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NOTE ON ASYMPTOTICALLY CONICAL EXPANDING RICCI SOLITONS

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Abstract. We show that at the level of formal expansions, any compact Riemannian manifold is the sphere at infinity of an asymptotically conical gradient expanding Ricci soliton.

1. Introduction

When looking at Ricci flow on noncompact manifolds, the asymptotically conical geometries are especially interesting. An asymptotically conical Riemannian manifold (M, g_0) is modelled at infinity by its asymptotic cone $C(Y)$. We take the link Y to be a compact manifold with Riemannian metric h. If $\star$ is the vertex of $C(Y)$, then the Riemannian metric on $C(Y) - \star$ is $dr^2 + r^2 h$, with $r \in (0, \infty)$.

Suppose that there is a Ricci flow solution $(M, g(t))$ on M that exists for all $t \geq 0$, with $g(0) = g_0$. One can analyze the large time and large distance behavior of the flow by parabolic blowdowns. With a suitable choice of basepoints, there is a subsequential blowdown limit flow $g_\infty(\cdot)$ that is defined at least on the subset of $C(Y) \times [0, \infty)$ given by $\{(r, \theta, t) \in (0, \infty) \times Y \times [0, \infty) \, : \, t \leq \epsilon r^2\}$, for some $\epsilon > 0$ [3, Proposition 5.6]. For each $t > 0$, the metric $g_\infty(t)$ is asymptotically conical, with asymptotic cone $C(Y)$.

Since $g_\infty(\cdot)$ is a blowdown limit, the simplest scenario is that it is self-similar in the sense that it is an expanding Ricci soliton flow coming out of the cone $C(Y)$. This raises the question of whether such an expanding soliton exists for arbitrary choice of (Y, h). Note that the relevant expanding solitons need not be smooth and complete. For example, if (M, g_0) is an asymptotically conical Ricci flat manifold, then the blowdown flow g_∞ is the static Ricci flat metric on $C(Y) - \star$; this is an expanding soliton, although $C(Y)$ may not be a manifold.

The equation for a gradient expanding Ricci soliton (M, g), with potential f, is

$$\mathrm{Ric} + \mathrm{Hess}(f) \ = \ - \, \frac{1}{2} \, g. \tag{1.1}$$

The main result of this paper says that any (Y, h) is the sphere at infinity of an asymptotically conical gradient expanding Ricci soliton, at least at the level of formal expansions.

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Theorem 1.1. *Given a compact Riemannian manifold (Y, h), there is a formal solution to* (1.1) *on $(0, \infty) \times Y$ of the form*

$$g = dr^2 + r^2 h + h_0 + r^{-2} h_2 + \cdots + r^{-2i} h_{2i} + \cdots , \tag{1.2}$$
$$f = -\frac{1}{4} r^2 + f_0 + r^{-2} f_2 + \cdots + r^{-2i} f_{2i} + \cdots ,$$

where h_{2i} is a symmetric 2-tensor field on Y and $f_{2i} \in C^\infty(Y)$. The solution is unique up to adding a constant to f_0.

When writing (1.1) in the $(0, \infty) \times Y$ decomposition, one obtains two evolution equations and a constraint equation. The main issue in proving Theorem 1.1 is to show that solutions of the evolution equations automatically satisfy the constraint equation.

There has been earlier work on asymptotically conical expanding solitons.

(1) Schulze and Simon considered the Ricci flow on an asymptotically conical manifold with nonnegative curvature operator [5]. They showed that there is a long-time solution and its blowdown limit is a gradient expanding soliton solution.
(2) Deruelle showed that if (Y, h) is simply connected and $C(Y) - \star$ has nonnegative curvature operator, then there is a smooth gradient expanding Ricci soliton (M, g, f) with asymptotic cone $C(Y)$ [1].
(3) In the Kähler case, the analog of Theorem 1.1 was proven by the first author and Zhang [3]. The Kähler case differs from the Riemannian case in two ways. First, in the Kähler case the Ricci soliton equation reduces to a scalar equation. Second, a Kähler cone has a natural holomorphic vector field that generates a rescaling of the complex coordinates. In [3, Propositions 5.40 and 5.50] it was shown that there is a formal expanding soliton based on this vector field and then that the vector field is the gradient of a soliton potential f. In the Riemannian case there is no *a priori* choice of vector field. Instead, we work directly with the gradient soliton equation (1.1).

In what follows, we use the Einstein summation convention freely.

2. Soliton equations

Put $\dim(Y) = n$. Consider a Riemannian metric on $(0, \infty) \times Y$ given in radial form by $g = dr^2 + H(r)$. Here for each $r \in (0, \infty)$, we have a Riemannian metric $H(r)$ on Y. Letting $\{x^i\}_{i=1}^n$ be local coordinates for Y, the gradient expanding soliton equation (1.1) becomes the equations

$$R^g_{jk} + (\mathrm{Hess}_g\, f)_{jk} + \frac{1}{2} g_{jk} = 0, \tag{2.1}$$
$$R^g_{rr} + (\mathrm{Hess}_g\, f)_{rr} + \frac{1}{2} = 0,$$
$$R^g_{rl} + (\mathrm{Hess}_g\, f)_{rl} = 0.$$

After multiplying by 2, these equations can be written explicitly as

$$- H_{jk,rr} + 2 R^H_{jk} - \frac{1}{2} H^{il} H_{il,r} H_{jk,r} + H^{il} H_{kl,r} H_{ij,r} \tag{2.2}$$
$$+ 2 (\mathrm{Hess}_H\, f)_{jk} + H_{jk,r} f_{,r} + H_{jk} = 0,$$

$$-H^{jk}H_{jk,rr} + \frac{1}{2}H^{ij}H_{jk,r}H^{kl}H_{li,r} + 2f_{,rr} + 1 = 0 \tag{2.3}$$

and

$$H^{im}\left(\nabla_i H_{ml,r} - \nabla_l H_{im,r}\right) + 2f_{,rl} - H^{mn}H_{nl,r}f_{,m} = 0, \tag{2.4}$$

where the covariant derivatives are with respect to the Levi-Civita connection of $H(r)$.

We now write

$$\begin{aligned} H &= r^2 h + h_0 + r^{-2}h_2 + \cdots + r^{-2i}h_{2i} + \cdots, \\ f &= -\frac{1}{4}r^2 + f_0 + r^{-2}f_2 + \cdots + r^{-2i}f_{2i} + \cdots. \end{aligned} \tag{2.5}$$

We substitute (2.5) into (2.2)-(2.4) and equate coefficients. Using (2.4), one finds that f_0 is a constant. For $i \geq 0$ we can determine h_{2i} in terms of the quantities $\{h, h_0, \ldots, h_{2i-2}, f_0, \ldots, f_{2i}\}$ from (2.2), since the $H_{jk,r}f_{,r}$-term and the H_{jk}-term combine to give a factor of $(i+1)r^{-2i}\left(h_{2i}\right)_{jk}$. (When $i = 0$, we determine h_0 in terms of h and f_0.) And we can determine f_{2i+2} in terms of $\{h, h_0, \ldots, h_{2i}\}$ from (2.3), thanks to the $f_{,rr}$-term. Iterating this procedure, one finds

(2.6)

$$\begin{aligned} H_{jk} = {} & r^2 h_{jk} - 2\left[R_{jk} - (n-1)h_{jk}\right] \\ & + r^{-2}\left[-\Delta_L R_{jk} + \frac{1}{3}(\mathrm{Hess}_h R)_{jk} + \frac{4}{3}R h_{jk} - 4R_{jk} - 4(\frac{n}{3} - 1)(n-1)h_{jk}\right] \\ & + O(r^{-4}), \end{aligned}$$

$$\begin{aligned} f = {} & -\frac{1}{4}r^2 + \text{const.} - \frac{1}{3}r^{-2}\left[\, R - n(n-1)\right] \\ & + \frac{1}{5}r^{-4}\left[\, -\Delta R - 2|\operatorname{Ric}|_h^2 + 2(3n-5)R - 4(n-2)(n-1)n\right] + O(r^{-6}), \end{aligned}$$

where all geometric quantities on the right-hand side of each equation are calculated with respect to h. Here Δ_L is the Lichnerowicz Laplacian.

As f can be changed by a constant without affecting (1.1), we will assume for later purposes that the r^0-term of f is $-(n-1)$. Then the asymptotic expansion is uniquely determined by h.

By construction, the expressions that we obtain for (2.5) satisfy (2.2) and (2.3) to all orders. It remains to show that (2.4) is satisfied to all orders. Using (2.6), one can check that the left-hand side of equation (2.4) is $O\left(r^{-7}\right)$.

3. Weighted contracted Bianchi identity

Consider a general Riemannian manifold (M, g) and a function $f \in C^\infty(M)$. We can consider the triple $\left(M, g, e^{-f}\,\mathrm{dvol}_g\right)$ to be a smooth metric-measure space. The analog of the Ricci tensor for such a space is the Bakry-Emery-Ricci tensor $\operatorname{Ric} + \operatorname{Hess}(f)$.

One can ask if there is a weighted analog of the contracted Bianchi identity $\nabla^a R_{ab} = \frac{1}{2}\nabla_b R$, in which the Ricci tensor is replaced by the Bakry-Emery-Ricci tensor. It turns out that

$$\nabla^a\left(R_{ab} + \nabla_a\nabla_b f\right) - \left(\nabla^a f\right)\left(R_{ab} + \nabla_a\nabla_b f\right) = \frac{1}{2}\nabla_b\left(R + 2\Delta f - |\nabla f|^2\right). \tag{3.1}$$

One recognizes $R + 2\Delta f - |\nabla f|^2$ to be Perelman's weighted scalar curvature [4, Section 1.3].

A slight variation of (3.1) is

$$\nabla^a \left(R_{ab} + \nabla_a \nabla_b f + \frac{1}{2} g_{ab} \right) - (\nabla^a f) \left(R_{ab} + \nabla_a \nabla_b f + \frac{1}{2} g_{ab} \right) = \frac{1}{2} \nabla_b \left(R + 2\Delta f - |\nabla f|^2 - f \right). \tag{3.2}$$

A corollary is the known fact that if (M, g, f) is a gradient expanding Ricci soliton, then $R + 2\Delta f - |\nabla f|^2 - f$ is a constant. By adding this constant back to f, we can assume that the soliton satisfies $R + 2\Delta f - |\nabla f|^2 - f = 0$.

4. Proof of Theorem 1.1

If we substitute an asymptotic expansion like (2.5) into (3.2), then it will be satisfied to all orders. Returning to the variables $\{r, x^1, \ldots, x^n\}$, let us write $X_{ir} = R_{ir} + \nabla_i \nabla_r f$ and $S = R + 2\Delta f - |\nabla f|^2 - f$. If we assume that equations (2.2) and (2.3) are satisfied, then (3.2) gives

$$\nabla^i X_{ir} - (\nabla^i f) X_{ir} = \frac{1}{2} \partial_r S \tag{4.1}$$

and

$$\nabla_r X_{ir} - (\partial_r f) X_{ir} = \frac{1}{2} \partial_i S, \tag{4.2}$$

where the covariant derivatives on the left-hand side are with respect to the Levi-Civita connection of g. Rewriting in terms of covariant derivatives with respect to the Levi-Civita connection of $H(r)$, the equations become

$$H^{ij} \left[\nabla_j X_{ir} - (\partial_j f) X_{ir} \right] = \frac{1}{2} \partial_r S \tag{4.3}$$

and

$$\partial_r X_{ir} - \frac{1}{2} H^{jk} H_{ki,r} X_{jr} - (\partial_r f) X_{ir} = \frac{1}{2} \partial_i S. \tag{4.4}$$

Lemma 4.1. *If S vanishes to all orders in r^{-1}, then X_{ir} vanishes to all orders in r^{-1}.*

Proof. Suppose, by way of contradiction, that $X_{ir} = r^{-N} \phi + O\left(r^{-N-1}\right)$ for some $N \geq 1$ and some nonzero $\phi \in \Omega^1(Y)$. Using the leading order asymptotics for H and f from (2.6), the left-hand side of (4.4) is $\frac{1}{2} r^{-N+1} \phi_i + O\left(r^{-N}\right)$. As the right-hand side of (4.4) vanishes to all orders, we conclude that $\phi = 0$, which is a contradiction. This proves the lemma. $\square$

We now prove Theorem 1.1. It suffices to show that X_{ir} vanishes to all orders. Suppose, by way of contradiction, that $X_{ir} = r^{-N} \phi + O\left(r^{-N-1}\right)$ for some $N \geq 1$ and some nonzero $\phi \in \Omega^1(Y)$. From Lemma 4.1, S does not vanish to all orders. Hence $S = r^{-M} \psi + O\left(r^{-M-1}\right)$ for some $M \geq 1$ and some nonzero $\psi \in C^\infty(Y)$. Using the leading order asymptotics for H and f from (2.6), the left-hand side of (4.4) is $\frac{1}{2} r^{-N+1} \phi_i + O\left(r^{-N}\right)$. The right-hand side of (4.4) is $\frac{1}{2} r^{-M} \partial_i \psi + O\left(r^{-M-1}\right)$. Since ϕ is nonzero, we can say that $M \leq N - 1$.

Next, the left-hand side of (4.3) is $r^{-N-2}h^{ij}\nabla_j\phi_i + O\left(r^{-N-3}\right)$, while the right-hand side of (4.3) is $-\frac{1}{2}Mr^{-M-1}\psi + O\left(r^{-M-2}\right)$. Since ψ is nonzero, we can say that $M \geq N+1$. This is a contradiction and proves the theorem.

Remark 4.2. Consider the quantities

$$R^g_{rr} + (\mathrm{Hess}_g\, f)_{rr} + \frac{1}{2} - \frac{1}{2}\left(R^g + 2\triangle_g f - |\nabla f|^2_g - f\right) \tag{4.5}$$

and $R^g_{rl} + (\mathrm{Hess}_g\, f)_{rl}$. Without assuming that the gradient expanding soliton equations are satisfied, one finds that these quantities only involve first derivatives of r. In this sense, the vanishing of these quantities on a level set of r is like the constraint equations in general relativity. As a nonasymptotic statement, if (2.2) and (2.3) hold and the aforementioned quantities all vanish on one level set of r, then from (4.3) and (4.4), they vanish identically.

Remark 4.3. Asymptotic expansions can also be constructed for asymptotically conical gradient shrinking solitons. The leading term in the function f becomes $\frac{1}{4}r^2$. The (nonasymptotic) uniqueness, in a neighborhood of the end, was shown in [2].

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References

[1] Alix Deruelle, *Smoothing out positively curved metric cones by Ricci expanders*, Geom. Funct. Anal. **26** (2016), no. 1, 188–249, DOI 10.1007/s00039-016-0360-0. MR3494489

[2] Brett Kotschwar and Lu Wang, *Rigidity of asymptotically conical shrinking gradient Ricci solitons*, J. Differential Geom. **100** (2015), no. 1, 55–108. MR3326574

[3] John Lott and Zhou Zhang, *Ricci flow on quasiprojective manifolds II*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 8, 1813–1854, DOI 10.4171/JEMS/630. MR3519542

[4] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, preprint, https://arxiv.org/abs/math/0211159 (2002).

[5] Felix Schulze and Miles Simon, *Expanding solitons with non-negative curvature operator coming out of cones*, Math. Z. **275** (2013), no. 1-2, 625–639, DOI 10.1007/s00209-013-1150-0. MR3101823

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