a separate pdf file containing John's transparencies. It wasn't necessary to take any notes here.

Tuesday, June 21 (John Lott)

Goal this week: Perelman's no local collapsing theorem

Proof involves monotonic quantities for the Ricci flow: (I.1) Entropy (today + tomorrow), (I.7) Reduced volume ($\rightsquigarrow$ proof)

Generalities about Ricci flow

M^n closed manifold, $g(t)$ family of Riemannian metrics parametrized by t (time). Ricci flow equation: $\frac{dg}{dt} = -2\operatorname{Ric}$.

Local coordinates $x^1, ..., x^n$: $g = g_{ij}(x^1, ..., x^n, t)$, $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}(x^1, ..., x^n, t)$ (involves g, $\partial_x g$, $\partial_x \partial_x g$).

Claim: If $g(t)$ is a Ricci flow solution, $\varphi \in \operatorname{Diff}(M)$, then $\varphi^* g(t)$ is a Ricci flow solution.

Proof: $\frac{d}{dt}\varphi^* g(t) = \varphi^* \frac{dg}{dt} = \varphi^*(-2\operatorname{Ric}(g)) = -2\operatorname{Ric}(\varphi^* g)$. $\square$

Formal picture of Ricci flow

$\mathcal{M} := \{$riemannian metrics on $M\}$, infinite-dimensional manifold (formally). If $g \in \mathcal{M}$, then think of $T_g\mathcal{M}$ as $\{v_{ij} : v$ a symmetric covariant 2-tensor$\}$.

Ricci vector field on $\mathcal{M}$: $g \in \mathcal{M} \mapsto -2\operatorname{Ric} \in T_g\mathcal{M}$; Ricci flow = flow of Ricci vector field.

$\operatorname{Diff}(M)$ acts on $\mathcal{M}$, $(\varphi, g) \mapsto \varphi^* g$. Ricci vector field is invariant under $\operatorname{Diff}(M)$; flow of Ricci vector field commutes with $\operatorname{Diff}(M) \rightsquigarrow$ induced flow on $\mathcal{M}/\operatorname{Diff}(M)$ (what we care about; curvature, volume, ... invariant under $\operatorname{Diff}(M)$).

Say $g(t)$ is a Ricci flow solution, $\varphi(t)$ a 1-parameter family of diffeos of M. Set $\hat{g}(t) = \varphi(t)^* g(t) \rightsquigarrow$ still a Ricci flow solution?

$$\begin{aligned}
\frac{d}{dt}\hat{g}(t) &= \frac{d}{dt}(\varphi^*(t)g(t)) = \left(\frac{d}{dt}\varphi(t)^*\right) g(t) + \varphi(t)^* \frac{dg}{dt} \\
&= \left(\frac{d}{dt}\varphi^*(t)\right)(\varphi^*(t))^{-1}\hat{g}(t) - 2\varphi(t)^* \operatorname{Ric}(g) \\
&= \left(\frac{d}{dt}\varphi^*(t)\right)(\varphi^*(t))^{-1}\hat{g}(t) - 2\operatorname{Ric}(\hat{g}(t)) \\
&= \mathcal{L}_{V(t)}\hat{g}(t) - 2\operatorname{Ric}(\hat{g}(t)),
\end{aligned}$$

$V(t)$ = time-dependent vector field that generates $\{\varphi(t)\}$, $(\mathcal{L}_V g)_{ij} = \nabla_i V_j + \nabla_j V_i$. **Modified** Ricci flow equation.

Upshot: modified Ricci flow and honest Ricci flow give same trajectories on $\mathcal{M}/\operatorname{Diff}(M)$, i.e. may add Lie derivatives.

\$64 question: Is Ricci flow the flow of a gradient field on $\mathcal{M}$ ($\mathcal{M}/\operatorname{Diff}(M)$)?

Say X smooth finite-dimensional Riemannian manifold, $f \in C^\infty(X)$, gradient flow: $\frac{dx}{dt} = \nabla f(x)$ (algorithm for climbing Mount Everest if f height function). Along flow line:

$$\frac{df}{dt} = \left\langle \nabla f, \frac{dx}{dt} \right\rangle = |\nabla f|^2(x) \geq 0$$

$\rightsquigarrow$ monotonic quantity.

Answer: not in an obvious way

Forget about Ricci flow!

Entropy function:

$$\mathcal{F}(g, f) = \int_M (|\nabla f|^2 + R) e^{-f} d\text{vol}$$

(g a metric, f a function, $d$vol = Riemannian volume form of $g = \sqrt{\det g_{ij}}\ dx^1 \wedge \ldots \wedge dx^n$). $\mathcal{F} : \mathcal{M} \times C^\infty(M) \to \mathbb{R}$.

Compute $d\mathcal{F}$ on $\mathcal{M} \times C^\infty(M)$, i.e. $\delta\mathcal{F}(v_{ij}, h)$, v_{ij} covariant symmetric 2-tensor, h function. Idea: $(v_{ij}, h) \in T_g\mathcal{M} \times T_f C^\infty(M)$, $v_{ij} = \delta g_{ij}$, $h = \delta f$.

Computation: see Ben Chow. $\delta\mathcal{F}(v_{ij}, h) =$

$$= \int_M \Big(-v_{ij}(R_{ij} + \nabla_i \nabla_j f) + \Big(\frac{v}{2} - h\Big)\left(R - |\nabla f|^2 + 2\Delta f\right)\Big) e^{-f}\ d\text{vol},$$

$v =$ trace of v_{ij}.

Recall:

$$\begin{aligned}
\delta(e^{-f}\ d\text{vol}) &= e^{-f}(\delta f)\ d\text{vol} + e^{-f}\delta(d\text{vol}) \\
&= -he^{-f}\ d\text{vol} + e^{-f}\frac{1}{2}\operatorname{tr}(g^{-1}\delta g)\ d\text{vol} \\
&= \Big(\frac{v}{2} - h\Big) e^{-f}\ d\text{vol}.
\end{aligned}$$

So let's require in our variations: $\frac{v}{2} - h = 0$, i.e. $e^{-f}\ d\text{vol}$ constant.

More coherently: fix a measure dm (nothing weird, just a smooth function times $d$vol), relate f and g by $e^{-f}\ d\text{vol} = dm$ to preserve constraint, $\frac{v}{2} - h = 0$.

Geometrically: had $\mathcal{F}$ on $\mathcal{M} \times C^\infty(M)$, computed $d\mathcal{F}$ on $\mathcal{M} \times C^\infty(M)$. Have section $s : \mathcal{M} \to \mathcal{M} \times C^\infty(M)$, $g \mapsto (g, -\ln(dm/d\text{vol}(g)))$. Have $\mathcal{F}^m = s^*\mathcal{F}$, a function on $\mathcal{M}$, $\mathcal{F}^m(g) = \mathcal{F}(g, -\ln(dm/d\text{vol}(g)))$.

$$(d\mathcal{F}^m)(v_{ij}) = -\int_M v_{ij} \bullet (R_{ij} + \nabla_i \nabla_j f) e^{-f}\ d\text{vol},$$

$\bullet$ = scalar product of tensors w.r.t. g.

dual vector field on $\mathcal{M}$?

Say $\langle v_{ij}, w_{ij}\rangle_g = \int_M v_{ij} \bullet w_{ij}\ dm$, Riemannian metric on $\mathcal{M}$ (formally). Then (up to a factor of 2) the gradient flow of $\mathcal{F}^m$ on $\mathcal{M}$ is

$$\frac{\partial g_{ij}}{\partial t} = -2(R_{ij} + \nabla_i \nabla_j f).$$

Get an equation for $\frac{\partial f}{\partial t}$:

$$\begin{aligned} 0 = \frac{d}{dt}(dm) &= \frac{d}{dt}(e^{-f}\ \mathrm{dvol}) = -e^{-f}\ \frac{df}{dt}\ \mathrm{dvol} + e^{-f}\ \frac{d}{dt}\ \mathrm{dvol} \\ &= -e^{-f}\ \frac{df}{dt}\ \mathrm{dvol} + e^{-f}\ \frac{1}{2}\ \mathrm{tr}\left(g^{-1}\ \frac{dg}{dt}\right)\ \mathrm{dvol} \\ \implies \frac{df}{dt} &= \frac{1}{2}\ \mathrm{tr}\left(g^{-1}\ \frac{dg}{dt}\right) = -(R + \Delta f). \end{aligned}$$

Finally:

$$\begin{aligned} \frac{d\mathcal{F}}{dt}(g(t), f(t)) &= -\int_M (-2)(R_{ij} + \nabla_i\nabla_j f) \bullet (R_{ij} + \nabla_i\nabla_j f)\ e^{-f}\ \mathrm{dvol} \\ &= 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f}\ \mathrm{dvol} \geq 0. \end{aligned}$$

Act by a 1-parameter family of diffeos to get honest Ricci flow:

$$\frac{dg_{ij}}{dt} = -2(R_{ij} + \nabla_i\nabla_j f) + \nabla_i V_j + \nabla_j V_i, \quad \frac{df}{dt} = -\Delta f - R + \mathcal{L}_V f$$

$\rightsquigarrow$ take $V_i = \nabla_i f$, $(\mathcal{L}_V g)_{ij} = 2\nabla_i\nabla_j f$, $\mathcal{L}_V f = \langle V, \nabla f\rangle = |\nabla f|^2$.

Get: $\frac{dg_{ij}}{dt} = -2R_{ij}$, $\frac{df}{dt} = -\Delta f + |\nabla f|^2 - R$.

What about the evolution of $\mathcal{F}(g(t), f(t))$? Still have

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |R_{ij} + \nabla_i\nabla_j f|^2 e^{-f}\ \mathrm{dvol} \geq 0,$$

because acting on g, f with a simultaneous diffeomorphism does not change $\mathcal{F}$.

But: no longer have e^{-f} dvol constant in t; at least we do have $\int_M e^{-f}\ \mathrm{dvol} = 1$ (total mass is not changed by diffeo).

Upshot: if we have a solution to the Ricci flow equation $\frac{dg}{dt} = -2\,\mathrm{Ric}$ and we have $(*)$ $\frac{df}{dt} = -\Delta f + |\nabla f|^2 - R$, then $\mathcal{F}(g(t), f(t))$ is nondecreasing in t.

Understand $(*)$ more clearly:

$$\begin{aligned} &\frac{d}{dt}(e^{-f}) = e^{-f}\left(-\frac{df}{dt}\right) = -e^{-f}(-\Delta f + |\nabla f|^2 - R), \\ &\Delta(e^{-f}) = \nabla_i\nabla_i e^{-f} = \nabla_i(-e^{-f}\nabla_i f) = e^{-f}|\nabla f|^2 - e^{-f}\Delta f, \\ \implies &\frac{d}{dt}(e^{-f}) = -\Delta(e^{-f}) + Re^{-f}, \end{aligned}$$

like a backward heat equation; possibly not solvable forward in time given initial condition.

Whole trick: will solve it backward in time!

Logic: Say we have a solution of $\frac{dg}{dt} = -2\,\mathrm{Ric}$ on $[t_1, t_2]$. Specify $f(t_2)$; solve equation $(*)$ backwards in time to get f on $[t_1, t_2]$. Conclusion: $\mathcal{F}(g(t), f(t))$ is nondecreasing.

Wednesday, June 22 (John Lott)

Comment: $\frac{\partial \tilde{f}}{\partial t} = -\Delta \tilde{f} + R\tilde{f}$, why do we call that a backward heat equation? Consider $\frac{\partial f}{\partial t} = \Delta f$.

$$\frac{d}{dt}\int_M \tilde{f}f \, d\mathrm{vol} = \int_M ((\Delta f)\cdot \tilde{f} + f\cdot(-\Delta\tilde{f} + R\tilde{f}) - Rf\tilde{f}) \, d\mathrm{vol} = 0$$

$\rightsquigarrow$ adjoint heat equation (in the time-dependent sense).

From last time: If we have a Ricci flow solution, $\frac{dg}{dt} = -2\,\mathrm{Ric}$, and we can solve $\frac{\partial}{\partial t}(e^{-f}) = (-\Delta + R)(e^{-f})$, then $\mathcal{F}(g(t), f(t))$ is nondecreasing in t.

Back to Riemannian geometry: $\mathcal{F}(g,f) = \int_M (|\nabla f|^2 + R)e^{-f} \, d\mathrm{vol}$,

$$\lambda(g) := \int_{f:\, \int_M e^{-f}\, d\mathrm{vol}=1} \mathcal{F}(g,f).$$

What is that? Put $h = e^{-f/2}$ (change of variable), $\nabla h = -\frac{1}{2}e^{-f/2}\nabla f$, $|\nabla h|^2 = \frac{1}{4}e^{-f}|\nabla f|^2$, $\mathcal{F} = \int_M (4|\nabla h|^2 + Rh^2) \, d\mathrm{vol}$, and hence

$$\lambda(g) = \inf_{h>0,\, \int h^2\, d\mathrm{vol}=1} \int_M (4|\nabla h|^2 + Rh^2) \, d\mathrm{vol} = \inf_{h>0,\, \int h^2\, d\mathrm{vol}=1} \int_M h(-4\Delta + R)h \, d\mathrm{vol}.$$

Recall: M self-adjoint real $n\times n$ matrix, eigenvalues $\lambda_1, ..., \lambda_n$,

$$\lambda_1 = \inf_{v\in\mathbb{R}^n:\, \|v\|^2=1} \langle v, Mv\rangle.$$

Similarly: smallest eigenvalue λ_1 of $-4\Delta + R$ is $\inf_{\int h^2=1}\int_M h(-4\Delta+R)h \, d\mathrm{vol}$.

Fact from Schrödinger operators: lowest eigenvalue is simple; corresponding eigenfunction h can be chosen uniquely to be normalized and positive.

Upshot: $\lambda(g)$ is the smallest eigenvalue of $-4\Delta + R$.

Theorem: If we have a Ricci flow solution on some time interval $[t_1, t_2]$, then $\lambda(g(t_1)) \le \lambda(g(t_2))$.

Proof: Look at time t_2. $\lambda(g(t_2))$ is the smallest eigenvalue of $-4\Delta + R$ at time t_2. Say: $h(t_2)$ corresponding normalized positive eigenfunction. $\lambda(g(t_2)) = \mathcal{F}(g(t_2), f(t_2))$ where $e^{-f(t_2)/2} = h(t_2)$.

Solve $\frac{\partial}{\partial t}(e^{-f} = -\Delta(e^{-f}) + Re^{-f}$ on $[t_1, t_2]$ backwards with given value of $f(t_2)$ at t_2. Solution exists for all times, positive.

Last time $\Rightarrow$ $(*)$ $\mathcal{F}(g(t_1), f(t_1)) \le \mathcal{F}(g(t_2), f(t_2)) = \lambda(g(t_2))$

By definition, $\lambda(g(t_1)) = \inf_{\int e^{-f}\, d\mathrm{vol}=1} \mathcal{F}(g(t_1), f)$.

Last time $\Rightarrow$ $\int_M e^{-f} \, d\mathrm{vol}$ is constant in t $\Rightarrow$ $f(t_1)$ is appropriately normalized, and $\lambda(g(t_1)) \le (*)$. $\square$

Application: A Ricci flow solution $g(t)$ on $[t_1, t_2]$ is a breather if $\exists \varphi \in \mathrm{Diff}(M)$ such that $g(t_2) = \varphi^* g(t_1)$.

Proposition: Any breather solution satisfies $R_{ij} + \nabla_i \nabla_j f = 0$ for some function $f(t)$.

Proof: Have solution $g(t)$. Look at $\lambda(g(t))$: $\lambda(g(t_2)) = \lambda(\varphi^* g(t_1)) = \lambda(g(t_1))$ $\Rightarrow \lambda(g(t))$ constant in t. Equality case of last proof: $\mathcal{F}(g(t), f(t))$ is constant in t. Recall: $\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |R_{ij} + \nabla_i \nabla_j f|^2 \, d\mathrm{vol}$. $\square$

Continue: $R_{ij} + \nabla_i \nabla_j f = 0$. Tracing: $R + \Delta f = 0$.

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2R_{ij} = 2\nabla_i \nabla_j f = \mathcal{L}_{\nabla f} g, \\ \frac{\partial f}{\partial t} &= -\Delta f + |\nabla f|^2 - R = |\nabla f|^2 = \mathcal{L}_{\nabla f} f, \\ \Longrightarrow g(t) &= \varphi_t^* g(0), \ f(t) = \varphi_t^* f(0), \end{aligned}$$

where φ_t is the flow of ∇f.

Definition: The steady gradient soliton equation is $R_{ij} + \nabla_i \nabla_j f = 0$ (Riemannian geometry).

Say φ_t is the flow of ∇f, $\varphi_{t_1} = \mathrm{id}$. $g(t) := \varphi_t^* g(t_1)$, $f(t) := \varphi_t^* f(t_1)$. Get a Ricci flow solution with $(R_{ij} + \nabla_i \nabla_j f)(t) = 0$.

Examples: 1) if we have a Ricci flat metric and we take f to be a constant
2) cigar soliton (2d example)
3) Bryant soliton (3d example)

Claim: On a closed manifold, any steady gradient soliton is constant in time, i.e. $\mathrm{Ric} \equiv 0$, $f \equiv \mathit{const}$.

Proof: From proof of monotonicity of λ, $e^{-f/2}$ is the normalized eigenfunction for the lowest eigenvalue of $-4\Delta + R$, i.e. $(-4\Delta + R)e^{-f/2} = \lambda e^{-f/2}$.

$$\begin{aligned} \Delta(e^{-f/2}) = \nabla_i \nabla_i (e^{-f/2}) = -\frac{1}{2}\nabla_i(e^{-f}\nabla_i f) = e^{-f/2}\left(\frac{1}{4}|\nabla f|^2 - \frac{1}{2}\Delta f\right), \\ (-4\Delta + R)(e^{-f/2}) = e^{-f/2}(-|\nabla f|^2 + 2\Delta f) + Re^{-f/2} = \lambda e^{-f/2}, \\ -|\nabla f|^2 + 2\Delta f + R = \lambda. \end{aligned}$$

Soliton equation: $R + \Delta f = 0$. Together: $-|\nabla f|^2 + \Delta f = \lambda$. Hence $\Delta(e^{-f}) = -\lambda e^{-f} \Rightarrow \lambda = 0 \Rightarrow \Delta(e^{-f}) = 0 \Rightarrow 0 = \int e^{-f}\Delta(e^{-f}) \, d\mathrm{vol} = \int |\nabla(e^{-f})|^2 \, d\mathrm{vol}$. $\square$

Ricci flow as a gradient flow

We showed that a modified Ricci flow is the gradient flow of $\mathcal{F}^m$ on $\mathcal{M}$. dm is a fixed smooth measure on M.

Claim: The Ricci flow on $\mathcal{M}/\operatorname{Diff}(M)$ is (up to a factor of 2) the gradient flow of $\lambda(g)$.

Precisions: 1) $\lambda(g) = \lambda(\varphi^*(g))$ for any $\varphi \in \operatorname{Diff}(M)$, so λ passes to a functional on the quotient space.

2) Riemannian metric on $\mathcal{M}/\operatorname{Diff}(M)$ is the quotient metric arising from following metric on $\mathcal{M}$:

$$\langle v_{ij}, w_{ij}\rangle_g = \int_M g^{ik} g^{jl} v_{ij} v_{kl}\, h^2\, d\mathrm{vol},$$

$h =$ normalized eigenfunction of $-4\Delta + R$ corresponding to $\lambda(g)$.

3) $\mathcal{M}/\operatorname{Diff}(M)$ is a stratified infinite-dimensional Riemannian manifold ($\exists$ nontrivial symmetries/isotropy groups).

Next time: (I.7)

Thursday, June 23 (John Lott)

(I.7) Reduced volume

Recall from Riemannian geometry: Fixed Riemannian metric on a manifold M. Fix $p \in M$, consider smooth curves $\gamma : [0, \bar{t}] \to M$, $\gamma(0) = p$.

Energy:

$$E(\gamma) = \int_0^{\bar{t}} \left| \frac{d\gamma}{dt} \right|^2 \, dt.$$

Say $q \in M$.

$$\min\{E(\gamma) | \gamma : [0, \bar{t}] \to M, \gamma(0) = p, \gamma(\bar{t}) = q\} = \frac{d^2(p,q)}{\bar{t}}.$$

Geodesic equation:

$$\nabla_X X = 0, \quad X = \frac{d\gamma}{dt}.$$

$\exp_{\bar{t}}$ map:

$$T_p M \to M, \quad \exp_{\bar{t}}(v) = \gamma(\bar{t}),$$

$\gamma = \bar{t}$-geodesic from $\gamma(0) = p$ in direction v.

Jacobi fields along γ; get estimates on $\mathrm{vol}(B_r(p))$, $\Delta \, \mathrm{dist}(p, \cdot)^2$. $\square$

Say we have a Ricci flow solution $g(t)$. M can be noncompact, but assume that sectional curvature is bounded on each time slice. Fix t_0, a possible time. Fix $p \in M$ living on time slice t_0. $\tau := t_0 - t$. Ricci flow equation: $\frac{dg}{d\tau} = 2\,\mathrm{Ric}$.

Consider $\gamma : [0, \overline{\tau}] \to M$, $\gamma(0) = p$. Idea: $\gamma(\tau)$ lives in time τ slice (honestly, it's a curve in M, space-time is only a word).

Definition: $\mathcal{L}$-length,

$$\mathcal{L}(\gamma) := \int_0^{\overline{\tau}} \sqrt{\tau} \left(\left| \frac{d\gamma}{d\tau} \right|^2_{g(\tau)} + R(\gamma(\tau)) \right) d\tau.$$

Definition: $L(q, \overline{\tau}) = \inf\{\mathcal{L}(\gamma) : \gamma : [0, \overline{\tau}] \to M \text{ with } \gamma(0) = p, \gamma(\overline{\tau}) = q\}$

Definition: reduced length,

$$l(q, \overline{\tau}) := \frac{L(q, \overline{\tau})}{2\sqrt{\overline{\tau}}}.$$

Definition: reduced volume,

$$\tilde{V}(\overline{\tau}) := \int_M \overline{\tau}^{-\frac{n}{2}} e^{-l(q, \overline{\tau})} d\mathrm{vol}_{\overline{\tau}}(q).$$

Theorem: $\tilde{V}(\overline{\tau})$ is nonincreasing in $\overline{\tau}$ (nondecreasing in the honest time t).

(no Riemannian counterpart; Ricci flow equation accounts for magic cancellations)

Example: $M = \mathbb{R}^n$, Ricci flow solution = Euclidean metric.

$$L(\gamma) = \int_0^{\overline{\tau}} \sqrt{\tau} \left| \frac{d\gamma}{d\tau} \right|^2 \, d\tau.$$

Change of variable: $s = \sqrt{\tau}$, $\frac{d\gamma}{d\tau} = \frac{1}{2\sqrt{\tau}} \frac{d\gamma}{ds}$, $d\tau = 2s \, ds$.

$$\mathcal{L}(\gamma) = \int_0^{\overline{s}^2} s \, \frac{1}{4s^2} \, \left| \frac{d\gamma}{ds} \right|^2 \, 2s \, ds = \frac{1}{2} \int_0^{\overline{s}^2} \left| \frac{d\gamma}{ds} \right|^2 \, ds.$$

Minimize over γ such that $\gamma(0) = p$, $\gamma(\overline{s}) = q$.

$$L(q, \overline{s}) = \frac{1}{2} \frac{d(p,q)^2}{\overline{s}}, \quad L(q, \overline{\tau}) = \frac{1}{2} \frac{d(p,q)^2}{\sqrt{\overline{\tau}}}, \quad l(q, \overline{\tau}) = \frac{d(p,q)^2}{4\overline{\tau}},$$

$$\tilde{V}(\overline{\tau}) = \int_{\mathbb{R}^n} \overline{\tau}^{-\frac{n}{2}} e^{-\frac{|q|^2}{4\overline{\tau}}} \, d^n q = (4\pi)^{-\frac{n}{2}},$$

constant in $\overline{\tau}$. □

Redo comparison geometry in order to prove the theorem.

$\gamma : [\tau_1, \tau_2] \to M$, $\mathcal{L}(\gamma) = \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left(\left| \frac{d\gamma}{d\tau} \right|^2 + R(\gamma(\tau)) \right) \, d\tau$-

Given $\gamma(\tau)$, consider variations, i.e. have a map $\tilde{\gamma} : [-\varepsilon, \varepsilon] \times [\tau_1, \tau_2] \to M$, $\tilde{\gamma}(0, \tau) = \gamma(\tau)$; $\gamma_s(\tau) := \tilde{\gamma}(s, \tau)$. Write $X = \frac{d\gamma}{d\tau}$, $Y = \frac{d\tilde{\gamma}(s,\tau)}{ds}|_{s=0}$ (variation vector field of $\tilde{\gamma}$ along $\gamma(\tau)$).

$\frac{d}{ds}|_{s=0} \mathcal{L}(\gamma_s) = (\delta_Y \mathcal{L})(0)$ (Perelman). $\delta_Y \gamma = X$, $\delta_Y X = \nabla_X Y = \nabla_X Y$ (because $[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$).

$$\delta_Y \mathcal{L} = \int_{\tau_1}^{\tau_2} \sqrt{\tau} (2 \langle X, \nabla_X Y \rangle + \langle \nabla R, Y \rangle \, d\tau.$$

Want to integrate by parts: $\frac{d}{d\tau} \langle X, Y \rangle = \langle \nabla_X X, Y \rangle + \langle X, \nabla_X Y \rangle + 2 \operatorname{Ric}(X, Y)$.

$$\begin{aligned}
\delta_Y \mathcal{L} &= \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left(2 \frac{d}{d\tau} \langle X, Y \rangle - 2 \langle \nabla_X Y, Y \rangle - 4 \operatorname{Ric}(X, Y) + \langle \nabla R, Y \rangle \right) \, d\tau \\
&= 2\sqrt{\tau} \langle X, Y \rangle \Big|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} \frac{1}{\sqrt{\tau}} \langle X, Y \rangle \, d\tau + \int_{\tau_1}^{\tau_2} \sqrt{\tau} (-2 \langle \nabla_X X, Y \rangle - 4 \operatorname{Ric}(X, Y) + \langle \nabla R, Y \rangle) \, d\tau \\
&= 2\sqrt{\tau} \langle X, Y \rangle \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \sqrt{\tau} \left\langle -\frac{1}{\tau} X - 2 \nabla_X X - 4 \operatorname{Ric}(X) + \nabla R, Y \right\rangle \, d\tau.
\end{aligned}$$

If γ is a minimizer among curves with $\gamma(\tau_1) = p$, $\gamma(\tau_2) = q$, then

$$\nabla_X X + \frac{1}{2\tau} X + 2 \operatorname{Ric}(X) - \frac{1}{2} \nabla R = 0$$

($\mathcal{L}$-geodesic equation).

Here: $\tau_1 > 0$. What happens if $\tau_1 = 0$? Put $s = \sqrt{\tau}$.

$$\mathcal{L}(\gamma) = \int_0^{\overline{s}} \left(\frac{1}{2} \left| \frac{d\gamma}{ds} \right|^2 + 2s^2 R(\gamma(s)) \right) \, ds.$$

In terms of s, a minimizer with $\gamma(0) = p$ goes like $\gamma(s) \sim_{s\to 0} p + 2s\vec{v}$, $\vec{v} \in T_pM$. So in terms of τ, $\gamma(\tau) \sim_{\tau\to 0} p + 2\sqrt{\tau}\vec{v}$,

$$\frac{d\gamma}{d\tau} \sim_{\tau\to 0} \frac{1}{\sqrt{\tau}}\vec{v}.$$

Define the initial vector of the $\mathcal{L}$-geodesic $\gamma : [0, \overline{\tau}] \to M$, $\gamma(0) = p$, as

$$\vec{v} = \lim_{\tau\to 0} \sqrt{\tau}\frac{d\gamma}{d\tau} \in T_pM.$$

Definition: $\mathcal{L}\exp_{\overline{\tau}} : T_pM \to M$, $\mathcal{L}\exp_{\overline{\tau}}(v) = \gamma(\overline{\tau})$, where γ is the $\mathcal{L}$-geodesic with $\gamma(0) = p$ and initial vector v.

Time $\overline{\tau}$ $\mathcal{L}$-cut locus: $\Omega_{\overline{\tau}} \subset T_pM$, $\Omega_{\overline{\tau}} = \{v \in T_pM :$ the $\mathcal{L}$-geodesic γ, $\gamma(0) = p$, initial vector v, is the unique minimizing curve among curves $\hat{\gamma} : [0, \overline{\tau}] \to M$, and $d\mathcal{L}\exp_{\overline{\tau}}$ is nonsingular at $v\}$. Time $\overline{\tau}$ $\mathcal{L}$-cut locus is $M - \mathcal{L}\exp_{\overline{\tau}}(\Omega_{\overline{\tau}})$.

Fact: The time $\overline{\tau}$ $\mathcal{L}$-cut locus is a closed measure zero subset of M. $\mathcal{L}\exp_{\overline{\tau}}$, restricted to $\Omega_{\overline{\tau}}$, is a diffeo onto its image. Proof cf. Notes.

Change of variables:

$$\tilde{V}(\overline{\tau}) = \int_M \overline{\tau}^{-\frac{n}{2}} e^{-l(q,\overline{\tau})}\, d\mathrm{vol}(q) = \int_{\Omega_{\overline{\tau}}} \overline{\tau}^{-\frac{n}{2}} e^{-l(\mathcal{L}\exp_{\overline{\tau}}(v),\overline{\tau})} \mathcal{J}(v, \overline{\tau})\, d^n v.$$

$\Omega_{\overline{\tau}}$ becomes smaller $\Rightarrow$ enough to show: $\overline{\tau}^{-\frac{n}{2}} e^{-l(\mathcal{L}\exp_{\overline{\tau}}(v),\overline{\tau})} \mathcal{J}(v, \overline{\tau})$ nonincreasing in $\overline{\tau}$ for each v, i.e. to show: $-\frac{n}{2}\ln\overline{\tau} - l(\mathcal{L}\exp_{\overline{\tau}}(v), \overline{\tau}) + \ln\mathcal{J}(v, \overline{\tau})$ nonincreasing.

Want to compute: $-\frac{n}{2\overline{\tau}} - \frac{d}{d\overline{\tau}}(\ldots) \leq 0$

Friday, June 24 (John Lott)

Dimensional discussion: time $\sim$ length^2 for Ricci flow. $L \sim$ length^1, $l \sim$ length^0, $\tilde{V} \sim$ length^0 ($(4\pi)^{-\frac{n}{2}}$ is the maximal value).

Say $\gamma(\tau) = \mathcal{L}\exp_{\overline{\tau}}(v)$. Need to understand $L(\gamma(\overline{\tau}, \overline{\tau}) = \int_0^{\overline{\tau}} \sqrt{\tau}(|\frac{d\gamma}{d\tau}|^2 + R)\, d\tau$.

$$\frac{dL(\gamma(\overline{\tau}), \overline{\tau})}{d\overline{\tau}} = \sqrt{\overline{\tau}} \left(\left| \frac{d\gamma}{d\overline{\tau}} \right|^2 + R(\gamma(\overline{\tau})) \right) = \,?$$

Case of $\mathbb{R}^n$: $L(\gamma, \tau) = \frac{d^2(p,q)}{2\sqrt{\tau}}$. $\mathcal{L}$-geodesics: $\gamma(\tau) = p + 2\sqrt{\tau}\vec{v}$. $L(\gamma(\tau), \tau) = 2\sqrt{\tau}|\vec{v}|^2$, $l(\gamma(\tau), \tau) = |\vec{v}|^2$. $\square$

$$(*) \qquad \frac{d}{d\tau}(|X|^2 + R(\gamma(\tau), \tau)) = 2\langle X, \nabla_X X\rangle + 2\operatorname{Ric}(X, X) + \langle \nabla R, X\rangle + \frac{\partial R}{\partial \tau}.$$

$\mathcal{L}$-geodesic equation ($\nabla_X X + 2\operatorname{Ric}(X) - \frac{1}{2}\nabla R + \frac{1}{2\tau}X = 0$) $\Rightarrow$

$$\begin{aligned}(*) &= -4\operatorname{Ric}(X,X) + \langle \nabla R, X\rangle - \frac{1}{\tau}|X|^2 + 2\operatorname{Ric}(X,X) + \langle \nabla R, X\rangle + \frac{\partial R}{\partial \tau} \\ &= \underbrace{\left(-2\operatorname{Ric}(X,X) + 2\langle \nabla R, X\rangle + \frac{\partial R}{\partial \tau} + \frac{R}{\tau}\right)}_{\text{Hamilton's trace Harnack expression } -H(X)} - \frac{1}{\tau}(|X|^2 + R)\end{aligned}$$

$$\Longrightarrow \frac{d}{d\tau}(|X|^2 + R) = -H(X) - \frac{1}{\tau}(|X|^2 + R)$$

$$\Longrightarrow \qquad \underbrace{\int_0^{\overline{\tau}} \tau^{\frac{3}{2}} \frac{d}{d\tau}(|X|^2 + R)\, d\tau}_{\overline{\tau}^{\frac{3}{2}}(|X|^2+R)(\tau=\overline{\tau}) - \frac{3}{2}\int_0^{\overline{\tau}} \tau^{\frac{1}{2}}(|X|^2+R)\, d\tau} = -\underbrace{\int_0^{\overline{\tau}} \tau^{\frac{3}{2}} H(X)\, d\tau}_{=:\,K} - \int_0^{\overline{\tau}} \tau^{\frac{1}{2}}(|X|^2 + R)\, d\tau$$

$$\Longrightarrow \overline{\tau}^{\frac{3}{2}}(|X|^2 + R)(\tau = \overline{\tau}) = -K + \frac{1}{2}L,$$

so

$$\frac{dL(\gamma(\tau), \tau)}{d\tau} = \tau^{\frac{1}{2}}(|X|^2 + R) = \frac{1}{\tau}\left(-K + \frac{1}{2}L\right).$$

Finally:

$$\frac{dl}{d\tau} = -\frac{1}{4}\tau^{-\frac{3}{2}}L + \frac{1}{2}\tau^{-\frac{1}{2}} \cdot \frac{1}{\tau} \cdot \left(-K + \frac{1}{2}L\right) = -\frac{1}{2}\tau^{-\frac{3}{2}}K$$

(this is how curvature influences $\frac{dl}{d\tau}$; compare to $\mathbb{R}^n$).

Next term: $\mathcal{J}(v, \tau) = \det(d\mathcal{L}\exp_\tau)_v$, want to see how this varies with τ.

Euclidean case: $\mathcal{L}\exp_\tau(v) = p + 2\sqrt{\tau}v$, $\mathcal{J}(v,\tau) = (2\sqrt{\tau})^n \Rightarrow \frac{d}{d\tau}\ln \mathcal{J}(v,\tau) = \frac{n}{2\tau}$.

General case: $\frac{d}{d\tau}\ln \mathcal{J}(v,\tau) \le \frac{n}{2\tau} - \frac{1}{2}\tau^{-\frac{3}{2}}K$ (proof is skipped; quite similar to Riemannian geometry volume comparison, need $\mathcal{L}$-Jacobi fields).

Put everything together: Care about $-\frac{n}{2}\ln\tau - l(\gamma(\tau),\tau)+\ln\mathcal{J}(v,\tau)$. $\frac{d}{d\tau}(...)\leq -\frac{n}{2\tau}+\frac{1}{2}\tau^{-\frac{3}{2}}K+(\frac{n}{2\tau}-\frac{1}{2}\tau^{-\frac{3}{2}}K)=0$.

Conclusion: $\tilde{V}(\overline{\tau})$ is nonincreasing in $\overline{\tau}$! □

Riemannian geometry: $p\in M$. Using Jacobi fields, you get estimates (in terms of curvature) on $\mathrm{vol}(B_r(p))$, $\Delta d^2(\cdot,p)$.

Say $\overline{L}(q,\overline{\tau})=2\sqrt{\overline{\tau}}L(q,\overline{\tau})$.

Claim: $\Delta\overline{L}+\overline{L}_{\overline{\tau}}\leq 2n$ (proof: Jacobi field methods).

$\Rightarrow \Delta(\overline{L}-2n\overline{\tau})+\frac{\partial}{\partial\overline{\tau}}(\overline{L}-2n\overline{\tau})\leq 0$. Maximum principle $\Rightarrow \min_q(\overline{L}(q,\overline{\tau})-2n\overline{\tau})$ is nonincreasing. Heuristics: $\overline{\tau}\to 0$, euclidean approximation. In $\mathbb{R}^n$: $\overline{L}(q,\overline{\tau})=d^2(p,q)$, $\min_q(\overline{L}(q,\overline{\tau})-2n\overline{\tau})=-2n\overline{\tau}$.

Corollary: For all $\overline{\tau}>0$, $\min_q(\overline{L}(q,\overline{\tau})-2n\overline{\tau})\leq 0$. $\Rightarrow \min_q l(q,\overline{\tau})\leq\frac{n}{2}$.

Corollary: $\forall\overline{\tau}>0\ \exists q\in M\ \exists\mathcal{L}$-geodesic $\gamma:[0,\overline{\tau}]\to M$, $\gamma(0)=p$, $\gamma(\overline{\tau})=q$, such that $\mathcal{L}(\gamma)\leq n\sqrt{\overline{\tau}}$.

Second variation arguments

$\mathcal{L}=\int_0^{\overline{\tau}}\sqrt{\tau}(|X|^2+R)\,d\tau$, $X=\frac{d\gamma}{d\tau}$, Y variation field.

$$\delta_Y\mathcal{L}=\int_0^{\overline{\tau}}\sqrt{\tau}(2\langle X,\nabla_Y X\rangle+\langle\nabla R,Y\rangle)\,d\tau$$

(integration by parts $\rightsquigarrow$ $\mathcal{L}$-geodesic equation).

$$\begin{aligned}\delta_Y^2\mathcal{L}&=\int_0^{\overline{\tau}}\sqrt{\tau}(2\langle\nabla_Y X,\nabla_Y X\rangle+2\langle X,\nabla_Y\nabla_X Y\rangle+(\mathrm{Hess}\,R)(Y,Y))\,d\tau\\ &=\int_0^{\overline{\tau}}\sqrt{\tau}(2|\nabla_X Y|^2+2\langle X,\nabla_X\nabla_Y Y\rangle+2\langle X,R(Y,X)Y\rangle+(\mathrm{Hess}\,R)(Y,Y))\,d\tau.\end{aligned}$$

Define index form: $Q(Y,Y)=\delta_Y^2\mathcal{L}-\delta_{\nabla_Y X}\mathcal{L}$ (Hessian of $\mathcal{L}$ on path space). Compute $\frac{d}{d\tau}\langle X,\nabla_Y Y\rangle$ to replace $\langle X,\nabla_X\nabla_Y Y\rangle$:

$$\begin{aligned}Q(Y,Y)=\int_0^{\overline{\tau}}\sqrt{\tau}(2|\nabla_X Y|^2&+2\langle X,R(Y,X)Y\rangle+(\mathrm{Hess}\,R)(Y,Y)+\\ &+4(\nabla_Y\mathrm{Ric})(Y,X)+2\nabla_X\mathrm{Ric}(Y,Y))\,d\tau.\end{aligned}$$

Minimizers of $Q(Y,Y)$ with fixed endpoints give $\mathcal{L}$-Jacobi fields: Say $\gamma(\tau)$ stays away from the cut locus. Fix $Y(\overline{\tau})$. Say $Y(\tau)$, $Y(0)=0$, is an $\mathcal{L}$-Jacobi field. For any other variation $\tilde{Y}$ with $\tilde{Y}(0)=0$, $\tilde{Y}(\overline{\tau})=\tilde{Y}(\overline{\tau})$, $Q(Y,Y)\leq Q(\tilde{Y},\tilde{Y})$.

Monday, June 27 (John Lott)

No local collapsing theorem

M^n, $g(t)$ Ricci flow defined for $t \in [0,T)$, $T \le \infty$

Definition (parabolic r-ball): Say $(x_0,t_0) \in M \times [0,T)$. Given $r > 0$, put $P(x_0,t_0,r) := \{(x,t) \in M \times [0,T) : \mathrm{dist}_{t_0}(x,x_0) < r, t \in [t_0 - r^2, t_0]\}$ (time t slices are not metric balls w.r.t. $g(t)$!) Denote t_0-ball by $B_{t_0}(x_0,r)$.

Definition: Our Ricci flow solution is κ-noncollapsed at a scale ρ if $\forall r < \rho$ $\forall (x_0,t_0)$: $|\,\mathrm{Rm}\,| \le \frac{1}{r^2}$ on $P(x_0,t_0,r) \Rightarrow \mathrm{vol}(B_{t_0}(x_0,r)) \ge \kappa r^n$ (invariant under parabolic rescaling). It is κ-noncollapsed at all scales if the same κ works for all ρ.

Examples: 1) $\mathbb{R}^n$, euclidean metric ($\forall t$), κ-noncollapsed for κ appropriate.
2) $T^{n-1} \times \mathbb{R}$, flat metric. $\forall \rho \exists \kappa_\rho$: κ_ρ-noncollapsed at scale ρ. However, not κ-noncollapsed at all scales no matter how you choose κ ($\rho \ll 1$: $\mathrm{vol}(B_r(x_0)) \sim r$).
3) shrinking cylinder $S^{n-1} \times \mathbb{R}$, $n \le 3$, on $[0,1)$, is κ-noncollapsed at all scales, for some κ.

Theorem: If M^n is closed and $T < \infty$, then $\forall \rho > 0$ there's some $\kappa(\rho) > 0$ such that it's κ-noncollapsed at scale ρ.

Remark: If solution defined on $[0,T]$, result obvious, because manifold compact. Real content: What's happening at small scales, near a singularity?

Proof: Fix $\rho > 0$. Suppose theorem is not true. Then we have a sequence $\{r_k\}$ in $(0,\rho]$, points $\{x_k\}$ in M, times $\{t_k\}$ in $[0,T)$, such that

- we do have $|\,\mathrm{Rm}\,| \le \frac{1}{r_k^2}$ on $P(x_k,t_k,r_k)$, but
- $\varepsilon_k := r_k^{-1} \mathrm{vol}(B_{t_k}(x_k,r_k))^{\frac{1}{n}} \to 0$.

Must have $t_k \to T$ (away from singularity time, theorem is trivially true). Write $B_k = B_{t_k}(x_k,r_k)$.

Take (p_k,t_k) as a base point to compute reduced volume $\tilde{V}_k(\tau)$, $\tau \equiv t_k - t$. Will show:

1) $\tilde{V}_k(\varepsilon_k r_k^2) \to 0$ $(k \to \infty)$,
2) $\tilde{V}_k(t_k) \ge c > 0$ (real time 0).

ad 1):

$$\begin{aligned}
\tilde{V}(\varepsilon_k r_k^2) &= \int_M (\varepsilon_k r_k^2)^{-\frac{n}{2}} e^{-l(q,\varepsilon_k r_k^2)}\, d\mathrm{vol}_{t_k - \varepsilon_k r_k^2}(q) \\
&= \int_{\Omega_{\varepsilon_k r_k^2}} (\varepsilon_k r_k^2)^{-\frac{n}{2}} e^{-l(\mathcal{L}\exp_{\varepsilon_k r_k^2}(v),\varepsilon_k r_k^2)}\, \mathcal{J}(v,\varepsilon_k r_k^2)\, d\mathrm{vol}(v) \\
&= (\tilde{V}_1 + \tilde{V}_2)(\varepsilon_k r_k^2),
\end{aligned}$$

where $\tilde{V}_1$ is result of integrating over v with $|v| \leq \frac{1}{10}\varepsilon_k^{-\frac{1}{2}}$.

Claim: If $\gamma(\tau) = \mathcal{L}\exp_\tau(v)$, $|v| \leq \frac{1}{10}\varepsilon_k^{-\frac{1}{2}}$, then $\gamma(\tau) \in B_k$ for $\tau \in [0, \varepsilon_k r_k^2]$.

Example: $\mathbb{R}^n$. $\gamma(\tau) = p_k + 2\sqrt{\tau}\vec{v}$,

$$d(\gamma(\tau), p_k) = 2\sqrt{\tau}|\vec{v}| \leq 2(\varepsilon_k r_k^2)^{\frac{1}{2}}(\frac{1}{10}\varepsilon_k^{-\frac{1}{2}}) \leq \frac{1}{5}r_k.$$

Also true in general (have curvature bound, so time is too short for curvature to matter much), so

$$\tilde{V}_1(\varepsilon_k r_k^2) \leq \int_{B_k} (\varepsilon_k r_k^2)^{-\frac{n}{2}} e^{-l(q,\varepsilon_k r_k^2)}\, d\mathrm{vol}(q).$$

Need: lower bound on

$$l(q, \varepsilon_k r_k^2) = \frac{1}{2\sqrt{\varepsilon_k r_k^2}} L(q, \varepsilon_k r_k^2) = \frac{1}{2\sqrt{\varepsilon_k r_k^2}} \int_0^{\varepsilon_k r_k^2} \sqrt{\tau}\left(\underbrace{\left|\frac{d\gamma}{d\tau}\right|^2}_{\geq 0} + \underbrace{R(\gamma(\tau))}_{\text{bounded in } B_k}\right) d\tau$$

$$\geq -\frac{1}{2\sqrt{\varepsilon_k r_k^2}} \int_0^{\varepsilon_k r_k^2} \sqrt{\tau} n(n-1)\frac{1}{r_k^2}\, d\tau = -\frac{1}{3}n(n-1)\varepsilon_k,$$

so

$$\tilde{V}_1(\varepsilon_k r_k^2) \leq (\varepsilon_k r_k^2)^{-\frac{n}{2}} e^{\frac{1}{3}n(n-1)\varepsilon_k} \mathrm{vol}_{t_k - \varepsilon_k r_k^2}(B_k),$$

but $\varepsilon_k = r_k^{-1}(\mathrm{vol}_{t_k}(B_k))^{\frac{1}{n}}$. Since we have curvature bound on $P(x_k, t_k, r_k)$, we get that

$$\mathrm{vol}_{t_k - \varepsilon_k r_k^2}(B_k) \leq e^{const\cdot(\varepsilon_k r_k^2) r_k^{-2}} \mathrm{vol}_{t_k}(B_k)$$

(just volume variation formula under Ricci flow), so

$$\tilde{V}_1(\varepsilon_k r_k^2) \leq (\varepsilon_k r_k^2)^{-\frac{n}{2}} e^{\frac{1}{3}n(n-1)\varepsilon_k} e^{const\cdot\varepsilon_k}(\varepsilon_k r_k)^n \leq 2\varepsilon_k^{\frac{n}{2}}$$

for large k.

$\tilde{V}_2(\varepsilon_k r_k^2)$: From before: $\tau^{-\frac{n}{2}} e^{-l(\mathcal{L}\exp_\tau(v),\tau)} \mathcal{J}(v,\tau)$ is nonincreasing in τ. When $\tau \to 0$: $\tau^{-\frac{n}{2}} e^{-|v|^2} \tau^{\frac{n}{2}} 2^n$ (euclidean approximation), so

$$\tilde{V}_2(\varepsilon_k r_k^2) \leq \int_{|v| \geq \frac{1}{10}\varepsilon_k^{-\frac{1}{2}}} 2^n e^{-|v|^2}\, dv \leq e^{-\frac{1}{1000\varepsilon_k}}$$

(k large).

Step 1 finished. It remains to show that $\tilde{V}_k(t_k) \geq c > 0$ for all k.

From last time: $\exists q_k$: $l(q_k, t_k - \frac{T}{2}) \leq \frac{n}{2}$ (real time $\frac{T}{2}$).

To bound $\tilde{V}_k(t_k)$, need upper bound on $l(q, t_k) = \frac{1}{2t_k} L(q, t_k)$, independent of k. Take $\hat{\gamma}$ a minimizing $\mathcal{L}$-geodesic, $\hat{\gamma} : [0, t_k - \frac{T}{2}] \to M$, with $\hat{\gamma}(0) = p_k$, $\hat{\gamma}(t_k - \frac{T}{2}) = q_k$. $\Rightarrow \mathcal{L}(\hat{\gamma}) \leq n\sqrt{t_k - \frac{T}{2}}$. Concatenate $\hat{\gamma}$ with curves from $(q_k, \frac{T}{2})$ to $(q, 0)$ (second coordinate = real time) $\rightsquigarrow$ compact parameters. $\square$

Remark: Also work if M is noncompact. Assume bounded curvature on each time slice and, at time 0, injectivity radius $\geq i_0 > 0$. Then $\forall \rho$ we have κ-noncollapse on scale ρ where $\kappa = \kappa(\rho, \sup_M |(\mathrm{g}(0))|, \mathrm{i}_0, \mathrm{T})$ can be estimated. Your bound gets worse as $T \to \infty$ (graph manifold parts in geometrization).

Corollary of NLC theorem: Any rescaling limit of a finite-time singularity is κ-noncollapsed on all scales, for some $\kappa > 0$.

Corollary: $\mathbb{R} \times$ cigar cannot arise as a blowup limit from a finite-time singularity (good, because otherwise could not do surgery).

Tuesday, June 28 (Bruce Kleiner)

κ-solutions (section I.11)

A Ricci flow $(N, h(\cdot))$ is a κ-solution if

- Time slices are complete.
- It is **ancient**: it is defined on an interval of the form $(-\infty, t]$ for some $t \in \mathbb{R}$.
- It has nonnegative curvature operator: $\mathrm{Rm} \geq 0$ ($\langle R(X,Y)Z, W\rangle$ is skew-symmetric under $X \leftrightarrow Y$, $Z \leftrightarrow W$, and symmetric under $(X,Y) \leftrightarrow (Z,W)$, so it defines a symmetric bilinear form Rm on $\wedge^2 T_pM$)
- The curvature $|\mathrm{Rm}|$ (or equivalently the scalar curvature R, since $R = \mathrm{tr}\,\mathrm{Rm}$ as a bilinear form on $\wedge^2 T_pM$) is bounded on each time slab.
- $(N, h(\cdot))$ has everywhere positive scalar curvature.
- $(N, h(\cdot))$ is κ-noncollapsed: if the normalized $|\mathrm{Rm}|$ of a parabolic ball $P(x,t,r)$ is ≤ 1, then the normalized volume of the ball $B(x,t,r)$ is at least κ.

  (An effectively equivalent definition of being κ-noncollapsed is: if the normalized curvature of a parabolic ball $P(x,t,r)$ is ≤ 1, then the normalized injectivity radius of $B(x,t,r)$ is at least κ.)

Where are we heading?

Main assertion (cf. I.12.1): Pick $\varepsilon > 0$ and $T < \infty$. Then there are constants $R_0 = R_0(\varepsilon, T)$, and $\kappa = \kappa(\varepsilon, T)$, such that if $(M, g(\cdot))$ is a 3-dimensional Ricci flow with normalized initial condition, and $R(x,t) \geq R_0$, then the pointed flow $(M, g(\cdot), x, t)$, after being parabolically rescaled by $R(x,t)$, is ε-close to a pointed κ-solution.

Remark: Due to a theorem of Hamilton-Ivey, for Ricci flows with normalized initial conditions, a point in space-time has large scalar curvature if and only if the curvature tensor has large norm.

Not at all clear that you can apply compactness; only have $R(x,t) \leq 1$ at a single point. Want to prove it for any sequence of base-points where curvature is blowing up, not only for those where maximum curvature is attained.

Logic: Need to study κ-solutions first.

Theorem I.11.4: If $(M, g(\cdot))$ is an n-dimensional ancient κ-solution defined at t, then $\mathcal{V}(M, g(t)) = 0$. (For any Riemannian manifold X, $\sec \geq 0$,

$$\mathcal{V} := \lim_{r \to \infty} \frac{\mathrm{vol}(B(p,r))}{r^n}$$

exists by Bishop-Gromov: asymptotic volume of X).

Furthermore, if M is noncompact, then the asymptotic scalar curvature ratio

$$\mathcal{R} := \lim_{r\to\infty} \sup\{R(x,t)d_t^2(x,p) | d_t(x,p) \geq r\} = \infty$$

and there is a sequence of points $x_i \in M$ such that $d_t(x_i, p) \to \infty$, and if we rescale the Ricci flows $(M, g(\cdot), (x_i, t))$ by $R(x_i, t)$, then they subconverge in the pointed smooth topology to a pointed κ-solution $(N, h(\cdot), (x_\infty, 0))$ such that N splits isometrically as a product $Y \times \mathbb{R}$ (Y is a κ'-solution). □

Now: discuss consequences, where you can see the tools at work which are used in the proof.

Consequences: A) The only 2D κ-solutions are round shrinking S^2, $\mathbb{R}P^2$.

B) If $(M, g(\cdot))$ is a noncompact 3D κ-solution, then rescaling $(M, g(\cdot), x_i, t)$ by scalar curvature we get a sequence of Ricci flows which subconverges to round cylindrical flow.

A) $\Rightarrow$ B): Apply (11.4) to the κ-solution; easy to check: Y is a κ'-solution. A) tells you that Y is a round shrinking sphere.

Sketch of proof of A), assuming (11.4):

Step 0: Any 2D κ-solution is compact (else apply (11.4), but product of 1-manifolds is flat). □ (Step 0)

Step 1: $\exists C = C(\kappa)$ such that if $(M, g(\cdot))$ is a κ-solution defined at time t, $R_{\max} :=$ maximum scalar curvature of $(M, g(t))$, then $R_{\max} \cdot \operatorname{diam}^2(M, g(t)) \leq C$.

Proof: If not, $\exists$ sequence $(M_i, g_i(x_i, t_i))$ of 2D κ-solutions such that $R_{\max}(t_i) = R_{g_i}(x_i)$, $R_{\max}(t_i) \cdot \operatorname{diam}^2(M_i, g_i(t_i)) \to \infty$, and, by rescaling, $R_{\max}(t_i) = 1$.

New tool: Hamilton's Harnack inequality

Suppose $(M, g(\cdot))$ is an ancient Ricci flow with $\mathrm{Rm} \geq 0$, and bounded curvature on compact time intervals. If $t_1 < t_2$, $x_1, x_2 \in M$, then

$$R(x_2, t_2) \geq \exp\left(-\frac{d_{t_1}^2(x_1, x_2)}{2(t_2 - t_1)}\right) R(x_1, t_1).$$

Applications:

- The functions $t \mapsto R(x,t)$ is a nondecreasing function for every $x \in M$.
- If $R(x,t) = 0$ somewhere, then $\mathrm{Rm} \equiv 0$.
- To apply the compactness theorem to a sequence of κ-solutions $(M_i, g_i(\cdot), (x_i, t_i))$, it suffices to show that for every $D < \infty$, there is a $C < \infty$ such that $R < C$ on $B(x_i, t_i, D)$.

  Note: A priori the limit will be an ancient Ricci flow which is noncollapsed at all scales and satisfies $\mathrm{Rm} \geq 0$, but one might not have bounded curvature on time slabs.

□ (Harnack)

Next thing we need to show is lower bound on volume (if B is a Riemannian unit ball, then a lower bound on $\mathrm{vol}(B)$ is equivalent to a lower bound on $\mathrm{injrad}(B, p)$). Apply HCG compactness to get a limit $(M_\infty, g_\infty(\cdot), (x_\infty, 0))$. This is a κ-solution. Noncompact, because $\mathrm{diam}(M_i, g_i(t_i)) \to \infty$. Contradiction to Step 0. $\square$ (Step 1)

Step 2: $\mathcal{W} := \{(M, g(\cdot), (x,t))\ \kappa\text{-solution} \mid R(x,t) = 1 = R_{\max}(t)\}$. $\mathcal{W}$ is compact in the pointed smooth topology (this is what Step 1 really tells us). Now let $\mathcal{F}$ be a scale invariant continuous functional on Riemannian metrics which is nondecreasing for Ricci flow, and constant only at shrinking solitons. $\mathcal{F}$ is bounded on time slices of κ-solutions $\rightsquigarrow$ limiting values at $-\infty$ and the final time ω. I.e. in the limit of blowing up towards $-\infty$ or ω, we get a shrinking soliton. $\Rightarrow S^2$, $\mathbb{R}P^2 \Rightarrow \mathcal{F}$ has the same limiting values as $t \to -\infty$, $t \to \omega$. Monotonicity of $\mathcal{F} \Rightarrow$ constant $\Rightarrow$ whole thing is a shrinking soliton. $\square$

Wednesday, June 29 (Bruce Kleiner)

Review of nonnegative curvature

Similarity with theory of convex sets $0 \in C \subset \mathbb{R}^n$ (via ∂C). Two phenomena:

- cone at infinity (rays in C starting at 0; same as scaling down)
- splitting: If C contains a complete line, then $C = C' \times \mathbb{R} \subset \mathbb{R}^{n-1} \times \mathbb{R}$.

In particular, given any C, $x_i \in C$, $x_i \to \infty$, then $\lambda_i \cdot (C - x_i) \to C_\infty$ splitting off a line if $d(0, x_i) \cdot \lambda_i \to \infty$.

Suppose now X is a complete Riemannian manifold of nonnegative sectional curvature. Then the following hold for X:

- Toponogov's triangle comparison theorem.
- $\Rightarrow$ Monotonicity of comparison angles (won't use it).
- If X contains a line (i.e. a complete geodesic γ which is minimizing between any two of its points; example: $S^2 \times \mathbb{R}$; non-examples: $\{y = 0\}$ in paraboloid $\{z = x^2 + y^2\}$, hyperboloid $\{z^2 = x^2 + y^2 + 1\}$), then X splits isometrically as a product of another manifold with $\mathbb{R}$.

Definition: An Alexandrov space of nonnegative curvature is a complete geodesic space which satisfies the conclusion of the Toponogov triangle comparison theorem.

Gromov-Hausdorff limits

Definition: Pick $\varepsilon > 0$. An ε-isometry (or Hausdorff approximation) is a (not necessarily continuous) map $f : X \to Y$ between metric spaces, such that

1. For every pair of points $x, x' \in X$, $|d(x, x') - d(f(x), f(x'))| < \varepsilon$.
2. For every $y \in Y$, the distance from y to $f(X)$ is at most ε.

Example: The inclusion of the integer lattice $\mathbb{Z}^2$ into $\mathbb{R}^2$ is an ε-isometry where $\varepsilon = \frac{\sqrt{2}}{2}$.

Definition: Let X_i, Y be metric spaces. Then X_i converges to Y in the Gromov-Hausdorff topology if for each i there is an ε_i-isometry $f_i : X_i \to Y$ where $\varepsilon_i \to 0$.

Example: If Z is a metric space, and $V_i \subset Z$ is a $\frac{1}{i}$-net in Z, then V_i converges to Z in the Gromov-Hausdorff topology (no good example, because classical notion of Hausdorff convergence already applies).

In noncompact case, appropriate to allow for convergence on compact subsets.

Definition: A sequence of pointed metric spaces (X_i, x_i) converges in the Gromov-Hausdorff topology to a pointed metric space (Y, y) if there is a sequence of ε_i-isometries $f_i : B(x_i, \varepsilon_i^{-1}) \to B(y, \varepsilon_i^{-1})$ for some sequence $\varepsilon_i \to 0$.

Example: Let X_i be the standard n-sphere of diameter i, equipped with the Riemannian distance, and pick $x_i \in X_i$. Then (X_i, x_i) converges to $(\mathbb{R}^n, 0)$ in the pointed Gromov-Hausdorff topology, but there is no limit in the usual Gromov-Hausdorff topology.

Analogy between nonnegatively curved manifolds and convex sets ($C \subset \mathbb{R}^n$ convex, ∂C smooth $\Rightarrow \mathrm{Rm}_{\partial C} \geq 0$):

Definition: A pointed metric space (Z, z) is a Euclidean cone if

- Z is a union of geodesic rays starting at z.
- Given any two geodesic rays γ_1, γ_2 starting at z, the union $\gamma_1 \cup \gamma_2$ isometrically embeds into the plane $\mathbb{R}^2$.

(only Riemannian manifold which is a Euclidean cone is flat $\mathbb{R}^n$)

Theorem: Suppose X is a Riemannian manifold of nonnegative sectional curvature.

1. If $p \in X$, and $\lambda_i \to 0$, then the sequence $(\lambda_i X, p)$ converges in the pointed Gromov-Hausdorff topology to a Euclidean cone (X_∞, p_∞). The limit is independent of the basepoint p, and is a locally compact Alexandrov space of nonnegative curvature.
2. If $p \in X$, $x_i \in X$, $\lambda_i > 0$, and $d(x_i, p) \to \infty$, $\lambda_i \cdot d(x_i, p) \to \infty$, then $(\lambda_i X, x_i)$ subconverges to a pointed Alexandrov space (Z, z) which splits off an $\mathbb{R}$-factor.

Examples: (solid) hyperboloid $\rightsquigarrow$ cone over a circle (disk). In general, the asymptotic cone is much less smooth: (solid) parabolid $\rightsquigarrow$ real half-line.

Indication for 1) $\Rightarrow$ 2): Pick y_i on the ray from p to x_i such that $d(p, x_i) = d(x_i, y_i)$. The segment $[p, y_i]$ will give a complete line in the limit. Apply Toponogov splitting. $\square$

Now turn to proof of (11.4).

$(M^n, g(\cdot))$ n-dimensional κ-solution, $(M^n, g(t))$ time-t slice, noncompact

Rescale the slice, get a limit. Maximum principle for curvature operator (Shi) $\Rightarrow$ splitting in the limit.

What's wrong? Limit is Alexandrov space! $\rightsquigarrow$ must rescale in such a way that points are disappearing quickly enough, but with curvature control.

Plan: $x_i \in M$, $x_i \to \infty$, $(M, g(\cdot), (x_i mt))$. Scale by $R(x_i, t)$, get $(M, h_i(\cdot), (x_i, 0))$ $(*)$, apply Hamilton's compactness. Need: $d_{h_i}(x_i, p) \to \infty \Leftrightarrow \liminf_{i \to \infty} R_{g_t}(x_i, t) \cdot d_t^2(x_i, p) = \infty \Leftrightarrow \mathcal{R} = \infty$. Call this Case I.

In $(*)$, have curvature control only pointwise $\rightsquigarrow$ need point selection: $x_i \in M$, $d_t(x_i,p) \to \infty$, $R_t(x_i,t)d_t^2(x_i,p) \to \infty$. Get a new sequence $\{y_i\}$ of points in M and $D_i \to \infty$ such that

1) $y_i \to \infty$
2) $R(y_i,t)d_t^2(y_i,p) \to \infty$
3) $R \le 2R(y_i,t)$ on the ball $(**)$ $B(y_i,t,D_i(R(y_i,t))^{-1/2})$.

Argument: First for some fixed D. Either x_i works already, or you find a bad point x_i' with too large scalar curvature; take ball of type $(**)$ around x_i' $\rightsquigarrow$ radius scaled down by $\frac{1}{\sqrt{2}}$! Either this works, or ... (have scalar curvature blowing up if process does not terminate).

Non-collapsing theorem $\Rightarrow$ injrad bound. Can now apply HCG compactness: $(M,h_i(\cdot),(y_i,0))$ subconverge in the pointed C^∞ topology to $(M_\infty,h_\infty(\cdot),(y_\infty,0))$, scalar curvature $R \le 2$, $|\operatorname{Rm}| \le C$, ancient. Also know $R(y_\infty,0)=1$. Noncollapsing passes to limits, $R>0$ somewhere, hence everywhere (Harnack).

$\Rightarrow$ in the limit, get κ-solution!

Considering time slices at 0: GH-subconverge to an Alexandrov space splitting off a factor.

Together: $(M_\infty,h_\infty(0))$ is Alexandrov of curvature ≥ 0, splitting as a metric space. Now maximum principle $\Rightarrow$ all earlier time slices split $\mathbb{R}$ factor. $\square$ (Case 1)

Thursday, June 30 (Bruce Kleiner)

Where are we? $(M, g(\cdot), (x,t))$ κ-solution; showed that if M is noncompact and $\mathcal{R} := \lim_{r\to\infty} \sup\{R(x,t)d^2(x,p) | d(x,p) \geq r\} = \infty$, then $\exists$ sequence $x_i \to \infty$ such that if we rescale by scalar curvature at (x_i, t) and take (x_i, t) as basepoints, there is a sublimit in the pointed smooth topology which splits off an $\mathbb{R}$ factor.

Now: case 2 ($0 < \mathcal{R} < \infty$), case 3 ($\mathcal{R} = 0$); want: contradiction!

Case 2. If $d_t(x_i, p) \to \infty$, then $R(x_i, t) \cdot d^2(x_i, p) \leq 2\mathcal{R}$ ($\exists l < \infty$ such that if $d(x,p) \geq l$, then $R(x,t) \cdot d^2(x,p) \leq 2\mathcal{R}$). Pick $x_i \in M$, $x_i \to \infty$, such that $R(x_i, t) \cdot d^2(x_i, p) = R(x_i, t) \cdot r_i^2 \geq \frac{\mathcal{R}}{2}$. Pick $0 < a < 1 < b$.

Claim: $\exists C = C(a)$ such that for large i: $R \leq C \cdot R(x_i, t)$ on $\mathrm{Ann}(p, ar_i, br_i)$.

Proof: $y \in \mathrm{Ann}(p, ar_i, br_i)$. i large $\Rightarrow l < ar_i \leq d(y,p) \leq br_i$.

$$R(y,p) \cdot d^2(y,p) \leq 2\mathcal{R} \implies R(y,p) \leq \frac{2\mathcal{R}}{d^2(y,p)} \leq \frac{2\mathcal{R}}{a^2 \cdot d^2(x_i,p)} \leq \frac{4}{a^2} R(x_i,t).$$

$\square$ (claim)

Rescale $(M, g(\cdot), (x_i, t))$ by $R(x_i, t) \to 0$; get $(M, h_i(\cdot), (x_i, 0))$.

$$\sqrt{\mathcal{R}/2} < r_i R(x_i, t_i)^{1/2} < \sqrt{2\mathcal{R}},$$

i.e. distance between x_i and p in $(M, h_i(0))$ remains bounded, i.e. limits pointed at x_i resp. p do not differ $\Rightarrow$ $(M, h_i(0), (x_i, 0))$ GH-subconverges to (X_∞, x_∞), euclidean cone ($x_\infty \neq$ vertex).

Look at $B(x_i, 0, \alpha) \subset (M, h_i(0))$: $(B(x_i, 0, \alpha), h_i(\cdot), (x_i, 0))$ subconverge in C^∞ to $(N, h_\infty(\cdot), (x_\infty, 0))$ (incomplete; reason: above claim).

$\Rightarrow$ $(N, h_\infty(0), (x_\infty, 0))$ locally is a Riemannian Euclidean cone, $R(x_\infty, 0) = 1$. Also: final time slice of a Ricci flow. $\mathrm{Rm} \geq 0$,

$$\frac{\partial \,\mathrm{Rm}}{\partial t} = \Delta \,\mathrm{Rm} + Q(\mathrm{Rm}).$$

Pick an $h_\infty(0)$-ONB for $T_{x_\infty}N$ such that e_1 = outward direction, e_2, e_3 span a 2-plane with strictly positive curvature (OK because any 2-plane spanned by e_1 and vector tangent to distance sphere has 0 curvature, and $R(x_\infty) = 1$). Want to show: $\langle \partial_t \,\mathrm{Rm}(e_1, e_2)e_2, e_1 \rangle > 0$ ($\Rightarrow$ contradiction to $\mathrm{Rm} \geq 0$, since e_1, e_2 span a 2-plane with zero curvature).

$$\langle \partial_t \,\mathrm{Rm}(e_1,e_2)e_2, e_1\rangle = \langle \Delta \,\mathrm{Rm}(e_1,e_2)e_2, e_1\rangle + \underbrace{\langle Q(\mathrm{Rm})(e_1,e_2)e_2,e_1\rangle}_{\geq 0 \text{ (basic property of } Q)}$$

$$\geq \sum_{i=1}^{n} \langle (\nabla^2_{e_i,e_i} \mathrm{Rm})(e_1,e_2)e_2, e_1\rangle \geq 0$$

(look at geodesics through x_∞ in direction e_i; function is ≥ 0 everywhere, $= 0$ at the point). Claim: $\langle(\nabla^2_{e_3,e_3}\operatorname{Rm})(e_1,e_2)e_2,e_1\rangle > 0$. Cf. Notes. $\square$ (Case 2)

Case 3. $\mathcal{R} = 0$ (Notes). Pick $x_i \to \infty$, scale by $d(x_i,p)^{-2} \rightsquigarrow (M, h_i(\cdot), (x_i, 0))$ $\rightsquigarrow$ incomplete, flat limit, which, as a metric space, can be completed by a point $\rightsquigarrow$ special structure, essentially determined by π_1. $\square$ (case 3)

Finally, let's check that $\mathcal{V} = 0$ for all κ-solutions.

- All 2D κ-solutions are compact.
- $n \geq 3$. Assume statement true for dim $\leq n-1$. $(M, g(\cdot))$ n-dimensional κ-solution, wlog noncompact. Know: $\exists x_i \in M$, $d_t(x_i,p) \to \infty$, such that rescaling $(M, g(\cdot), (x_i, t))$ by $R(x_i,t)$, we get limit $(M_\infty, g_\infty(\cdot), (x_\infty, 0))$ splitting as $\mathbb{R}\times Y$. $\mathcal{V}(Y) = 0$ by induction, hence $\mathcal{V}(\mathbb{R}\times Y) = \mathcal{V}(M_\infty) = 0$. Now if $\mathcal{V}(M) > 0$, then all ratios $\operatorname{vol}(B_r(q)/r^n$, $q \in M$, would be bounded from below $\rightsquigarrow$ contradiction in the limit.

$\square$ (11.4)

Consequences of I.11.4

Volume controls curvature: For all $A > 0$ there is a $B < \infty$ such that if $B(x,t,r)$ is an r-ball in a κ-solution and $\operatorname{vol}(B(x,t,r))/r^n \geq A$, then $R(x,t)r^2 < B$.

Proof: Equivalent: $\operatorname{vol}(B(x,t,1)) \geq A \Rightarrow R(x,t) < B$. If false, $\exists\, (M_i, g_i(\cdot), (x_i,t_i))$ pointed κ-solutions such that $\operatorname{vol}(B(x_i,t_i,1)) \geq A$, but $R(x_i,t_i) \to \infty$. Point selection $\Rightarrow$ can find sequence $y_i \in B(x_i,t_i,1)$, $D_i \to \infty$, such that $R(y_i,t_i) \to \infty$, $R \leq 2R(y_i,t_i)$ on $B(y_i,t_i,D_i(R(y_i,t_i))^{-1/2})$.

Scale by $R(y_i,t_i)$: $(M_i, h_i(\cdot), (y_i, 0))$, $R \leq 2$ on $B(y_i, 0, D_i)$. Harnack $\Rightarrow$ control backwards in time. HCG $\Rightarrow \exists$ limit $(M_\infty, h_\infty(\cdot), (y_\infty, 0))$, $\operatorname{Rm} \geq 0$, κ-noncollapsed, $R(y_\infty, 0) = 1$. (11.4) $\Rightarrow \mathcal{V}(M_\infty) = 0$.

$$\begin{aligned} 0 \leftarrow \frac{1}{r^n}\operatorname{vol}(B(y_\infty,0,r)) &= \lim_{i\to\infty} \frac{\operatorname{vol}(B(y_i,t_i,r\cdot(R(y_i,t_i))^{-1/2}))}{(r\cdot R(y_i,t_i)^{-1/2})^n} \\ &\geq \lim_{i\to\infty} \frac{\operatorname{vol}(B(y_i,t_i,D_i(R(y_i,t_i))^{-1/2}))}{(D_i(R(y_i,t_i))^{-1/2})^n} \\ &\geq \operatorname{vol}(B(y_i,t_i,1)) \geq A : \end{aligned}$$

contradiction. $\square$

Friday, July 1 (Bruce Kleiner)

Precompactness: For all $A > 0$, the collection of pointed κ-solutions $(M, g(\cdot), (x,t))$ with $\mathrm{vol}(V(x,t,1)) > A$ is precompact in the pointed smooth topology. Any sequence from this family has a subsequence which converges to an ancient κ-noncollapsed Ricci flow (which a priori may not have bounded curvature on time slabs or strictly positive scalar curvature). However, the limit will satisfy Hamilton's Harnack inequality,

$$R(x_2,t_2) \geq \exp\left(-\frac{d_{t_1}^2(x_1,x_2)}{2(t_2-t_2)}\right) R(x_1,t_1).$$

Proof: If $y \in M$, let $\rho := d_t(y,x)$; then by Bishop-Gromov volume comparison,

$$\frac{\mathrm{vol}(B(y,t,1))}{1^n} \geq \frac{\mathrm{vol}(B(y,t,\rho+1))}{(\rho+1)^n} \geq \frac{A}{(\rho+1)^n}.$$

Therefore $R(y,t)$ is bounded above by a function of ρ. By applying the Harnack inequality, we get that $R(y,t')$ is bounded above by the same function of ρ, for all $t' \leq t$. Now HCG-compactness applies, and we're done. ☐

Derivative estimates: There is an $\eta < \infty$ such that the following inequalities hold at any point in a κ-solution:

$$\frac{\partial R}{\partial t} \leq \eta R^2, \quad |\nabla R| \leq \eta R^{\frac{3}{2}}.$$

Proof: Contradiction argument; exploit scale invariance of the estimates and apply a precompactness lemma (see below). ☐

Curvature controls volume: For all $A < \infty$ there is a $B > 0$ such that if $R(x,t)\cdot r^2 < A$ then $\mathrm{vol}(B(x,t,r)) \cdot r^{-n} > B$.

Proof: If not, for some $A < \infty$ there is a sequence $(M_i, g_i(\cdot), (x_i,t_i))$ of κ-solutions such that $R(x_i,t_i) < A$ and $\mathrm{vol}(B(x_i,t_i,1)) \to 0$. Pick $r_i > 0$ such that

$$\frac{\mathrm{vol}(B(x_i,t_i,r_i))}{r_i^n} = \frac{c_n}{2}$$

where c_n is the volume of a Euclidean unit ball $B(0,1) \subset \mathbb{R}^n$ (exists for large enough i, because of relative volume comparison). Note that

$$\frac{c_n}{2} r_i^n = \mathrm{vol}(B(x_i,t_i,r_i)) \leq \mathrm{vol}(B(x_i,t_i,1)),$$

so $r_i \to 0$.

Rescale the pointed sequence $(M_i, g_i(\cdot), (x_i,t_i))$ by r_i^{-2}, to get a new pointed sequence $(M_i, h_i(\cdot), (x_i, 0))$. Then

$$\mathrm{vol}_{h_i(0)}(B(x_i,0,1)) = \frac{\mathrm{vol}(B(x_i,t_i,r_i))}{r_i^n} = \frac{c_n}{2}.$$

Therefore the precompactness theorem applies (note: curvature $\sim r_i^2 \to 0$), and so a subsequence converges in the pointed smooth topology to some $(M_\infty, g_\infty(\cdot), (x_\infty, 0))$, which is ancient, κ-noncollapsed, has $\mathrm{Rm} \geq 0$, and satisfies Hamilton's Harnack inequality.

Suppose $(M_\infty, g_\infty(0))$ were flat. Since

$$\mathrm{vol}(B(x_\infty, 0, 1)) = \lim_{i\to\infty} \frac{\mathrm{vol}(B(x_i, t_i, r_i))}{r_i^n} = \frac{c_n}{2},$$

it could not be flat $\mathbb{R}^n$.

Fact: A complete flat manifold other than $\mathbb{R}^n$ has zero asymptotic volume.

Thus a parabolic ball $P(x_\infty, 0, r)$ would have $\mathrm{Rm} \equiv 0$, but its final time slice satisfies $\mathrm{vol}(B(x_\infty, 0, r))/r^n \to 0$. This contradicts the fact that (M_∞, g_∞) is κ-noncollapsed.

Therefore $(M_\infty, g_\infty(0))$ is not flat, and by the Harnack inequality, we must have $R(x_\infty, 0) > 0$.

Therefore $0 < R(x_\infty, 0) = \lim_{i\to\infty} R(x_i, t_i) r_i^2 = 0$, which is a contradiction. $\square$

Precompactness II: For any $A < \infty$, the collection of pointed κ-solutions $(M, g(\cdot), (x, t))$ with $R(x, t) < A$ is precompact in the pointed smooth topology. In particular for such normalized κ-solutions the curvature $R(y, t)$ is uniformly bounded by a function of the distance $d_t(y, x)$.

Proof: By the preceding result, we know that $\mathrm{vol}(B(x, t, 1))$ is bounded away from zero uniformly. Therefore the earlier precompactness theorem applies. $\square$

• For all $A < \infty$ there is a $B < \infty$ such that if $R(x, t) \cdot d_t^2(x, y) < A$, then $R(y, t) \cdot d_t^2(x, y) < B$.

Proof: If not, after rescaling by $R(x_i, t_i)$, we would have sequences $(M_i, g_i(\cdot), (x_i, t_i))$, $y_i \in M_i$, where $R(x_i, t_i) = 1$, $d_{t_i}(x_i, y_i) < A$, but $R(y_i, t_i) \to \infty$. $\square$

Compactness of 3-dimensional κ-solutions

• The collection of 3-dimensional κ-solutions $(M, g(\cdot), (x, t))$ with $R(x, t) = 1$ is **compact** in the pointed smooth topology.

Sketch of proof: Let $(M_i, g_i(\cdot), (x_i, t_i))$ be a sequence of pointed 3-dimensional κ-solutions. By the precompactness result, a subsequence will converge in the pointed smooth topology to a limitng Ricci flow $(M_\infty, g_\infty(\cdot), (x_\infty, 0))$ which is ancient, has $\mathrm{Rm} \geq 0$, is κ-noncollapsed, satisfies the Harnack inequality, and has $R(x_\infty, 0) = 1$. Therefore, it suffices to show (by contradiction) that R is boundes on the final time slice $(M_\infty, g_\infty(0))$.

If there exists $y_k \in M_\infty$ such that $R(y_k, 0) \to \infty$, then by repeating the $\mathcal{R} = \infty$ case of the proof of 11.4, one concludes that there is a possibly different sequence $z_k \in M_\infty$ such that $d_\infty(z_k, x_\infty) \to \infty$, $R(z_k, 0) \to \infty$, and the pointed sequence of

Ricci flows $(M_\infty, g_\infty(\cdot), (z_k, 0))$ converges modulo scaling to round cylindrical flow. Therefore the geometry of $(M_\infty, g_\infty(0))$ near z_k is like a very small round neck. It turns out $(*)$ that no Riemannian manifold of nonnegative curvature can contain such a sequence of necks, and hence we have a contradiction. $\square$

$(*)$: X complete Riemannian manifold, $K_X \geq 0$. Via Busemann functions: $\exists$ exhaustion $\{C_t\}_{t\geq 0}$ by compact convex sets such that $(**)$ $C_{t-s} = \{x \in C_t | d(x, \partial C_t) \geq s\}$.

In our case, ∂C_t has only **one** component: Else we could connect points in different boundary components by geodesics in C_t and get a line as $t \to \infty$, because the geodesic segments have to intersect C_0 and hence can't disappear. Hence would have round cylindrical flow, and then R would be bounded.

Hence $\forall z_k$ $\exists t_k$: ∂C_{t_k} is a cross-sectional sphere in the neck near z_k (follows from $(**)$ and $\#\pi_0(\partial C_t) = 1$) $\Rightarrow$ ∂C_{t_k} has smaller and smaller diameter; contradiction to Sharafutdinov retraction (cf. Cheeger). $\square$ $(*)$

Neck structure in 3d κ-solutions

Definition: Say that a point (x, t) is the center of an ε-neck if after parabolic rescaling by $R(x, t)$, the flow is ε-close to round cylindrical flow on $S^2 \times \mathbb{R}$ or $\mathbb{R}P^2 \times \mathbb{R}$.

For all $\varepsilon > 0$ there is a $D = D(k, \varepsilon) < \infty$ such that

- If $(M^3, g(\cdot))$ is a noncompact κ-solution defined at time $t \in \mathbb{R}$, then there is a point $x \in M$ such that all points lying outside the ball $B(x, t, DR(x, t)^{-1/2})$ are centers of ε-necks at time t. Furthermore, unless $(M, g(\cdot))$ is round cylindrical flow on $S^2 \times \mathbb{R}$, then x can be chosen so that the metric ball $B(x, t, DR(x, t)^{-1/2})$ is a 3-ball or a twisted line bundle over $\mathbb{R}P^2$.
- If $(M^3, g(\cdot))$ is a compact κ-solution defined at time t, then there is a pair of points $x_1, x_2 \in M$ such that points in M lying outside the union of the two balls $B(x_1, t, DR(x_1, t)^{-1/2}) \cup B(x_2, t, DR(x_2, t)^{-1/2})$ are centers of ε-necks at time t. $\mathrm{diam}(M, g(t)) \leq Cd(x_1, x_2)$. Note that M is diffeomorphic to a spherical space form by Hamilton's theorem on 3-manifolds with positive Ricci curvature.

For the proof, first need strengthened version of (11.4): statement is true for **any** sequence of points diverging to infinity. Then: Let $(M^3, g(\cdot))$ b a κ-solution, $x_i \in M$, $x_i \to \infty$. Rescale $(M, g(\cdot), (x_i, t))$ by $R(x_i, t)$; get $(M, h_i(\cdot), (x_i, 0))$ with $R(x_i) = 1$. Compactness $\Rightarrow$ subconverges to a κ-solution $(M_\infty, h_\infty(\cdot), (x_\infty, 0))$. Claim: With any basepoint $p \in M$, $R(x_i, t) \cdot d_t^2(x_i, p) \to \infty$ ... time is over ...

Monday, July 4 (Bruce Kleiner)

Corollary: Pick $\varepsilon > 0$. Then there exists $D_1 < \infty$ such that for all $A < \infty$ there is a $D_2 < \infty$ such that if $(M, g(\cdot), (x,t))$ is a κ-solution, then one of the following holds:

- (x,t) is the center of an ε-neck.
- $R(x,t) \cdot \mathrm{diam}^2(M, g(t)) < D_2$.
- Every point in $\mathrm{Ann}(M, g(t), (x,t); D_1 R(x,t)^{-1/2}, AR(x,t)^{-1/2})$ is an ε-neck.

Definition: A Ricci flow $(M, g(\cdot))$ on a compact n-manifold M has a normalized initial condition if

- It is defined on a time interval of the form $[0,T)$, where $0 < T \leq \infty$ is the blowup time.
- $|\operatorname{Rm}| \leq 1$ on the time zero slice.
- The volume of every unit ball at time zero is at least half the volume of a Euclidean unit ball, $\mathrm{vol}(B(x,0,1)) \geq \frac{c_n}{2}$.

If $g(0)$ is any smooth Riemannian metric on a compact manifold M, if we scale $g(0)$ enough it will be a normalized initial condition.

By the noncollapsing result of section I.7, we know that there is a function $\kappa = \kappa(n,T)$ such that any n-dimensional Ricci flow with normalized initial condition is κ-noncollapsed at scales < 1.

Henceforth we will be considering only Ricci flows on orientable 3-manifolds.

Main assertion (cf. I.12.1): Pick $\varepsilon > 0$ and $T < \infty$. Then there are constants $R_0 = R_0(\varepsilon, T)$, and $\kappa = \kappa(\varepsilon, T)$, such that if $(M, g(\cdot))$ is a 3-dimensional Ricci flow with normalized initial condition, and $R(x,t) \geq R_0$, then the pointed flow $(M, g(\cdot), x, t)$, after being parabolically rescaled by $R(x,t)$, is ε-close to a pointed κ-solution.

Remark: Due to a theorem of Hamilton-Ivey, for 3D Ricci flows with normalized initial conditions, a point in space-time has large scalar curvature if and only if the curvature tensor has large norm.

The global picture of the large curvature part

Fix $T < \infty$, $\varepsilon > 0$. Then there are constants $D > 0$, $C < \infty$, etc, such that for any (3-d!!) Ricci flow $(M, g(\cdot))$ with normalized initial condition, and any $0 \leq t < T$, there are subsets $M_A, M_B \subset M$ such that

- $M = M_A \cup M_B$.
- For all $x \in M_A$, the parabolic ball $P(x,t,D)$ has controlled geometry: $|\operatorname{Rm}| < D$ and $\mathrm{injrad}(M, g(t), x) \geq C^{-1} > 0$.
- M_B is a union of connected components $C_1, ..., C_k$ such that for each i, one of the following holds:

a. (Neck) Every point in C_i is the center of an ε-neck, and the pair $(C_i, \partial C_i)$ is diffeomorphic to $(S^2 \times [0,1], S^2 \times \{0,1\})$. Recall that a point (x,t) in a Ricci flow is the center of an ε-neck if, modulo scaling, the pointed Ricci flow $(M, g(\cdot), (x,t))$ is ε-close to round cylindrical flow.

b. (Capped neck) There is a point $(p,t) \in C_i$ such that every point in $C_i \setminus B(p, C(R(p,t))^{-1/2})$ is the center of an ε-neck, and the pair $(C_i, \partial C_i)$ is diffeomorphic to either (D^3, S^2) or $(\mathbb{R}P^3 \setminus B^3 =$ normal bundle of the cut locus in $\mathbb{R}P^3 =$ twisted line bundle over $\mathbb{R}P^2, S^2)$.

c. (Tube) Every point in C_i is the center of an ε-neck, and C_i is diffeomorphic to $S^2 \times S^1$.

d. (Worm) There are two points (p_1,t), $(p_2,t) \in C_i$, such that every point in $C_i \setminus (B(p_1, t, CR(p_1,t)^{-1/2}) \cup B(p_2, t, CR(p_2,t)^{-1/2})$ is the center of an ε-neck, and C_i is diffeomorphic to S^3, $\mathbb{R}P^3$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Note: In all these cases, the neck structure around a point only appears after parabolic rescaling by scalar curvature (i.e. $R_{\max}/R_{\min}$ might be unbounded along a neck).

Justification of global picture

Pick $\varepsilon > 0$, $T < \infty$, and assume that $t < T$. Then:

- There is an $\eta = \eta(T)$, $R_0 = R_0(T)$, such that if $R(x,t) \geq R_0$ then
$$\left| \frac{\partial R}{\partial t} \right| \leq \eta R^2 \quad \text{and} \quad |\nabla R| \leq \eta R^{\frac{3}{2}}$$
(closeness to κ-solution, for which these estimates hold. $\eta = \eta(\varepsilon)$ improves with $\varepsilon \to 0$, but we don't care about that and just fix **some** ε).
- $\Rightarrow$ For all $R_1 < \infty$ there is a $Q < \infty$ such that if $R(x,t) \leq R_1$ then $R \leq Q$ on the parabolic ball $P(x,t,Q^{-1})$, and $\mathrm{injrad}(M, g(t), (x,t)) \geq Q^{-1}$.
  Reason: curvature bound – either curvature is smaller than $R_0(T)$, or the gradient estimate applies; injrad bound – from no local collapsing.

Consequence: Either we have a parabolic ball of controlled geometry, or scalar curvature at the given point is really large and (12.1) applies. On the latter part of the manifold, get an S^2-fibration with boundary, which is closed by caps of controlled geometry.

Ingredients in the proof of I.12.1

If $(M, g(\cdot))$ is a 3-dimensional Ricci flow with normalized initial condition, then:

- (Noncollapsing) By section I.7, we know that there is a function $\kappa = \kappa(T)$ such that $(M, g(\cdot))$ κ-noncollapsed at scales < 1 at any (x,t) with $t \leq T$.
- $R \geq -6$ everywhere, since the minimum of the scalar curvature can only increase with time.
- (Hamilton-Ivey curvature pinching) There is a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\lim_{s \to \infty} \varphi(s)/s = 0$, and for any point (x,t) in a 3-d Ricci flow: $\mathrm{Rm}(x,t) \geq \varphi(R(x,t))$.

- $\Rightarrow$ To control $|\operatorname{Rm}|$ at a point (x,t), it suffices to control $R(x,t)$.
- If $(M, g(\cdot), (x,t))$ happens to be ε-close to a κ-solution, then we may apply the conclusions of section I.11 near (x,t).

Remark: Note that the assertion I.12.1 for smaller values of ε implies the assertion for larger values; therefore it suffices to prove the statement when ε is sufficiently small.

Proof of I.12.1

Suppose I.12.1. were false for some $T < \infty$, where κ is the constant coming from section I.7. Then there would be an $\varepsilon > 0$ and a sequence of pointed 3-d Ricci flows $(M_i, g_i(\cdot), (x_i, t_i))$ with normalized initial conditions, such that $t_i \leq T$, and $R(x_i, t_i) \to \infty$, but after rescaling by $R(x_i, t_i)$, none of the resulting pointed flows $(M_i, h_i(\cdot), (x_i, t_i))$ is ε-close to a κ-solution.

Goal: Extract a sublimit which **is** a κ-solution, $\Rightarrow$ immediate contradiction.

Step 1: Point selection. We may assume that there is a sequence $D_i \to \infty$ such that if $(y,t) \in P(x_i, t_i, D_i R(x_i,t_i)^{-1/2})$, and $R(y,t) \geq 2R(x_i,t_i)$, then the conclusion of I.12.1 does hold at (y,t) (i.e. after rescaling, we are ε-close to a κ-solution). Let $\mathcal{G}$ be the set of points in spacetime where the conclusion of I.12.1 holds.

Let $(M_i, h_i(\cdot), (x_i, 0))$ be the result of rescaling the Ricci flow by $R(x_i, t_i)$.

Step 2: For all $\rho < \infty$ there is a $Q < \infty$ such that if $(y,t) \in B(x_i, 0, \rho)$ then $R(y,t) \leq Q$. (i.e.: final time slices of the rescaled flows have bounded curvature at bounded distances)

Definition: Let $\rho_0 \geq 0$ be the largest number such that R is controlled on the balls $B(x_i, 0, r) \subset (M_i, h_i, 0)$ for $r \in [0, \rho_0)$. Then $\rho_0 > 0$.

Tuesday, July 5 (Bruce Kleiner)

Proof of I.12.1 – Outline

Step 1: Point selection.

Step 2: The final time slices $(M_i, h_i(0), (x_i, 0))$ have uniformly bounded curvature at bounded distance, i.e. for all $\rho < \infty$ there is a $Q < \infty$ such that $R \le Q$ on $B(x_i, 0, \rho)$ for all i.

Step 3: One may extract a limit $(M_\infty, h_\infty(0), (x_\infty, 0))$ of the final time slices, and this has bounded curvature.

Step 4: One may extract a limitig Ricci flow $(M_\infty, h_\infty(\cdot), (x_\infty, 0))$ which is ancient, and therefore a κ-solution. $\square$ (outline)

• **Step 1: Point selection.** We may assume that there is a sequence $D_i \to \infty$ such that if $(y,t) \in P(x_i, t_i, D_i R(x_i,t_i)^{-1/2})$, and $R(y,t) \ge 2R(x_i,t_i)$, then the conclusion of I.12.1 does hold at (y,t) (i.e. after rescaling, we are ε-close to a κ-solution). Let $\mathcal{G}_i$ be the set of points in spacetime where the conclusion of I.12.1 holds.

Therefore for some constant $\eta < \infty$, for all $(x,t) \in \mathcal{G}_i$ one has the derivative estimates

$$\left|\frac{\partial R}{\partial t}\right| \le \eta R^2 \quad \text{and} \quad |\nabla R| \le \eta R^{\frac{3}{2}}.$$

Using ODE comparisons, this implies that if $(y,t) \in P(x_i, t_i, D_i R(x_i,t_i)^{-1/2})$, then there is a parabolic ball $P(y,t,r)$ where $r \gtrsim \min(R(y,t)^{-1/2}, R(x_i,t_i)^{-1/2})$ such that $R \lesssim \max(R(y,t), R(x_i,t_i))$ on $P(y,t,r)$, provided $P(y,t,r) \subset P(x_i, t_i, D_i R(x_i,t_i)^{-1/2})$. $\square$ (Step 1)

Let $(M_i, h_i(\cdot), (x_i, 0))$ be the result of rescaling the Ricci flow by $R(x_i, t_i)$.

• **Step 2:** For all $\rho < \infty$ there is a constant $Q = Q(\rho) < \infty$ such that $R < Q$ on $B(x_i, 0, \rho) \subset (M_i, h_i(0), (x_i, 0))$. For all $\rho < \infty$, define $Q = Q(\rho)$ to be

$$\sup_i \sup\{R(y,0) | y \in B(x_i, \rho, 0)\} \in \mathbb{R} \cup \infty.$$

Let $\rho_0 := \sup\{\rho | Q(\rho) < \infty\}$. The goal in step 2 is to show that $\rho_0 = \infty$. So assume $\rho_0 < \infty$. By passing to a subsequence, if necessary, we may assume that $\lim_{i\to\infty} \sup\{R(y,0) | (y,0) \in B(x_i, 0, \rho_0)\} = \infty$.

Note that for every $(y,t) \in P(x_i, 0, \rho_0)$ there is a parabolic ball $P(y,t,r)$ of radius $r \gtrsim \min(R(y,t)^{1/2}, 1)$ such that $R \lesssim \max(R(y,t), 1)$ on $P(y,t,r)$. Therefore, $R \lesssim Q(\rho)$ on the parabolic region $B(x_i, 0, \rho) \times [t(\rho), 0]$ where $t(\rho) \lesssim -Q(\rho)^{-1}$. By Hamilton-Ivey pinching, we also have $|\operatorname{Rm}| \lesssim Q(\rho)$ on the same parabolic region.

Due to the noncollapsing estimate, we may extract a limiting incomplete Ricci flow $(M_\infty, h_\infty(\cdot), (x_\infty, 0))$ with the following properties:

- It is defined on an open ball of radius ρ_0 at time 0, and for each $(y, 0) \in (M_\infty, h_\infty(0))$ it is defined on a parabolic ball of radius $\gtrsim \min(R(y,0)^{-1/2}, 1)$, on which $R \lesssim \max(R(y,0), 1)$.
- $R(x_\infty, 0) = 1$.
- By Hamilton-Ivey curvature pinching, (M_∞, h_∞) has $\mathrm{Rm} \geq 0$ everywhere.
- $\sup\{R(y,0) | (y,0) \in B(x_\infty, 0, \rho_0)\} = \infty$.
- For each $(y,0) \in B(x_\infty, 0, \rho_0)$ with $R(y,0) \geq 2$, if we rescale by $R(y,0)$ we get a Ricci flow which is 2ε-close to a κ-solution, at least over a backward time interval of thickness $\gtrsim 1$.

Choose a minimizing geodesic path $\gamma : [0,1) \to (M_\infty, h_\infty(0))$ such that $R(\gamma(s), 0) \to \infty$ as $s \to 1$. Then there is an $s_0 \in (0,1)$ such that for all $s \in [s_0, 1)$, $R(\gamma(s), 0) \geq 2$. There is an $s_1 \in [s_0, 1)$ such that for all $s \in [s_1, 1)$, the point $(\gamma(s), 0)$ is the center of a 2ε-neck (why not annulus or cap? have a geodesic segment through $\gamma(s)$ which becomes very long after rescaling by $R(\gamma(s), 0)$).

Now add a completion point to $(M_\infty, d_{h_\infty(0)})$ so that the limit $y_\infty := \lim_{s\to 1}(\gamma(s), 0)$ exists. Then a small closed ball $\overline{B}(y_\infty, r) \subset Y_\infty := (M_\infty \cup \{y_\infty\}, d_{h_\infty(0)})$ is an Alexandrov space of nonnegative curvature, i.e. the conclusion of Toponogov's triangle comparison theorem holds (enough to show: $\forall \alpha$ unit speed geodesic, $\forall p$: $(d_p^2 \circ \alpha)'' \leq 2$; in our space, α cannot pass through y_∞, since, after rescaling, we have long thin necks around each point of γ, so there would be shortcuts for α).

Then the tangent cone of Y_∞ at y_∞ exists: if $\lambda_k \to \infty$, then the sequence of rescalings $(Y_\infty, \lambda_k d_{h_\infty(0)}, y_\infty)$ converges in the pointed Gromov-Hausdorff topology to a nonnegatively curved metric cone (Z, z).

By using triangle comparison, one can argue that the radius of the 2ε-neck at $(\gamma(s), 0)$ is $\gtrsim 1 - s$, and therefore $R(\gamma(s), 0) \lesssim (1-s)^{-2}$. Therefore the metric cone Z is 3-dimensional, and is a cone over a metric 2-sphere.

If we take a sequence $s_k \to 1$, and rescale the Ricci flows $(M_\infty, h_\infty(\cdot), (\gamma(s_k), 0))$ (these are actually defined only near $(\gamma(s_k), 0)$, and the size of the respective parabolic neighborhood shrinks to zero) by $R(\gamma(s), 0)$, we get a sequence of locally defined Ricci flows which subconverge to a locally defined Ricci flow which has $\mathrm{Rm} \geq 0$, and at time 0 is locally isometric to a Riemannian cone.

Repeating the argument from the proof of I.11.4, we get a contradiction. □

Wednesday, July 6 (John Lott)

Today:

- distance-distortion estimates (I.8.3),
- review proof of I.12.1,
- finish off I.12.1 using point 1.

$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}$, $x, y \in M$, $d_t(x,y) =$ time-t-distance

Curvature not so positive $\rightsquigarrow$ contraction not so fast: naive/clever estimate.

Naively: $\gamma : [0,a] \to M$ smooth curve, $L(\gamma) = \int_0^a \sqrt{\langle \frac{d\gamma}{ds}, \frac{d\gamma}{ds}\rangle}\, ds$,

$$\frac{dL(\gamma)}{dt} = \int_0^a \frac{1}{2\sqrt{\langle \frac{d\gamma}{ds}, \frac{d\gamma}{ds}\rangle}} \cdot \left(-2\,\mathrm{Ric}\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)\right) ds = -\int_0^a \mathrm{Ric}\left(\frac{\frac{d\gamma}{ds}}{\left|\frac{d\gamma}{ds}\right|}, \frac{\frac{d\gamma}{ds}}{\left|\frac{d\gamma}{ds}\right|}\right) \left|\frac{d\gamma}{ds}\right|\, ds$$

Suppose $\mathrm{Ric} \le (n-1)Kg$. Then $\frac{d}{dt}L(\gamma) \ge -(n-1)K \cdot L(\gamma)$, hence

$$\frac{L(\gamma)(t_1)}{L(\gamma)(t_0)} \ge e^{-(n-1)K(t_1-t_0)}.$$

Say γ is a time-t_1-geodesic from x to y: $d_{t_1}(x,y) \ge e^{-(n-1)K(t_2-t_1)} d_{t_0}(x,y)$. I.e.: If $\mathrm{Ric} \le (n-1)Kg$ always and everywhere, then

$$\frac{d}{dt} d_t(x,y) \ge -(n-1)K \cdot d_t(x,y).$$

Cleverly: Assume $d_t(x_0,x_1) > 2r_0$ and $\mathrm{Ric} \le (n-1)Kg$ on $B_t(x_0,r_0) \cup B_t(x_1,r_0)$. γ minimizing geodesic between x_0 and x_1 at time t. As above:

$$\frac{d}{dt} d_t(x_0,x_1) = -\int_0^d \mathrm{Ric}(X,X)\, ds, \quad X = \frac{d\gamma}{ds}.$$

How to estimate $\mathrm{Ric}(X,X)$? From Riemannian geometry (2nd variation):

$$0 \le \int_0^d \left(|\nabla_X V|^2 + \langle R(X,V)X, V\rangle\right) ds$$

for any vector field V along γ that vanishes at the endpoints. Test it cleverly: Say e_i is a parallel unit vector field along γ, and put $V_i(s) := f(s)e_i(s)$, where

$$f(s) := \begin{cases} \frac{s}{r_0} & \text{if } 0 \le r \le r_0, \\ \frac{d-s}{r_0} & \text{if } d - r_0 \le s \le d, \\ 1 & \text{else.} \end{cases}$$

Then:

$$\int_0^d |\nabla_X V_i|^2\, ds = \int_0^d \left|\frac{df}{ds}\right|^2 ds = \frac{2}{r_0},$$

$$\int_0^d \langle R(X,V_i)X, V_i\rangle\, ds = \int_0^d f(s)^2 \langle R(X,e_i)X, e_i\rangle\, ds.$$

Sum over $i = 1, ..., n-1$:

$$0 \leq \sum_{i=1}^{n-1} \int_0^d (|\nabla_X V_i|^2 + \langle R(X,V_i)X, V_I\rangle)\ ds = \frac{2(n-1)}{r_0} - \int_0^d f^2(s)\ \mathrm{Ric}(X,X)\ ds$$

$$= \frac{2(n-1)}{r_0} - \int_0^d \mathrm{Ric}(X,X)\ ds + \int_0^{r_0} \left(1 - \frac{s^2}{r_0^2}\right) \mathrm{Ric}(X,X)\ ds +$$

$$+ \int_{d-r_0}^d \left(1 - \frac{(d-s)^2}{r_0^2}\right) \mathrm{Ric}(X,X)\ ds$$

$$\leq \frac{2(n-1)}{r_0} - \int_0^d \mathrm{Ric}(X,X)\ ds + \frac{2}{3} r_0 (n-1) K.$$

Upshot:

$$\frac{d}{dt} d_t(x_0, x_1) \geq -\frac{2(n-1)}{r_0} - \frac{2}{3} r_0 (n-1) K$$

(**much** sharper).

Corollary (Hamilton): If $\mathrm{Ric} \leq Kg$ for some $K > 0$, then

$$\frac{d}{dt} d_t(x_0, y_0) \geq -C(n)\sqrt{K}.$$

Proof: Put $r_0 = K^{-1/2}$. If $d_t(x_0, y_0) > 2r_0$, then previous computation gives $\frac{d}{dt} d_t(x_0, y_0) \geq -C(n)\sqrt{K}$. If $d_t(x_0, y_0) \leq 2r_0$, first estimate gives $\frac{d}{dt} d_t(x_0, y_0) \geq -CK d_t \geq -C\sqrt{K}$. $\square$

I.12.1: Say we have a Ricci flow with normalized initial conditions. $\exists\ R_0$ such that up to a given time $t_{\max}$, any point (x,t) with $R(x,t) \geq R_0$ is, after rescaling, ε-close to a chunk of a κ-solution.

Baby case: Let's assume in addition that $R(x,t') \leq R(x,t)$ $(\forall t' \leq t)$. Suppose (12.1) was false. $(M_i, g_i(\cdot), (x_i, t_i))$ with $t_i \leq t_{\max}$, $R(x_i, t_i) \to \infty$. Rescale by curvatures, take a sublimit (curvature bounds **automatic** in this case, κ-noncollapsing $\Rightarrow$ volume bound). Get an ancient solution, κ-noncollapsed. Bruce $\Rightarrow$ bounded curvature on time slices. Nonnegative Rm: from Hamilton-Ivey 3D estimate. $\square$

General case

Step 1: Point picking ("induction on the curvature scale")

Can assume $D_i \to \infty$ such that if $(y,t) \in P(x_i, t_i, D_i R(x_i,t_i)^{-1/2})$ has $R(y,t) \geq 2R(x_i,t_i)$, then (y,t) is ε-close to a κ-solution $\square$ (Step 1).

Step 2

Lemma: Say $(y, \overline{t}) \in P(x_i, t_i, D_i R(x_i,t_i)^{-1/2})$. Put $\overline{Q} = R(y, \overline{t}) + R(x_i, t_i)$. Then $R \leq 4\overline{Q}$ on $P(y, \overline{t}, C \cdot \overline{Q}^{-1/2})$ for some $C = C(\kappa)$.

Proof: Say $(x,t) \in P(y, \overline{t}, C \cdot \overline{Q}^{-1/2})$. If $R(x,t) \leq 2R(x_i,t_i)$, done. If $R(x,t) > 2R(x_i,t_i)$, take a spacetime path, first linearly from (x,t) to $(x, \overline{t})$, then over to $(y, \overline{t})$. Pick p the nearest point on the path with $R(p) = 2R(x_i, t_i)$, if there is one, or else put $p = (y, \overline{t})$. From $(x, \overline{t})$ to p, gradient bounds hold. $\square$ (Lemma)

• Rescale by $R(x_i, t_i)$, get $(M_i, h_i(\cdot), (x_i, 0))$. Want to extract a convergent subsequence on time-zero-slice. Take ρ to be biggest radius so that we can take a limit to some $B(x_\infty, \rho)$. Limit smooth, because locally, we can go backwards in time a little bit, by lemma.

• Want to show $\rho = \infty$. If $\rho < \infty$, add a limit point v to M_∞ where curvature blows up. Have blowup cone at v. Arguing like in (11.4) ($0 < \mathcal{R} < \infty$), get contradiction. $\square$ (Step 2)

Step 3

• Actually have global upper bound $R(x, 0) \leq Q$ on time-zero-slice. Otherwise would have necklike points going to ∞.

• From κ-noncollapsing, get $\mathrm{injrad}(M_\infty) \geq C > 0$.

• Can we go backwards in time? $\square$ (Step 3)

Step 4

Our "time 0 slice" extends backwards in time for some interval $\Delta t > 0$. Reason: Locally, we could always go back to a certain amout of time, bounded by curvature, but now have a global curvature bound.

How to go to $-\infty$? Not that $|\frac{\partial R^{-1}}{\partial t}| \leq C$ is of no use to bound curvature. Say $(t', 0]$ is the maximal time interval on which we can extend our time 0 slice backwards using CGH compactness. Suppose $t' > -\infty$. Curvature should be blowing up, but how much?

• Use trace Harnack inequality: $\frac{\partial R}{\partial t} + \frac{R}{t-t'} \geq 0$ (assumptions: $\mathrm{Rm} \geq 0$, bounded curvature on time slices, flow exists for $t' \leq t$).

$$\Rightarrow \frac{d}{dt}\left((t-t')R(x,t)\right) \geq 0 \;\Rightarrow\; (t-t')R(x,t) \nearrow \Rightarrow\; R(x,t) \leq \frac{(-t')Q}{t-t'} \;(t \in (t', 0]).$$

• Plug into the distance-distortion estimates:

$$\frac{d}{dt} d_t(x,y) \geq -C\sqrt{\frac{(-t')Q}{t-t'}},$$

integrable in t!

• Get: $\exists C$: $|d_0(x,y) - d_t(x,y)| \leq C$ ($\forall t \in (t', 0]$).

First case: M_∞ is compact. Get $\mathrm{diam}_t(M_\infty, h(t)) \leq \mathit{const}$ ($\forall t \in (t', 0]$). From maximum principle, $R_{\min}(t) \leq R_{\min}(0)$ for all such t. Repeat step 2, get uniform curvature bound back to t' (at bounded distances – which we have). Could again go back further: contradiction. $\square$ (compact case)

Thursday, July 7 (John Lott)

Second case: M_∞ **noncompact.** First want curvature bound outside large ball around the basepoint: If D is big enough, then $\forall y \notin B_0(x_\infty, D)\ \exists x : d_0(x_\infty, y) = d_0(y, x), d_0(x_\infty, x) \geq \frac{3}{2} d_0(x_\infty, y)$ (reason: obviously true for the tangent cone at ∞ of M_∞).

Also have these inequalities up to constant C for all $t \in (t', 0)$ (choose D larger).

To get curvature bounds on $M_\infty - B(x_\infty, D)$, uniform in t, suppose not. Then have points y where $R(y,t) \to \infty$. If $R(y,t)$ big enough, close to κ-solution (by point-picking). Given x_∞ and y, form point x such that inequalities hold up to C.

Claim: There's a region U around y with diameter $\sim R(y,t)^{-1/2}$ that separates x_∞ and x (either M_∞ has two ends, then it splits as $\mathbb{R} \times$ surface, or it has one end).

Fix U, evolve the picture up to time 0. $\operatorname{diam}_0(U) \leq \operatorname{diam}_t(U)$.

Take the sequence of y's with $R(y,t) \to \infty$. In time 0 slice, get sequence of U's with diamter $\to 0$, but each of them separates the manifold.

Contradicts the fact that time 0 slice has bounded geometry (upper bound on sectional curvature, strictly positive injectivity radius).

Now extend curvature bound over $B(x_\infty, D)$ by repeating step 2 again (bounded curvature at bounded distances). $\square$ (I.12.1)

Know now: The more you know about κ-solutions, the better off you are! Heading for another characterization of κ-solutions.

Gradient shrinking soliton on $(-\infty, 0)$**:** $R_{ij} + \nabla_i \nabla_j f + \frac{1}{2t} g_{ij} = 0$,

$$\frac{\partial g}{\partial t} = -2 \operatorname{Ric} = 2 \operatorname{Hess} f + \frac{1}{t} g = \mathcal{L}_{\nabla f} g + \frac{1}{t} g.$$

Say that $\frac{\partial f}{\partial t} = |\nabla f|^2 = \mathcal{L}_{\nabla f} f$. Then $g(t) = (-t)\varphi_t^* g(-1)$, $\varphi_t =$ flow of $\nabla f(t)$.

Examples: 0) S^n shrinking, $f = 0$
1) flat $\mathbb{R}^n$, $f = -\frac{|x|^2}{4t}$
2) $\mathbb{R} \times S^{n-1}$, $f = -\frac{x^2}{4t}$ $(x \in \mathbb{R})$

I.11.2: M^n a κ-solution on $(-\infty, 0]$. Pick $p \in M$ on final time slice. Define $l(q, \tau)$, where $\tau = -t$. Choose $q(\tau)$ on time $-\tau$ slice with $l(q(\tau), \tau) \leq \frac{n}{2}$.

Claim: Then there's a sequence $\tau_i \to \infty$ such that $(M, \frac{1}{\tau} g_{ij}(-\tau_i), (q(\tau_i), \tau_i))$ converge to a gradient shrinking soliton (**"asymptotic soliton"**).

It will be κ-noncollapsed, but does not a priori have bounded curvature on time slices (if $n = 3$, it does).

Compare with Bruce: Go out in space and split off a line. Here: Go back in time and get a soliton.

Example: M^3 Bryant soliton, an ancient solution, which is a steady soliton. $p =$ vertex. Is $(q(\overline{\tau}), \overline{\tau}) = (p, \overline{\tau})$? Take $\gamma(\tau) \equiv p$. By symmetry, $R(\gamma(\tau)) \equiv R(p, 0)$. Hence $L(\gamma) = \frac{2}{3}R(p)\overline{\tau}^{3/2}$, $l = \frac{1}{3}R(p)\overline{\tau}$: contracdiction. Must go out to avoid high curvature! I.e., in this case, asymptotic cylinder = asymptotic soliton.

Don't prove 11.2, because it uses stuff from §7 which we skipped.

In order to apply it, need to understand the possible solutions arising.

3D oriented gradient shrinking soliton that's a κ-solution

I) If sectional curvature is not strictly positive, have a zero curvature, get $\tilde{M} = S^2 \times \mathbb{R}$ (standard flow) via Hamilton reduction of holonomy. Hence M must be $\mathbb{R} \times S^2$ or $\mathbb{R} \times_{\mathbb{Z}_2} S^2 \to \mathbb{R}P^2$, a line bundle over $\mathbb{R}P^2$, topologically a neighborhood of $\mathbb{R}P^2$ in $\mathbb{R}P^3$, i.e. $\mathbb{R}P^3$ − 3-ball. **Cannot** have $S^1 \times S^2$, since this is not κ-noncollapsed (go back in time to see it).

II) If sectional curvatue IS strictly positive:

A) M is compact. By pinching-improves estimate, M is isometric to a shrinking quotient of S^3.

B) M not compact. **Claim:** $\nexists$

Lemma II.1.2: $\nexists$ 3D noncompact κ-noncollapsed gradient shrinking soliton with positive bounded curvature.

Proof: $\mathrm{Ric} + \mathrm{Hess}\, f + \frac{1}{2t}g = 0$. Taking the divergence:

$$-\frac{1}{2}\nabla_i R + [\nabla_j, \nabla_i(\nabla_j f) + \nabla_i(\Delta f) = 0.$$

Also: $R + \Delta f + \frac{3}{2t} = 0$, hence $\nabla_i R + \nabla_i(\Delta f) = 0$. Combine:

$$\nabla_i R = 2R_{ij}\nabla_j f.$$

Idea of proof:

- f increases quadratically at infinity,
- see how R varies along gradient flow of f,
- get contradiction ith Gauß-Bonnet in the end.

γ unit speed geodesic in time -1 slice, minimizing, $X = \frac{d\gamma}{ds}$.

$$0 \leq \int_0^{\overline{s}} (|\nabla_X V|^2 + \langle V, R(X, V)X\rangle)\, ds,$$

if V vanishes at the endpoints.

Claim: There's a bound $\int_0^{\overline{s}} \mathrm{Ric}(X,X)\, ds \le const$, independent of $\overline{s}$ (proof like distance-distortion estimate). □ (Claim)

Say $\{Y_i\}_{i=1}^3$ are parallel ON vector fields along γ.

$$\left(\int_0^{\overline{s}} |\,\mathrm{Ric}(X,Y_i)|^2\, ds\right)^2 \le \overline{s}\int_0^{\overline{s}} \mathrm{Ric}(X,Y_i)^2\, ds \le \overline{s}\int_0^{\overline{s}} \sum_{i=1}^3 \mathrm{Ric}(X,Y_i)^2\, ds.$$

Say $\{e_i\}$ is an ON eigenframe for Ric, $\mathrm{Ric}(e_i) = \lambda_i e_i$. Write $x = \sum_{i=1}^3 X_i e_i$.

$$\sum_{i=1}^3 \mathrm{Ric}(X,Y_i)^2 = \langle X, \overset{2}{\mathrm{Ric}}\, X\rangle = \sum_i \lambda_i^2 x_i^2 \le \sum \lambda_i \sum \lambda_i x_i^2 = R\cdot \mathrm{Ric}(X,X).$$

Plugging in back:

$$\left(\int_0^{\overline{s}} |\,\mathrm{Ric}(X,Y_i)|\, ds\right)^2 \le \overline{s}\left(\sup_M R\right)\int_0^{\overline{s}} \mathrm{Ric}(X,X)\, ds \le const\cdot \overline{s}.$$

What's the use of that?

- $X^iX^j(R_{ij} + \nabla_i\nabla_j f + \frac{1}{2t}g_{ij}) = 0$, hence

$$\mathrm{Ric}(X,X) + \frac{d^2}{ds^2}f(\gamma(s)) - \frac{1}{2} = 0$$
$$\Rightarrow\ \frac{d}{ds}f(\gamma(s))\Big|_{\overline{s}} - \frac{d}{ds}f(\gamma(s))\Big|_0 = -\int_0^{\overline{s}} \mathrm{Ric}(X,X)\, ds + \frac{1}{2}\overline{s},$$

  so (compare to shrinking cylinder!)

$$\frac{d}{ds}f(\gamma(s))\Big|_{s=\overline{s}} \ge \frac{1}{2}\overline{s} - const.$$

- $X^iY^j(R_{ij} + \nabla_i\nabla_j f + \frac{1}{2t}g_{ij}) = 0$, hence

$$\mathrm{Ric}(X,Y) + \frac{d}{ds}(Y\cdot f)(\gamma(s)) = 0$$
$$\Rightarrow\ (Y\cdot f)(\gamma(\overline{s})) - (Y\cdot f)(\gamma(0)) = -\int_0^{\overline{s}} \mathrm{Ric}(Y,X)\, ds \le const\cdot\left(\sqrt{\overline{s}}+1\right).$$

In particular, far away from the basepoint, f has no critical points, and secondly, ∇f lines up with the gradient of the distance function at time -1 to x_0 as we go out to spatial infinity.

Friday, July 8 (John Lott)

Now look at level surfaces of f and apply Gauß-Bonnet.

Gradient flow of f: $\frac{dp}{dt} = \nabla f(p)$.

Behavior of R along this flow:

$$\frac{d}{dt}R(p) = \langle \nabla R, \nabla f\rangle = 2\,\mathrm{Ric}(\nabla f, \nabla f)$$

(from soliton equation); this is **positive** away from the critical points of f.

Say $\overline{R} = \lim_{D\to\infty} \sup_{x:d_{-1}(x_0,x)=D} R(x) \in (0,\infty]$. What is $\overline{R}$?

Take a sequence $\{x_i\}_{i=1}^{\infty}$ going to infinity so that $\lim_{i\to\infty} R(x_i) = \overline{R}$. Now take a pointed limit of our Ricci flow $(M, (\cdot), (x_i, -1))$. Bruce $\Rightarrow$ converges to a shrinking cylinder $\mathbb{R}\times S^2$. At time -1, this S^2 has radius r, and $\overline{R}$ = scalar curvature at time -1 of $\mathbb{R}\times S^2 = \frac{2}{r^2}$. Time it takes for shrinking cylinder to disappear is $\frac{r^2}{2}$. But we know that metric $g(t)$ has to disappear at time 0, i.e. we get

$$\overline{R} = 1.$$

Now say $\{x_i\}$ is a sequence going to infinity in time -1 slice, and if $R(x_i,-1)$ converges, then the limit is also 1 (surely not >1, but if it was <1, then would get analogous contradiction via rescaling). Hence $\lim_{p\to\infty} R(p) = 1$, and $\exists K\subset M$ compact such that if $p\in M-K$, then $R(p) < 1$.

Now look at level surfaces of f: N connected component of a level surface of f, and at $p\in N$, let $X = e_3$ be a normal vector and e_1, e_2 tangent vectors. Gauß-Codazzi:

$$R^N = 2K^N(e_1,e_2) = 2(K^M(e_1,e_2) + \det S),$$

S = shape operator of N.

$$\begin{aligned} R &= 2(K^M(e_1,e_2) + K^M(e_1,e_3) + K^M(e_2,e_3)),\\ \mathrm{Ric}(X,X) &= K^M(e_1,e_3) + K^M(e_2,e_3). \end{aligned}$$

Together: $R^N = R - 2\,\mathrm{Ric}(X,X) + 2\det(S)$. What is the last term?

$$S = \frac{(\mathrm{Hess}\, f)|_{TN}}{|\nabla f|_N}, \quad \mathrm{Hess}\, f = \frac{1}{2} - \mathrm{Ric}\,.$$

Diagonalize Ric on TN:

$$\mathrm{Ric} = \begin{pmatrix} r_1 & 0 & c_1 \\ 0 & r_2 & c_2 \\ c_1 & c_2 & r_3 \end{pmatrix},$$

$\Rightarrow \det(\mathrm{Hess}\, f) \le \frac{1}{4}(1-r_1r_2)^2 = \frac{1}{4}(1-R+\mathrm{Ric}(X,X))^2$. Upshot:

$$R^N \le R - 2\,\mathrm{Ric}(X,X) + \frac{1}{2|\nabla f|^2}(1-R+\mathrm{Ric}(X,X))^2.$$

If $|\nabla f|$ is large (far out), then $1 - R + \mathrm{Ric}(X,X) < 2|\nabla f|^2$. Also, far out: $1 - R + \mathrm{Ric}(X,X) \geq 0$. Hence:

$$\begin{aligned}(1 - R + \mathrm{Ric}(X,X))^2 &< 2|\nabla f|^2(1 - R + \mathrm{Ric}(X,X)) \\ &\leq 2|\nabla f|^2(1 - R + \mathrm{Ric}(X,X)) + 2|\nabla f|^2 \mathrm{Ric}(X,X),\end{aligned}$$

so far out

$$\frac{(1 - R + \mathrm{Ric}(X,X))^2}{2|\nabla f|^2} < 1 - R + 2\,\mathrm{Ric}(X,X).$$

Conclusion: $R^N < 1$ for level surfaces far away from x_0.

Say Y is tangential to N. $\nabla_Y \nabla_Y f = \frac{1}{2} - \mathrm{Ric}(Y,Y) \geq \frac{1}{2} - \frac{R}{2} > 0$ far away from x_0, hence N is strictly convex. As the level increases, the area of N is strictly increasing. In the limit, area(N) approaches area(S^2 with radius $\sqrt{2}$) (same limiting argument as before), and topologically, N is an S^2, if we go far enough out.

From Gauß-Bonnet: $8\pi = \int_N R^N\, dA$, but $R^N < 1$ and $\int_N\, dA < 8\pi$ for N far enough out. □ (Lemma)

(Intuition: Show that the guy looks like a cylinder, but a cylinder cannot have positive curvature.)

Classification of orientable 3D κ-solutions

I) Sectional curvature not strictly positive: $\tilde{M}$ splits off a line, i.e. it is standard shrinking cylinder. Hence M is $\mathbb{R} \times S^2$, $\mathbb{R} \times_{\mathbb{Z}_2} S^2$, NOT $S^1 \times S^2$, $S^1 \times_{\mathbb{Z}_2} S^2$ (not κ-noncollapsed; go back in time!).

II) Sectional curvature > 0.

- A) noncompact case: Cheeger $\Rightarrow$ diffeomorphic to $\mathbb{R}^3$, Bruce $\Rightarrow$ B^3 cap + collar consisting of necklike regions. Example: Bryant soliton.
- B) compact: diffeomorphic to S^3/Γ
  - a) asymptotic soliton also compact. Then we have S^3/Γ, isometrically, as an asymptotic soliton, so also M itself must be a round shrinking S^3/Γ (way back in the past ($t \ll 0$), M looks awfully round, flowing forward makes it even rounder, and taking $t \to -\infty$, we see that actually M itself is round).
  - b) asymptotic soliton is noncompact.
    - i) it is $\mathbb{R} \times S^2$: M looks like a 3D Rosenau solution (each cap is either B^3 or $\mathbb{R}P^3 - B^3$, but not both caps are $\mathbb{R}P^3 - B^3$, since otherwise $\#\pi_1 = \infty$).
    - ii) $\mathbb{R} \times_{\mathbb{Z}_2} S^2$

Corollary: Any compact oriented κ-solution is

- a) diffeomorphic to S^3
- b) diffeomorphic to $\mathbb{R}P^3$
- c) isometric to S^3/Γ. □ (Corollary)

Compact M^3, oriented; run Ricci flow. Suppose it goes singular at time $T < \infty$. $\lim_{t\to T} \sup_x |\operatorname{Rm}|(x,t) = \infty$, Hamilton-Ivey $\Rightarrow \lim_{t\to T} \sup_x R(x,t) = \infty$.

(12.1): If $R(x,t) \geq R_0$, a parabolic neighborhood of (x,t) is modelled by a κ-solution.

Suppose that $\lim_{t\to\infty} R(x,t) = \infty$ for **all** $x \in M$. Right before the singularity time t, $R(x,t) \gg 0$ for all x. Possibilities for the geometry near (x,T): modelled by a parabolic neighborhood in a κ-solution.

1.) M is contained in that neighborhood: M diffeomorphic to S^3/Γ.
2.) otherwise, neighborhood looks like ε-cap or like ε-neck. What could M look like? $S^2 \times S^1$, or tube with two ends, closed by caps.

Conclusion: M is diffeomorphic to $S^1 \times S^2$ or S^3/Γ or $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Otherwise have to do surgery. Eventually, everything should become singular everywhere $\Rightarrow$ know what it is!

Claim: If M oriented compact is a # of nonaspherical irreducible 3-manifolds, then it is diffeomorphic to a # of $S^1 \times S^2$'s and S^3/Γ's.

Corollary: Poincaré conjecture.

Still have to discuss surgery procedure.

– no lectures by Bruce and John on July 11, July 12 –

Wednesday, July 13 (John Lott)

Today: $\mathcal{W}$ (I.3) (because Tian needs it)

$$\mathcal{F}(g,f) = \int_M (|\nabla f|^2 + R)e^{-f}\,d\mathrm{vol};$$

this is constant in time along a steady soliton. Converse also true.

Now $\tau > 0$,

$$\mathcal{W}(g,f,\tau) = \int_M \big(\tau(|\nabla f|^2 + R) + f - n\big)\,(4\pi\tau)^{-\frac{n}{2}} e^{-f}\,d\mathrm{vol}.$$

Note: $\mathcal{W}(\varphi^* g, \varphi^* f, \tau) = \mathcal{W}(g,f,\tau)$, $\mathcal{W}(cg, f, c\tau) = \mathcal{W}(g,f,\tau)$ $(c>0)$.

Consequence: Along a gradient shrinking soliton defined on $(-\infty,0)$, $\mathcal{W}(g(t), f(t), -t)$ is constant in t $(g(t) = -t\varphi_t^* g(-1),\ f(t) = \varphi_t^*(-1),\ \tau(t) = -t)$.

Example: flat $\mathbb{R}^n$, gradient shrinking soliton, $f = \frac{|x|^2}{4\tau}$. Compute:

$$\mathcal{W} = -(4\pi\tau)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \mathrm{div}\left(e^{-\frac{|x|^2}{4\tau}}\vec{x}\right) = 0.$$

$\square$

$\mathcal{W}(g,f,\tau)$, $\delta g_{ij} = v_{ij}$, $\delta\tau = \sigma$,

$$\delta\mathcal{W} = \int_M \Bigg(\sigma(|\nabla f|^2 + R) - \tau v_{ij}(R_{ij} + \nabla_i\nabla_j f) + h + \\ + \Big(\tau(2\Delta f - |\nabla f|^2 + R) + f - n\Big)\left(\frac{v}{2} - h - \frac{n\sigma}{2\tau}\right)\Bigg)(4\pi\tau)^{-\frac{n}{2}} e^{-f}\,d\mathrm{vol}.$$

To kill the bad term: $\delta\left((4\pi\tau)^{-\frac{n}{2}} e^{-f}\,d\mathrm{vol}\right) = ?$

$$\begin{aligned}
\delta \ln\left((4\pi\tau)^{-\frac{n}{2}} e^{-f}\,d\mathrm{vol}\right) &= \delta\left(-\frac{n}{2}\ln(4\pi\tau) - f + \ln d\mathrm{vol}\right) \\
&= -\frac{n\sigma}{2\tau} - h + \frac{1}{2}v \\
\Rightarrow \delta\left((4\pi\tau)^{-\frac{n}{2}} e^{-f}\,d\mathrm{vol}\right) &= \left(-\frac{n\sigma}{2\tau} - h + \frac{v}{2}\right)(4\pi\tau)^{-\frac{n}{2}} e^{-f}\,d\mathrm{vol}.
\end{aligned}$$

Let's now fix $(4\pi\tau)^{-\frac{n}{2}} e^{-f}\,d\mathrm{vol} \equiv dm$, smooth measure with total mass $= 1$. Then

$$\delta\mathcal{W} = \int_M \big(\sigma(|\nabla f|^2 + R) - \tau\langle v, \mathrm{Ric} + \nabla^2 f\rangle + g\big)\,(4\pi\tau)^{-\frac{n}{2}} e^{-f}\,d\mathrm{vol}.$$

Take $\frac{\partial g}{\partial t} = -2(\mathrm{Ric} + \nabla^2 f)$, $\frac{\partial f}{\partial t} = -\Delta f - R + \frac{n}{2\tau}$, $\frac{d\tau}{dt} = -1$. Then

$$\begin{aligned}
\frac{d\mathcal{W}}{dt} &= \int_M \left((-1)(|\nabla f|^2 + R) + 2\tau\left|\mathrm{Ric} + \nabla^2 f\right|^2 - \Delta f - R + \frac{n}{2\tau}\right)(4\pi\tau)^{-\frac{n}{2}} e^{-f}\,d\mathrm{vol} \\
&= \int_M 2\tau\left|\mathrm{Ric} + \nabla^2 f - \frac{g}{2\tau}\right|^2 (4\pi\tau)^{-\frac{n}{2}} e^{-f}\,d\mathrm{vol}.
\end{aligned}$$

I.e.: If $\mathcal{W}$ is constant along the flow, then we are on a gradient shrinking soliton.

Modify flow by Lie derivatives:

$$\frac{\partial g}{\partial t} = -2\operatorname{Ric}, \quad \frac{\partial f}{\partial t} = |\nabla f|^2 - \Delta f - R + \frac{n}{2\tau}, \quad \frac{d\tau}{dt} = -1$$

(i.e. $(-\frac{\partial}{\partial t} - \Delta + R)((4\pi\tau)^{-\frac{n}{2}}e^{-f}) = 0$, backwards heat equation), then

$$\frac{d\mathcal{W}}{dt} = \int_M 2\tau(4\pi\tau)^{-\frac{n}{2}}e^{-f}\left|\cdots\right|^2 \, d\mathrm{vol}.$$

No longer have $dm \equiv (4\pi\tau)^{-\frac{n}{2}}e^{-f}\, d\mathrm{vol}$, but the mass of RHS still equals 1.

Definition: $\mu(g,\tau) := \inf_f\{\mathcal{W}(g,f,\tau) : \int_M (4\pi\tau)^{-\frac{n}{2}}e^{-f}\, d\mathrm{vol} = 1\}$

Fact: $\exists$ unique minimizing $f \in C^\infty$!

Proposition: shrinking breather $\Rightarrow$ shrinking gradient soliton.

Pre-Prop: $\mu(g(t), t_0 - t)$ is monotonically nondecreasing in t along a Ricci flow.

Proof: Say $t_1 < t_2 < t_0$. Find $f(t_2)$ which minimizes $\mathcal{W}(g(t_2), \cdot, t_0 - t_2)$, etc. $\square$

Proof of prop: How to choose t_0? If the shrinking factor is $c < 1$, and if we choose t_0 such that the linear function which is $= 1$ at t_1 and $= c$ at t_2 vanishes at t_0, then $\mu(g(t_2), \tau_2) = \mu(c\varphi^* g(t_1), c\tau_1) = \mu(\varphi^* g(t_0), \tau_1) = \mu(g(t_1), \tau_1)$. $\square$

Say $g(\cdot)$ is a Ricci flow solution defined for $t \in [0, T)$.

Definition: g is κ-noncollapsed at scale ρ if $\forall r < \rho$:

$$|\operatorname{Rm}| \le \frac{1}{r^2} \text{ on } B_{t_0}(x_0, r) \;\Rightarrow\; \mathrm{vol}(B_{t_0}(x_0, r)) \ge \kappa r^n.$$

Theorem: For some closed $(M, g(0))$, say we have Ricci flow on $0 \le t < T < \infty$. Then for all $\rho > 0$ there is $\kappa = \kappa(\rho)$: solution is κ-noncollapsed at scale ρ. ($\kappa = \kappa(\rho, g(0), T) \to 0$ as $T \to \infty$).

Proof: Suppose not. Then have a sequence $\{r_k\}_{k=1}^\infty$ in $(0, \rho)$ and (x_k, t_k) so that $|\operatorname{Rm}| \le r_k^{-2}$ on $B_{t_k}(x_k, r_k) =: B_k$, but $r_k^{-1}\mathrm{vol}(B_k)^{\frac{1}{n}} = \varepsilon_k \to 0$ $(k \to \infty)$.

Note: Can assume $\lim_{k\to\infty} t_k = T$.

1.) Want to show: $\lim_{k\to\infty} \mu(g(t_k), r_k^2) = -\infty$.

Let's find some f so that $\mathcal{W}(g(t_k), f, r_k^2)$ is very negative. Idea: Take f so that $e^{-f} = e^{-c_k}e^{-d_{t_k}(\cdot, x_k)^2/4r_k^2}$, more precisely,

$$e^{-f} = e^{-c_k}\varphi\left(\frac{d_{t_k}(\cdot, x_k)^2}{4r_k^2}\right).$$

c_k is determined by $1 = \int_M (4\pi r_k)^{-\frac{n}{2}}e^{-c_k}e^{-\frac{d_{t_k}(x,x_k)^2}{4r_k^2}}\, d\mathrm{vol}(x)$. Compute this radially around x_k, get $1 \approx e^{-c_k}r_k^{-n}\mathrm{vol}(B_k)$, i.e. $e^{c_k} \approx \varepsilon_k^n$, so $c_k \to -\infty$.

Now: $\mathcal{W}(g(t_k), f, r_k^2$, computed radially around x_k. Leading term:

$$\int_M (4\pi r_k^2)^{-\frac{n}{2}} f_k e^{-f_k}\ d\mathrm{vol} \approx c_k \to -\infty.$$

2.) Monotonicity $\Rightarrow \mu(g(0), t_k + r_k^2) \leq \mu(g(t_k), r_k^2)$. Take $k \to \infty$. LHS stays bounded, since $t_k + r_k^2 \in [0, T + \rho^2]$. $\square$ (Theorem)

More equations about $\mathcal{W}$

(I.5): Go back to picture where $(4\pi\tau)^{-\frac{n}{2}} e^{-f}\ d\mathrm{vol} \equiv dm$,

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric} + \nabla^2 f), \quad \frac{\partial f}{\partial t} = -\Delta f - R + \frac{1}{2t}.$$

Then:

$$\mathcal{W} = \frac{d}{d\tau}\left(\tau \int_M \left(f - \frac{n}{2}\right)\ dm\right).$$

Now go to picture where

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}, \quad \frac{\partial f}{\partial t} = |\nabla f|^2 - R + \frac{n}{2\tau}.$$

Then:

$$\mathcal{W} = \frac{d}{d\tau}\left(\tau \int_M \left(f - \frac{n}{2}\right)(4\pi\tau)^{-\frac{n}{2}} e^{-f}\ d\mathrm{vol}\right).$$

$\square$

(I.9): Differential Harnack inequality for backward heat flow. T fixed. $u = (4\pi(T-t))^{-\frac{n}{2}} e^{-f}$ satisfies $\square^* u = -u_t - \Delta u + Ru = 0$ (conjugate heat operator) if we have Ricci flow on $[0, T)$. Put

$$v = \big((T-t)(2\Delta f - |\nabla f|^2 + R) + f - n\big) \cdot u.$$

Then $\mathcal{W} = \int_M v\ d\mathrm{vol}$, and

$$\square^* v = -2(T-t)\left|\mathrm{Ric} + \nabla^2 f - \frac{1}{2(T-t)} g\right|^2 \cdot u,$$

and hence $\frac{d\mathcal{W}}{dt} \geq 0$ (another type of deriving monotonicity; local version). $\square$

Thursday, July 14 (John Lott)

Minor point related to Tian: $\frac{d\hat{g}}{ds} = -2\,\mathrm{Ric}(\hat{g})$,

$$g(t) = e^t \hat{g}(\frac{1}{2}(1 - e^{-t}),$$

then $\frac{dg}{dt} = -\mathrm{Ric} + g$. Had: $\mu(\hat{g}(u), \frac{1}{2} - u)$ nondecreasing in u. Take $u = \frac{1}{2}(1 - e^{-t})$: $\mu(e^{-t}g(t), \frac{1}{2}e^{-t}) = \mu(g(t), \frac{1}{2})$ nondecreasing in t. $\square$

Proposition: $\exists \kappa_0 > 0$ such that any κ-solution is either a round shrinking S^3/Γ or a κ_0-solution.

Proof: Look at asymptotic soliton. If it's compact, it's S^3/Γ. If not, it's $\mathbb{R} \times S^2$ or $\mathbb{R} \times_{\mathbb{Z}_2} S^2$. Take $(p, 0)$ in the solution. $\forall \tau > 0$ $\exists q(\tau)$ in the time $-\tau$ slice such that $l(q(\tau), \tau) \leq \frac{n}{2}$. Asymptotic soliton is rescaling of solution around points $(q(\tau_i), -\tau_i)$, so we have a backward neighborhood close to $\mathbb{R} \times S^2$. In proof of κ-noncollapse, only this neighborhood matters. $\square$

Corollary: $\exists \eta > 0$ so that if (x, t) is a point in any κ-solution, then $|\frac{\partial R^{-1}}{\partial t}| \leq \eta$, $|\nabla R^{-1/2}(x, t)| \leq \eta$.

Proof: Obvious in S^3/Γ case. Else use precompactness of κ_0-solutions. $\square$

Definitions: $B(x, t, r)$ = ball of radius r around x at time t. $P(x, t, r, \Delta t) = B(x, t, r) \times [t, t + \Delta t]$ $(\Delta t > 0)$ (or $\times [t + \Delta t, t]$ if $\Delta t < 0$). "$B(x, t, \varepsilon^{-1} r)$ is an ε-neck" means: after multiplying metric by r^{-2}, it's ε-close to $(-\varepsilon^{-1}, \varepsilon^{-1}) \times S^2$ (S^2 of scalar curvature 1). $P(x, t, \varepsilon^{-1} r, r^2)$ is a "strong ε-neck", if, after rescaling, it's ε-close to $(-\varepsilon^{-1}, \varepsilon^{-1}) \times S^2 \times [-1, 0]$.

Definition: Say we have a metric on $S^2 \times (-1, 1)$ so that each point is in an ε-neck. The metric is

1) an ε-tube, if R is bounded,
2) an ε-horn, if $R \to \infty$ on one end,
3) a double ε-horn, if $R \to \infty$ on both ends.

Definition: Say we have a metric on B^3 or $\mathbb{R}P^3 - \overline{B}^3$ so that outside of a compact set, each point is in an ε-neck. It is

1) an ε-cap, if R is bounded,
2) a capped ε-horn, if $R \to \infty$ on one end.

Proposition: $\forall$ small $\varepsilon > 0$ $\exists$ $C = C(\varepsilon) > 0$: if M is a κ-solution and (x, t) is a spacetime point, then $\exists$ $r \in (C^{-1}R(x, t)^{-1/2}, CR(x, t)^{-1/2})$ and a neighborhood B of x with $B(x, t, r) \subset B \subset B(x, t, 2r)$ so that

1) B is a strong ε-neck,
2) B is an ε-cap,
3) B is diffeomorphic to S^3 or $\mathbb{R}P^3$,
4) B is a round shrinking S^3/Γ.

Proof: Can assume κ_0-solution. Suppose not. Get a sequence $(M_i, (x_i, t_i))$ of κ_0-solutions so that for $r \in (C_i^{-1} B(x_i, t_i)^{-1/2}, C_i R(x_i, t_i)^{-1/2})$, there's no B squeezed between $B(x_i, r, t_i)$ and $B(x_i, 2r, t_i)$ which satisfies one of 1)–4).

Rescale so that $R(x_i, t_i) = 1$. Take a sublimit $(M_\infty, (x_\infty, 0))$. For all $r \in (0, \infty)$, there's no B squeezed between $B(x_\infty, 0, r)$ and $B(x_\infty, 0, 2r)$ which satisfies one of 1)–4).

If M_∞ is compact, take $r = 2\,\mathrm{diam}(M_\infty)$. $B = M_\infty$ satisfies 3) or 4).

If M_∞ is not compact, from I.11.8, $\exists r$ so that there's a B satisfying 1) or 2). □

M^3 compact oriented; run Ricci flow. Say we hit a first singularity at time $T < \infty$. From (12.1) $\exists r > 0$ so that for (x, t) with $R(x, t) \geq r^{-2}$, a rescaled neighborhood of (x, t) is

1) a strong ε-neck,
2) an ε-cap,
3) diffeomorphic to S^3/Γ.

If 3), drop it.

Know: $|\frac{\partial R^{-1}}{\partial t}| \leq \eta$, $|\nabla R^{-1/2}| \leq \eta$.

Definition: $\Omega := \{x \in M : \lim_{t \to T^-} R(x, t) < \infty\}$

Proposition: Ω is open in M.

Proof: Follows from gradient estimate. □

If $\Omega = \emptyset$, at times close to T, M is covered by ε-necks and ε-caps. Each component of M is $S^1 \times S^2$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$ (threw away any S^3/Γ's).

Say $\Omega \neq \emptyset$. Put $\overline{g} := \lim_{t \to T^-} g(t)|_\Omega$, a smooth metric on Ω.

Definition: Given $\rho < r$, put $\Omega_\rho := \{x \in \Omega : R(x, t) \leq \frac{1}{\rho^2}\}$, a compact subset of M.

Take an ε-neck in Ω, look at one boundary component. If it hits Ω_ρ, stop. Otherwise, there's an adjacent ε-neck or ε-cap. If it goes on forever, ε-neck is adjacent to an ε-horn.

Conclusion: Each ε-neck in $\Omega - \Omega_\rho$ is contained in an

(a) ε-tube with boundary hitting Ω_ρ,
(b) ε-cap with boundary hitting Ω_ρ,
(c) ε-horn with boundary hitting Ω_ρ,
(d) capped ε-horn,
(e) double ε-horn.

Start with $\Omega_\rho \subset M$ compact. It hits a finite # of components of Ω (because of volume bounds). The ends of the components that it his are all ε-horns. The other components of Ω (that don't hit Ω_ρ) are either capped ε-horns or double ε horns.

Surgery

Start with Ω_ρ. Look at components of Ω that hit Ω_ρ. Truncate each ε-horn and add a ball (topologically – which is not yet precise enough!). Call result M'. Then M can be reconstructed from M' by taking # of components of M', or connected sums with additional $S^1 \times S^2$'s or $\mathbb{R}P^3$'s (reason: look at M right before T). I.e., if we could geometrize M', we could geometrize M.

- How to continue the flow?
- Can avoid accumulation of surgery times?

Want to specify how to do the surgery analytically.

Example: $dr^2 + r^{2\alpha} g_{S^2}$, $\alpha > 1$

ε-horns have a self-improving property as you go into the cusp.

Lemma: $\forall \delta > 0\ \exists h > 0$ so that if x is in an ε-horn of $(\Omega, \overline{g})$ that hits Ω_ρ, and if $h(x) \equiv \overline{R}(x)^{-1/2} < h$, then $P(x, t, \delta^{-1}h(x), -h^2(x))$ is a strong δ-neck.

Proof: Suppose not. Take x_i going into horn which are counterexamples, $h(x_i) \to 0$. Rescale at (x_i, t) and take sublimit (get curvature bounds from step 2 of I.12.1). Get limit (M_∞, x_∞), nonnegative curvature.

x_∞ is in an ε-neck of M_∞. Have a geodesic in Ω going from Ω_ρ into the horn. This becomes a **line** in M_∞ (!), so $M_\infty = \mathbb{R} \times S^2$, for some metric on S^2. Extend backwards in time (easier than in (12.1), because of gradient bounds), hence M_∞ is a shrinking cylinder: contradiction. $\square$

Friday, July 15 (John Lott)

Surgery procedure

Surgery: on ε-horns of components of Ω that hit Ω_ρ. Pick x so that $R(x) = h^{-2}$.

(PICTURE)

on $[0, \lambda]$: original metric g
on $[\lambda, 2\lambda]$: $e^{-2f}g$, $f = f(z)$ to be specified later
on $[2\lambda, 3\lambda]$, $\varphi e^{-2f}g + (1-\varphi)h^2 g_{\text{standard}}(0)$ ($\varphi \equiv 1$ on $[0, 2\lambda]$, $\equiv 0$ on $[3\lambda, \infty)$)
further out: $g_{\text{standard}}(0)$

Parameters f, λ to be adjusted in such a way that we don't mess up the curvature pinching. Take $f = c_0 e^{-p/(z-\lambda)}$ for some c_0, p.

Claim: If you choose λ, p, c_0 carefully, Hamilton-Ivey only improves (done by Hamilton in 4D).

Also, a ball of radius $(\delta')^{-1}h$ around the tip is δ'-close to ball in standard solution, $\delta' = \delta'(\delta)$, $\delta' \to 0$ as $\delta \to 0$.

Continue Ricci flow, hit first singularity, do surgery, continue, ...

One issue: How do you know surgery times don't accumulate?

Use volume! Normalized initial conditions $\Rightarrow$

$$R(x,t) \geq -\frac{3}{2}\frac{1}{t+\frac{1}{4}} \;\Rightarrow\; \frac{d}{dt}\text{vol} = -\int R\, d\text{vol} \leq \frac{3}{2}\frac{\text{vol}}{t+\frac{1}{4}} \;\Rightarrow\; \text{vol}(t) \leq \text{vol}(0)(1+4t)^{\frac{3}{2}};$$

also true when we have surgeries (they're vol-decreasing). Need to estimate the loss of volume in a surgery from below.

- Doing a surgery removes vol $\geq \mathit{const} \cdot h^3$

If we know that $h = h(t) \geq \mathit{const} \cdot h^3$, we could conclude that there's only a finite number of surgeries (claim in I, probably unjustified). On the other hand, if $h = h(t)$ is any continuous function of t, you have a finite number of surgeries on any time interval $[t_1, t_2]$ (claim in II).

What conditions do we need in order to do surgeries?

- Need κ-noncollapsing up to any given time.

In smooth case (i.e. without singularities), showed κ-noncollapsing at (p,t) by saying: $\exists q$ so that $l(q, T-1) \leq \frac{3}{2}$, $\overline{L}_\tau + \Delta\overline{L} \leq 6$ $(*)$.

With surgeries: In order to apply maximum principle to $(*)$, need to know: for some $\varepsilon > 0$, any curve γ going through a surgery cap has $\frac{1}{\sqrt{2\overline{\tau}}} L(\gamma) \geq \frac{3}{2} + \varepsilon$.

$$L(\gamma) = \int_0^{\overline{\tau}} \sqrt{\tau} \left(\left| \frac{d\gamma}{d\tau} \right|^2 + R(\gamma(\tau)) \right) d\tau.$$

Idea: If surgery radius is small enough, then R very big near a surgery cap, so minimizing γ should avoid it. Choose $h = h(t)$ to make this work!

Problem: If we fix surgery radii up to time t, if (p, t_0) has $t_0 \gg t$, curves from (p, t_0) back to time 0 can hit large regions with $R < 0$.

Perelman's idea to resolve this circle: Divide $[0, \infty)$ into time intervals $[2^i \varepsilon, 2^{i+1} \varepsilon)$. To prove κ-noncollapsedness in a given time interval, estimate l in previous time interval.

More precisely: Get time-dependent surgery parameters

1) $r(t)$, "canonical neighborhood scale": If $R(x,t) \geq r(t)^{-2}$, then a parabolic neighborhood of (x,t) is close to corresponding neighborhood of an ancient solution.
2) $\rho(t) = \delta(t) r(t)$, used to define Ω_ρ.
3) $h(t) \leq \delta(t) \rho(t)$, surgery radius.
4) $\kappa(t)$, noncollapsing parameter.

Claim (Perelman II): $\exists$ continuous functions $r(t), \delta(t), \kappa(t)$ so that we get a well-defined Ricci flow with surgery starting from any metric satisfying normalized initial conditions. It exists for all time unless solution goes extinct.

Induction on time: $2^i \varepsilon$, $2^{i+1} \varepsilon$, ... I.e., you **have** to redefine parameters, but only taking care of previous time interval (one step back – two forward).

Sufficient conditions for finite time extinction:

1) $R > 0$, $\Rightarrow$ $\frac{dR_{\min}}{dt} \geq \frac{2}{3} R_{\min}^2$ $\Rightarrow$ $R_{\min}$ goes to ∞ before time $\frac{2}{3} R_{\min}(0)$ (unaffected by surgeries, which don't change $R_{\min}$).
2) M is a # of nonaspherical irreducible 3-manifolds (Perelman III, Colding-Minicozzi).

In either case, M is diffeomorphic to a # of S^3/Γ's, $S^1 \times S^2$'s, whence Poincaré.

Suppose Ricci flow does not go extinct.

Claim: For large t, there's a decomposition $M_{\text{thick}} \cup M_{\text{thin}} = M$ so that

1) $\operatorname{int}(M_{\text{thick}})$ admits (!) a complete finite volume hyperbolic metric
2) M_{thin} is locally collapsed: $\forall x \in M_{\text{thin}}$ $\exists \rho = \rho(x)$ such that $\mathrm{Rm} \geq -\rho^{-2}$ on $B(x, \rho)$ and $\operatorname{vol}(B(x, \rho)) \leq w \rho^3$, w a small number
3) decomposition is along incompressible 2-tori.

Note: Could have an infinite number of surgeries, but they only happen in the thin part and are topologically trivial!

1), 3) use earlier work of Hamilton: M_{thick} is characterized by two-sided curvature bounds.

Proposition (Perelman, Shioya-Yamaguchi): $\exists$ small w such that in case 2), M_{thin} is a graph manifold. (Well known for 2-sided curvature bounds!)

$\Rightarrow$ geometrization conjecture