Mathematische Annalen (2024) 388:2787–2806
https://doi.org/10.1007/s00208-023-02596-9

Mathematische Annalen

On 3-manifolds with pointwise pinched nonnegative Ricci curvature

John Lott[1]

Received: 6 September 2019 / Revised: 17 February 2023 / Accepted: 18 February 2023 /
Published online: 3 March 2023

Abstract

There is a conjecture that a complete Riemannian 3-manifold with bounded sectional curvature, and pointwise pinched nonnegative Ricci curvature, must be flat or compact. We show that this is true when the negative part (if any) of the sectional curvature decays quadratically.

1 Introduction

Let (M, g) be a complete connected Riemannian 3-manifold. Suppose that $\mathrm{Ric}(M, g) \geq 0$. At a point $m \in M$, the Ricci tensor on $T_m M$ can be diagonalized relative to $g(m)$. Let $r_1 \leq r_2 \leq r_3$ be its eigenvalues. Given $c \in (0, 1]$, we say that (M, g) is *c-Ricci pinched* if at all $m \in M$, we have $r_1 \geq c r_3$.

Conjecture 1.1 *Let (M, g) be a complete connected Riemannian manifold of dimension three, with bounded sectional curvature and nonnegative Ricci curvature. Suppose that (M, g) is c-Ricci pinched for some $c \in (0, 1]$. Then (M, g) is flat or M is compact.*

Using basic properties of Ricci flow, one can show that Conjecture 1.1 is equivalent to the following conjecture.

Conjecture 1.2 *Let (M, g) be a complete connected Riemannian manifold of dimension three, with bounded sectional curvature and positive Ricci curvature. Suppose that (M, g) is c-Ricci pinched for some $c \in (0, 1]$. Then M is compact.*

We will think of Conjectures 1.1 and 1.2 interchangeably. They are apparently due to Hamilton, who proved a result similar to Conjecture 1.2 for hypersurfaces

Research partially supported by NSF grant DMS-1810700.

✉ John Lott
lott@berkeley.edu

1 Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

in Euclidean space [16]. Conjecture 1.2 can be considered to be a scale-invariant version of the Bonnet-Myers theorem. The latter says that if a complete Riemannian n-manifold (M, g) has $\mathrm{Ric} \geq (n-1)k^2 g$, with $k > 0$, then M is compact with diameter at most $\frac{\pi}{k}$. In Conjecture 1.2, rather than an explicit bound for the diameter, the claim is that the diameter is finite.

To get a feeling why Conjecture 1.1 might be true, consider a Riemannian manifold (M, g) with nonnegative Ricci curvature that is strictly conical outside of a compact subset. The Ricci curvature vanishes in the radial direction of the cone. The c-Ricci pinching then implies that M is Ricci-flat on the conical region and hence flat there, since the dimension is three. Then the link of the cone consists of copies of round S^2's and $\mathbb{R}P^2$'s. From the splitting theorem, the link must be connected. Since it bounds a compact 3-manifold, it must be S^2. The global nonnegativity of the Ricci curvature now implies that M is isometric to $\mathbb{R}^3$. This intuition will enter into the proof of Theorem 1.4 below.

We show that Conjectures 1.1 and 1.2 are true under an extra curvature assumption.

Theorem 1.3 *Conjecture 1.1 is true if*

(a) (M, g) *has nonnegative sectional curvature, or*
(b) (M, g) *has quadratic curvature decay.*

Theorem 1.3(a) was proven earlier in [8] using Ricci flow. Our proof also uses Ricci flow but is technically different.

Theorem 1.4 *Conjecture 1.1 is true if there is some $A < \infty$ so that the sectional curvatures of (M, g) satisfy $K(m) \geq -\frac{A}{d(m, m_0)^2}$, where m_0 is some basepoint.*

Theorem 1.4 implies Theorem 1.3 but we state it separately, since the proof of Theorem 1.4 uses results from the preprint [23].

Besides the particular results in Theorems 1.3 and 1.4, we prove more general results that may lead to a proof of Conjecture 1.1. The next proposition says that if (M, g_0) is noncompact and satisfies the hypotheses of Conjecture 1.1 then the ensuing Ricci flow exists for all positive time and is type-III.

Proposition 1.5 *Given (M, g_0) as in Conjecture 1.1 with M noncompact, there is a smooth Ricci flow solution $(M, g(\cdot))$ with $g(0) = g_0$ that exists for all $t \geq 0$. There is a constant $C < \infty$ so that $\| \mathrm{Rm}(g(t)) \|_\infty \leq \frac{C}{t}$ for all $t \geq 0$.*

The main technical result of this paper is that a three dimensional Ricci flow solution $(M, g(t))$ with positive Ricci curvature, that satisfies the conclusion of Proposition 1.5, admits a three-dimensional blowdown limit.

Proposition 1.6 *Let (M, g_0, m_0) be a complete connected pointed Riemannian manifold of dimension three, with bounded sectional curvature and positive Ricci curvature. Suppose that the ensuing Ricci flow exists for all $t \geq 0$, and that there is some $C < \infty$ so that $\| \mathrm{Rm}(g(t)) \|_\infty \leq \frac{C}{t}$ for all $t \geq 0$. For $s > 0$, put $g_s(t) = s^{-1} g(st)$. Then for some sequence $s_i \to \infty$, there is a limit $\lim_{i \to \infty} g_{s_i}(\cdot) = g_\infty(\cdot)$ in the pointed Cheeger-Hamilton topology. The Ricci flow solution $g_\infty(u)$ lives on a three dimensional manifold and is defined for $u > 0$.*

The issue in proving Proposition 1.6 is to rule out collapsing at large time. Examples of Proposition 1.6 come from expanding gradient solitons, for which the tangent cone at infinity can be the cone over any two-sphere with Gaussian curvature greater than one [11]. Of course, these are not c-Ricci pinched (Lemma 4.6).

Using distance distortion estimates, Proposition 1.6 has the following implication about the initial metric.

Corollary 1.7 *Under the hypotheses of Proposition 1.6, the Riemannian manifold* (M, g_0) *has cubic volume growth.*

The proof of Theorem 1.3(a) then uses a Ricci flow result of Simon-Schulze [35]. To prove Theorem 1.3(b) we apply a spatial rescaling argument to (M, g).

The proof of Theorem 1.4 uses Corollary 1.7 and results of [23] about weak convergence of curvature operators. Assuming that M is noncompact, we apply a spatial rescaling to (M, g) to get an locally Alexandrov three dimensional tangent cone at infinity. If (M, g) is nonflat then the weak convergence of curvature operators, along with the c-Ricci pinching, forces the tangent cone at infinity of (M, g) to be $\mathbb{R}^3$, which contradicts the nonflatness assumption.

One could ask about generalizations of Conjectures 1.1 and 1.2 without the uniform curvature bound, or in higher dimension. However, in this paper we stick with three dimensions and bounded sectional curvature. The paper [29] shows compactness in general dimension for complete Riemannian manifolds that have bounded sectional curvature and pointwise pinched positive curvature operator. Having a positive curvature operator allows the papers [8, 29] to apply the Gromoll-Meyer injectivity radius bound and Hamilton's differential Harnack inequality. As we do not make an assumption of positive curvature operator, we have to develop different tools.

The structure of the paper is the following. In Sect. 2 we prove Proposition 1.5 and give some distance distortion estimates. In Sect. 3 we prove Proposition 1.6. Section 4 has the proof of Corollary 1.7. In Sect. 5 we prove Theorem 1.3 and in Sect. 6 we prove Theorem 1.4. More detailed descriptions are at the beginnings of the sections.

I thank Nina Lebedeva for sending me a preliminary version of [23]. I also thank Christoph Böhm, Simon Brendle, Yi Lai, Man-Chun Lee, Yu Li and the referee for comments on an earlier version of this paper.

Note: After this paper was submitted for publication, Conjecture 1.1 was proved by Deruelle et al. [12]. The proof uses Propositions 1.5 and 1.6 of the present paper. The assumption of bounded sectional curvature was then removed by Lee and Topping [24].

2 Long-time existence and curvature decay

In this section we prove Proposition 1.5. We first show that the Ricci flow exists for all $t > 0$. The proof is similar to an argument in Hamilton's original Ricci flow paper [15] about what could possibly happen at a curvature blowup under the Ricci pinching assumption. When applied to long-time solutions, essentially the same argument is used to rule out type-II solutions, thereby proving the curvature bound in Proposition

1.5. Using the curvature bound, we give some distance distortion estimates that will be important in Sect. 3.

We begin by recalling some facts about Ricci flow. Let (M, g_0) be a Riemannian manifold as in the statement of Conjecture 1.1. Let $(M, g(\cdot))$ denote the unique maximal Ricci flow solution with initial time slice $g(0) = g_0$, having complete time slices and bounded curvature on compact time intervals. The condition $\mathrm{Ric} \geq 0$ is preserved under Ricci flow. Using the weak maximum principle, one can show that being c-Ricci pinched is preserved under Ricci flow. Using the strong maximum principle, if (M, g_0) is nonflat then for $t > 0$, the Ricci curvature is positive. Hence we can assume that (M, g_0) has positive Ricci curvature. This shows the equivalence between Conjecture 1.1 and Conjecture 1.2.

Under the hypotheses of Conjecture 1.2, to argue by contradiction, hereafter we also assume that M is noncompact. Then it is diffeomorphic to $\mathbb{R}^3$ [34].

Proposition 2.1 *The Ricci flow solution* $(M, g(\cdot))$ *exists for all* $t \geq 0$.

Proof We have

$$r_1 \geq c r_3 \Rightarrow r_1 \geq \frac{1}{2} c (r_2 + r_3) \Rightarrow \left(1 + \frac{1}{2} c\right) r_1 \geq \frac{1}{2} c (r_1 + r_2 + r_3). \tag{2.2}$$

Hence $\mathrm{Ric} \geq \rho R$, where $\rho = \frac{c}{2+c} \in \left(0, \frac{1}{3}\right]$, and R denotes the scalar curvature. Put $\sigma = \rho^2$.

Suppose that the maximal Ricci flow solution is on a finite time interval $[0, T)$. We claim first that for all $t \in [0, T)$, we have

$$R^{\sigma - 2} \left| \mathrm{Ric} - \frac{1}{3} R g(t) \right|^2 \leq \left(\frac{3}{2t} \right)^{\sigma} \tag{2.3}$$

everywhere on M. To prove this, we combine methods from [2, Pf. of Proposition 3] and [8, Pf of Lemma 6.1]. Put

$$f = R^{\sigma - 2} \left| \mathrm{Ric} - \frac{1}{3} R g(t) \right|^2. \tag{2.4}$$

From the bounded curvature assumption, f is uniformly bounded above at time zero. From [2, p. 539] and [8, Eqn. (76)], which are based on [15, Lemma 10.5],

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \triangle \right) f \leq\ & 2(1 - \sigma) \left\langle \frac{\nabla R}{R}, \nabla f \right\rangle - \sigma (1 - \sigma) R^{\sigma - 4} \left| \mathrm{Ric} - \frac{1}{3} R g(t) \right|^2 |\nabla R|^2 \\ & - \frac{2}{3} \sigma f^{1 + \frac{1}{\sigma}}. \end{aligned} \tag{2.5}$$

If M were compact then we could immediately derive (2.3) using the weak maximum principle, as in [2, Proposition 3]. If M is noncompact then the possible unboundedness of $\frac{\nabla R}{R}$ is an issue. To get around this, using

$$2\left\langle \frac{\nabla R}{R}, \nabla f \right\rangle \leq \sigma f \left| \frac{\nabla R}{R} \right|^2 + \frac{|\nabla f|^2}{\sigma f}, \tag{2.6}$$

we obtain

$$\left(\frac{\partial}{\partial t} - \triangle \right) f \leq \frac{1-\sigma}{\sigma} \frac{|\nabla f|^2}{f} - \frac{2}{3} \sigma f^{1+\frac{1}{\sigma}}. \tag{2.7}$$

Equivalently,

$$\left(\frac{\partial}{\partial t} - \triangle \right) f^{\frac{1}{\sigma}} \leq -\frac{2}{3} f^{\frac{2}{\sigma}} \tag{2.8}$$

in the barrier sense. From the weak maximum principle,

$$\sup_{m \in M} f^{\frac{1}{\sigma}}(m, t) \leq \frac{3}{2t}, \tag{2.9}$$

which proves the claim.

There is a sequence $\{t_i\}_{i=1}^{\infty}$ of times increasing to T, and points $\{m_i\}_{i=1}^{\infty}$ in M so that $\lim_{i\to\infty} |\operatorname{Rm}(m_i, t_i)| = \infty$ and $|\operatorname{Rm}(m_i, t_i)| \geq \frac{1}{2} \sup_{(m,t)\in M\times[0,t_i]} |\operatorname{Rm}(m, t)|$. Put $Q_i = |\operatorname{Rm}(m_i, t_i)|$ and $g_i(x, u) = Q_i g(x, t_i + Q_i^{-1}u)$. Then g_i is a Ricci flow solution with curvature norm equal to one at $(m_i, 0)$, and curvature norm uniformly bounded above by two for $u \in [-Q_i t_i, 0]$.

Suppose first that for some $i_0 > 0$ and all i, we have $Q_i \operatorname{inj}_{g(t_i)}(m_i)^2 \geq i_0$. (This does not follow from Perelman's no local collapsing result, since we do not assume that the initial metric has positive injectivity radius.) After passing to a subsequence, there is a pointed Cheeger-Hamilton limit

$$\lim_{i\to\infty} (M, g_i(\cdot), m_i) = (M_\infty, g_\infty(\cdot), m_\infty), \tag{2.10}$$

where $g_\infty(u)$ is defined for $u \in (-\infty, 0]$. The property of having nonnegative Ricci curvature passes to the limit. By construction, g_∞ has curvature norm one at $(m_\infty, 0)$. Hence g_∞ has positive scalar curvature at $(m_\infty, 0)$. By the strong maximum principle, it follows that g_∞ has positive scalar curvature everywhere.

Given $m' \in M_\infty$, the point $(m', 0)$ is the limit of a sequence of points $\{(m'_i, 0)\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} R_{g_i}(m'_i, 0) = R_{g_\infty}(m', 0) > 0$. As $\lim_{i\to\infty} Q_i = \infty$, after undoing the rescaling it follows that $\lim_{i\to\infty} R_g(m'_i, t_i) = \infty$. As $\lim_{i\to\infty} t_i = T$, we also have $\lim_{i\to\infty} t_i R_g(m'_i, t_i) = \infty$. Applying (2.3) to g_i and taking the limit as $i \to \infty$, it follows that the metric $g_\infty(0)$ satisfies $\operatorname{Ric} - \frac{1}{3} R g_\infty(0) = 0$. As $g_\infty(0)$ has positive scalar curvature at $(m_\infty, 0)$, it follows that M_∞ is a spherical space form. Then M is compact, which is a contradiction.

Even if there is no uniform positive lower bound for $Q_i \operatorname{inj}_{g(t_i)}(m_i)^2$, after passing to a subsequence, there is a pointed limit

$$\lim_{i\to\infty}(M, g_i(\cdot), m_i) = (\mathcal{G}_\infty, g_\infty(\cdot), \mathcal{O}_{x_\infty}). \tag{2.11}$$

Here $\mathcal{G}_\infty$ is a three dimensional closed Hausdorff étale groupoid and $g_\infty(\cdot)$ is a family of invariant Riemannian metrics on the unit space of $\mathcal{G}_\infty$ [25, Section 5]. Let X_∞ denote he orbit space of $\mathcal{G}_\infty$; then $\mathcal{O}_{x_\infty} \in X_\infty$ is a basepoint. The Ricci flow $g_\infty(u)$ is defined for $u \in (-\infty, 0]$. For each u, the metric $g_\infty(u)$ induces a metric on X_∞ that makes it into a complete metric space. As before, $\lim_{i\to\infty} R_g(m_i, t_i) = \infty$ and (2.3) again implies that the metric $g_\infty(0)$ satisfies $\operatorname{Ric} - \frac{1}{3} R g_\infty(0) = 0$. As $g_\infty(0)$ has positive scalar curvature along the orbit $\mathcal{O}_{x_\infty}$ in the unit space, the metric $g_\infty(0)$ has constant positive Ricci curvature. The argument for the Bonnet-Myers theorem implies that X_∞ is compact; c.f. [18, Section 2.9]. Then M is compact, which is a contradiction. □

Remark 2.12 One could avoid the use of étale groupoids by first looking at the pull-back flows on $T_{m_i}M$ and taking a limit, to argue that for large i, the metric $g(t_i)$ has almost constant positive sectional curvature on $B\left(m_i, R(m_i, t_i)^{-\frac{1}{2}}\right)$. One could then shift basepoints and repeat the argument, to obtain that for any $A < \infty$ and for large i, the metric $g(t_i)$ has almost constant positive sectional curvature on $B\left(m_i, AR(m_i, t_i)^{-\frac{1}{2}}\right)$. From Bonnet-Myers, one concludes that M is compact, which is a contradiction.

Proposition 2.13 *There is some $C < \infty$ so that for all $t \in [0, \infty)$, we have $\| \operatorname{Rm}(g(t))\|_\infty \le \frac{C}{t}$.*

Proof Suppose that the proposition is not true. After doing a type-II point picking [9, Chapter 8, Section 2.1.3], there are points (m_i, t_i) so that $\lim_{i\to\infty} t_i |\operatorname{Rm}(m_i, t_i)| = \infty$ and $|\operatorname{Rm}| \le 2|\operatorname{Rm}(m_i, t_i)|$ on $M \times [a_i, b_i]$, with $\lim_{i\to\infty} |\operatorname{Rm}(m_i, t_i)|(t_i - a_i) = \lim_{i\to\infty} |\operatorname{Rm}(m_i, t_i)|(b_i - t_i) = \infty$. Put $Q_i = |\operatorname{Rm}(m_i, t_i)|$ and $g_i(x, u) = Q_i g(x, t_i + Q_i^{-1}u)$.

Suppose first that for some $i_0 > 0$ and all i, we have $Q_i \operatorname{inj}_{g(t_i)}(m_i)^2 \ge i_0$. After passing to a subsequence, we get a limiting Ricci flow solution $\lim_{i\to\infty}(M, g_i(\cdot), m_i) = (M_\infty, g_\infty(\cdot), m_\infty)$ defined for times $u \in \mathbb{R}$. Here M_∞ is a 3-manifold and $|\operatorname{Rm}(m_\infty, 0)| = 1$. As in the proof of Proposition 2.1, for each $m' \in M_\infty$, the point $(m', 0)$ is the limit of a sequence of points $(m'_i, 0)$ with $\lim_{i\to\infty} t_i R_g(m'_i, t_i) = \infty$, where the latter statement now comes from the type-II rescaling. From (2.3), we get $\operatorname{Ric} - \frac{1}{3} R g_\infty = 0$. Then (M_∞, g_∞) has constant positive curvature time slices, which implies that M_∞ is compact. Then M is also compact, which is a contradiction.

If $\liminf_{i\to\infty} Q_i \operatorname{inj}_{g(t_i)}(m_i)^2 = 0$, we can still take a limit as in (2.11). As in the argument after (2.11), we again conclude that M is compact, which is a contradiction. □

Corollary 2.14 *There are numbers* $\{A_k\}_{k=0}^{\infty}$ *that for all* $t \in [0, \infty)$ *and all multi-indices* I*, we have* $\| \nabla^I \operatorname{Rm} \|_{g(t)} \leq A_{|I|} t^{-\frac{|I|}{2}-1}$.

Proof This follows from Proposition 2.13, along with derivative estimates for the Ricci flow [9, Theorem 6.9]. □

Let $d_t : M \times M \to \mathbb{R}$ be the distance function on M with respect to the Riemannian metric $g(t)$. In particular, d_0 is the distance function with respect to g_0.

Lemma 2.15 *There is some* $C' < \infty$ *so that whenever* $0 \leq t_1 \leq t_2 < \infty$*, we have*

$$d_{t_1} - C' \left(\sqrt{t_2} - \sqrt{t_1} \right) \leq d_{t_2} \leq d_{t_1}. \tag{2.16}$$

Proof This follows from distance distortion estimates for Ricci flow, as in [20, Remark 27.5 and Corollary 27.16]. □

Fix $m_0 \in M$. Given $s > 0$, put $g_s(u) = s^{-1} g(su)$. Then $(M, g_s(\cdot))$ is also a Ricci flow solution, with $\| \operatorname{Rm}(g_s(u)) \| \leq \frac{C}{u}$ and $\| \nabla^I \operatorname{Rm} \|_{g_s(u)} \leq A_{|I|} u^{-\frac{|I|}{2}-1}$. Its distance function at time u is $\widehat{d}_{s,u} = s^{-\frac{1}{2}} d_{su}$. From (2.16), we have

$$\frac{1}{\sqrt{s}} d_0 - C' \sqrt{u} \leq \widehat{d}_{s,u} \leq \frac{1}{\sqrt{s}} d_0. \tag{2.17}$$

Given $\rho > 0$, it follows that

$$B_{\widehat{d}_{s,u}}(m_0, \rho - C'\sqrt{u}) \subset B_{d_0}(m_0, \rho\sqrt{s}) \subset B_{\widehat{d}_{s,u}}(m_0, \rho) \tag{2.18}$$

Also, if $0 \leq s_1 \leq s_2 < \infty$ then

$$\sqrt{\frac{s_1}{s_2}} \widehat{d}_{s_1,u} - C' \left(1 - \sqrt{\frac{s_1}{s_2}} \right) \sqrt{u} \leq \widehat{d}_{s_2,u} \leq \sqrt{\frac{s_1}{s_2}} \widehat{d}_{s_1,u}. \tag{2.19}$$

Given $\rho > 0$, it follows that

$$B_{\widehat{d}_{s_2,u}} \left(m_0, \sqrt{\frac{s_1}{s_2}} \rho - C' \left(1 - \sqrt{\frac{s_1}{s_2}} \right) \sqrt{u} \right) \subset B_{\widehat{d}_{s_1,u}} (m_0, \rho) \subset B_{\widehat{d}_{s_2,u}} \left(m_0, \sqrt{\frac{s_1}{s_2}} \rho \right). \tag{2.20}$$

Given a sequence $\{s_i\}_{i=1}^{\infty}$ tending to infinity and $u > 0$, after passing to a subsequence we can assume that there is a limit of $\lim_{i \to \infty} (M, g_{s_i}(u), m_0)$ in the pointed Gromov-Hausdorff topology. We claim that we can choose the subsequence so that the limit exists simultaneously for each u, and as u varies the limiting metric spaces are all biLipschitz equivalent to each other. To see this, after passing to a subsequence we can assume that there is a limit $\lim_{i \to \infty} (M, g_{s_i}(\cdot), m_0) = (\mathcal{G}_{\infty}, g_{\infty}(\cdot), \mathcal{O}_{x_{\infty}})$. Here $g_{\infty}(\cdot)$ is a Ricci flow solution on the étale groupoid $\mathcal{G}_{\infty}$, that exists for $u > 0$. As u varies, the pointed Gromov-Hausdorff limit $\lim_{i \to \infty} (M, g_{s_i}(u), m_0)$ always has the same underlying pointed topological space, namely the pointed orbit space (X_∞, x_∞) of $\mathcal{G}_\infty$. The metric on the limit depends on u, and is the quotient metric $\widehat{d}_{\infty,u}$ coming from $g_\infty(u)$. It follows that the various quotient metrics, as u varies, are biLipschitz to each other.

Since M is noncompact, X_∞ is also noncompact. In particular, $\dim(X_\infty) > 0$.

3 Noncollapsing at large time

In this section we show that the Ricci flow solution from Sect. 2 is noncollapsed for large time, in a scale-invariant sense. More precisely, we show that there is a blowdown limit on a three dimensional manifold, where the emphasis is on the three dimensionality.

We recall that the Ricci flow solution from Sect. 2 has positive Ricci curvature and lives on a noncompact manifold, which is necessarily then diffeomorphic to $\mathbb{R}^3$. After passing to a subsequence, we can extract a blowdown limit X_∞ (corresponding to a fixed rescaled time) in the sense of pointed Gromov-Hausdorff convergence. The issue is to show that $\dim(X_\infty) = 3$. Since X_∞ is noncompact, we must exclude that $\dim(X_\infty)$ is one or two. We note that $\mathbb{R}^3$ can collapse with bounded sectional curvature [5, Example 1.4] due to a graph manifold structure, so the result is not immediate.

The following statement is the main result of this section.

Proposition 3.1 *There is some sequence $\{s_i\}_{i=1}^\infty$ tending to infinity so that the pointed limit $\lim_{i\to\infty}(M, g_{s_i}(\cdot), m_0)$ exists as a Ricci flow $(M_\infty, g_\infty(\cdot), m_\infty)$ on a pointed 3-manifold (M_∞, m_∞).*

Before giving the details of the proof, we sketch the main ideas. Suppose that the proposition is not true. Fixing u, for large s_0 the metric space $\left(M, \widehat{d}_{s_0,u}, m_0\right)$ is pointed Gromov-Hausdorff close to a noncompact Alexandrov space of dimension one or two. In either case, there is a short loop γ at m_0 that is not contractible in $B_{\widehat{d}_{s_0,u}}(m_0, 1)$. Since M is contractible, γ can be contracted in $B_{\widehat{d}_{s_0,u}}(m_0, \Delta)$ for some $\Delta > 1$. From the distance shrinking in (2.19), there will be some $s_1 \geq s_0$ so that γ cannot be contracted in $B_{\widehat{d}_{s_1,u}}(m_0, 1)$ but can be contracted in $B_{\widehat{d}_{s,u}}(m_0, 1)$ for all $s > s_1$. From continuity and the definition of s_1, the loop γ can be contracted in $B_{\widehat{d}_{s_1,u}}(m_0, 2)$ but cannot be contracted in $B_{\widehat{d}_{s_1,u}}(m_0, 1/10)$. When changing metrics from $\widehat{d}_{s_0,u}$ to $\widehat{d}_{s_1,u}$, the length of γ can only go down. Then $(M, \widehat{d}_{s_1,u}, m_0)$ will be pointed Gromov-Hausdorff close to a ray, with m_0 corresponding to a point of distance approximately one from the tip of the ray. The part of M in which γ contracts is a very collapsed solid torus. From the geometry of such solid tori, if a very short loop at m_0 is contractible in $B_{\widehat{d}_{s_1,u}}(m_0, 2)$ then it is already contractible in $B_{\widehat{d}_{s_1,u}}(m_0, 1/10)$, which is a contradiction.

In what follows, const. will denote a constant that is independent of the other parameters in the statement. Given a sequence $\{K_i\}_{i=0}^\infty$, we will say that a Riemannian manifold is $\vec{K}$-regular if $\| \nabla^I \operatorname{Rm} \| \leq K_{|I|}$ for all multi-indices I. For simplicity, we will take $\| \operatorname{Rm} \| \leq K_0$ to mean that the sectional curvatures are bounded by K_0 in magnitude. We will say that a loop γ at a basepoint m_0 is contractible in a ball

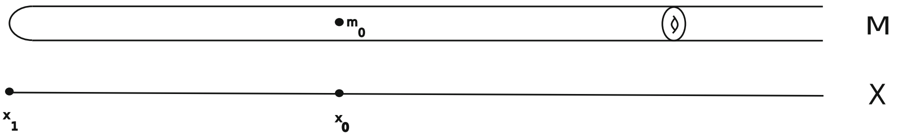

Fig. 1 Close to a ray

$B(m_0, r)$ if it can be contracted to a point by a family of loops in $B(m_0, r)$ that go through m_0, i.e. that γ represents a trivial element of $\pi_1(B(m_0, r), m_0)$. We begin by proving some lemmas about collapsed 3-manifolds. The first one is relevant when the manifold is pointed Gromov-Hausdorff close to a one dimensional space. (See Figure 1.)

Lemma 3.2 *Given $\vec{K}$ and $L < \infty$, there is some $\overline{\delta_1} = \overline{\delta_1}(\vec{K}, L) > 0$ with the following property. Suppose that (M, m_0) is a complete pointed $\vec{K}$-regular oriented Riemannian 3-manifold so that (M, m_0) has pointed Gromov-Hausdorff distance $\delta_1 \leq \overline{\delta_1}$ from a line or ray, which we denote by (X, x_0). Then there is a loop γ through m_0 with length $l(\gamma) \leq \mathrm{const.}_1\, \delta_1$ so that γ is not contractible in $B(m_0, L)$.*

Proof It is enough to consider δ_1's that are much smaller than L^{-1}. We apply [4, Theorem 1.7]. In our case, the relevant nilpotent Lie groups N to describe the local geometry near a point $m \in M$, from [4, p. 331], are $\mathbb{R}^2$ if X is a line and $\mathbb{R} \times U(1)$ if X is a ray.

If X is a line and δ_1 is sufficiently small then the pointed Gromov-Hausdorff approximation can be realized by a map that restricts to a fibration $\pi : U \to (-\delta_1^{-1}/2, \delta_1^{-1}/2)$, where U is an open subset of M, $\pi(m_0) = 0 = x_0$, the fibers of π have diameter at most const. δ_1, and the fibers are 2-tori or Klein bottles. Since an oriented 3-manifold cannot contain a two-sided Klein bottle, the fibers must be 2-tori. Up to a small biLipschitz perturbation, we can assume that the fibers are flat [4, Theorem 1.3]. We can then take γ to be a shortest nontrivial closed geodesic in $\pi^{-1}(x_0)$ passing through m_0.

If X is a ray, let x_1 denote its tip. If $d(x_0, x_1) > 2L$ then we are effectively in the previous case. If $d(x_0, x_1) \leq 2L$ then there is a singular fibration $\pi : U \to B(x_0, \delta_1^{-1}/2)$, where U is an open subset of M and the preimages of π have diameter at most const. δ_1. If $x \neq x_1$ then $\pi^{-1}(x)$ is a 2-torus, while $\pi^{-1}(x_1)$ is a circle. From [4, Theorem 1.7], a local biLipschitz model around $\pi^{-1}(x_1)$ is $(\mathbb{R} \times B^2)/\Lambda$, where $\mathbb{R} \times B^2$ has a metric that is $\mathbb{R} \times U(1)$-invariant and Λ is a discrete subgroup of $\mathbb{R} \times U(1)$ that is isomorphic to $\mathbb{Z}$. More precisely, for any given $\epsilon > 0$, there are $\rho > 0$ and $k \in \mathbb{Z}^+$ so that there is a (ρ, k)-round model metric, in the sense of [4, Section 1], that is e^ϵ-biLipschitz close to the geometry near $\pi^{-1}(x_1)$; in our case the model metric takes the form $(\mathbb{R} \times B^2)/\Lambda$. To be close to a ray, a generator $(\tau, e^{i\phi})$ of Λ has ϕ a small nonzero number and τ very small, relative to ϕ. Topologically, $(\mathbb{R} \times B^2)/\Lambda$ is a solid torus, as can be seen by isotoping ϕ to zero. Away from this model around $\pi^{-1}(x_1)$, there is a topological product structure of an interval with a 2-torus, so $\pi^{-1}(B(x_0, 3L))$ is a solid torus.

As before, up to a small biLipschitz perturbation, we can assume that the generic preimage of π is a flat 2-torus. If $x_0 = x_1$ then we can take γ to be the circle $\pi^{-1}(x_0)$. If $x_0 \neq x_1$, let γ_0 be a shortest nontrivial closed geodesic in the 2-torus $\pi^{-1}(x_0)$ passing through m_0 and let γ_1 be a shortest closed geodesic in $\pi^{-1}(x_0)$ passing through m_0 that is not homotopic to a multiple of γ_0. They both have length at most $\mathrm{const.}_1\, \delta_1$. At least one of γ_0 and γ_1 is noncontractible in the solid torus $\pi^{-1}(B(x_0, 3L))$ and we take γ to be that curve. □

Given $K, \delta_1 > 0$, let A_{K,δ_1} be the set of pointed two dimensional complete Alexandrov spaces $(X, d_X, \star_X)$ with Alexandrov curvature bounded below by $-2K$ that have pointed Gromov-Hausdorff distance at least $\delta_1/2$ from any pointed one dimensional complete Alexandrov space. It is compact in the pointed Gromov-Hausdorff topology.

The next lemma is relevant when the manifold is pointed Gromov-Hausdorff close to a two dimensional space but is not pointed Gromov-Hausdorff close to a one dimensional space.

Lemma 3.3 *Given $L < \infty$, $\delta_1 > 0$ and $\vec{K}$, there is some $\overline{\delta_2} = \overline{\delta_2}(\vec{K}, L, \delta_1) > 0$ with the following property. Suppose that (M, m_0) is a complete pointed $\vec{K}$-regular Riemannian 3-manifold diffeomorphic to $\mathbb{R}^3$ so that (M, m_0) has pointed Gromov-Hausdorff distance $\delta_2 \leq \overline{\delta_2}$ from an element (X, x_0) of A_{K_0,δ_1}. Then there is a loop γ through m_0 with length $l(\gamma) \leq \mathrm{const.}_2\, \delta_2$ so that γ is not contractible in $B(m_0, L)$.*

Proof We can assume that $\delta_2^{-1} \gg L$. We can apply [4, Theorem 1.7], with the relevant nilpotent Lie group N being $\mathbb{R}$. The conclusion is that if δ_2 is sufficiently small then the pointed Gromov-Hausdorff approximation can be realized by a map that restricts to an orbifold circle fibration $\pi : U \to B(x_0, \delta_2^{-1}/2)$ where U is an open subset of M and the fibers of π have diameter at most $\mathrm{const.}_2\, \delta_2$. (As A_{K_0,δ_1} is compact, we can choose δ_2 independent of $(X, x_0) \in A_{K_0,\delta_1}$.) As M is orientable, the orbifold $B(x_0, \delta_2^{-1}/2)$ has isolated singular points, i.e. there are no reflector lines. Since M does not have any embedded Klein bottles, the circle fibration is orientable and so describes a Seifert fibration. As $B(x_0, 2L)$ is noncompact, from [36, Lemma 3.2] there is an exact sequence

$$1 \to \mathbb{Z} \to \pi_1\left(\pi^{-1}(B(x_0, 2L))\right) \to \pi_1\left(B(x_0, 2L)\right) \to 1, \tag{3.4}$$

where the image of a generator of $\mathbb{Z}$ is represented by a regular fiber of the Seifert fibration, and $\pi_1\left(B(x_0, 2L)\right)$ denotes the orbifold fundamental group. (Since the $\mathbb{Z}$-subgroup is central in $\pi_1(\pi^{-1}(B(x_0, 2L)))$, it is well-defined independent of basepoint.) If δ_2 is small then $B(m_0, L) \subset \pi^{-1}(B(x_0, 2L))$ and we can take $\gamma = \pi^{-1}(x_0)$. □

The next lemma describes the geometry of a collapsed space with a short loop at the basepoint that can be contracted in a ball of radius $2L$ around the basepoint, but not in a ball of radius $L/10$ around the basepoint.

Lemma 3.5 *Given $L < \infty$, $\delta_3 \ll L^{-1}$ and $\vec{K}$, there is some $\delta_4 = \delta_4(\vec{K}, L, \delta_3) > 0$ with the following property. Suppose that (M, m_0) is a complete noncompact pointed $\vec{K}$-regular oriented Riemannian 3-manifold so that (M, m_0) has pointed Gromov-Hausdorff distance at most δ_4 from a one or two dimensional complete pointed Alexandrov space with Alexandrov curvature bounded below by $-2K_0$. Suppose that* *there is a loop γ through m_0 of length $l(\gamma) < \delta_4$ that can be contracted in $B(m_0, 2L)$ but not in $B(m_0, L/10)$. Then (M, m_0) has pointed Gromov-Hausdorff distance at most δ_3 from a ray (X, x_0).*

Proof Given $\delta_1 > 0$, which we will adjust, as in the proof of Lemma 3.3 if δ_4 is sufficiently small and (M, m_0) is δ_4-close to some $(X, x_0) \in A_{K_0,\delta_1}$ then the pointed Gromov-Hausdorff approximation can be realized by an orbifold circle fibration $\pi : U \to B(x_0, \delta_4^{-1})$.

We claim that there is an $r = r(\vec{K}, \delta_1) \ll L/100$ so that the pointed Gromov-Hausdorff approximation can be chosen such that $B(x_0, r)$ has at most one singular point. To see this, suppose by way of contradiction that there are sequences $\{M_i\}_{i=1}^{\infty}$ and $\{\delta_{4,i}\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} \delta_{4,i} = 0$ so that $\lim_{i\to\infty} M_i = X_\infty$ for some $(X_\infty, x_{0,\infty}) \in A_{K_0,\delta_1}$, but if $\pi_i : M_i \to X_i$ is an orbifold circle bundle that is a pointed $\delta_{4,i}$-approximation to an $(X_i, x_{0,i}) \in A_{K_0,\delta_1}$ then $B(x_{0,i}, 1/i)$ has more than one orbifold point. After passing to a subsequence, the frame bundles $\{FM_i\}_{i=1}^{\infty}$ converge in the pointed smooth SO(3)-equivariant topology to a 5-manifold $\mathcal{M}$ with a locally free SO(3)-action, and $X_\infty = \mathcal{M}/\mathrm{SO}(3)$ [13, Proposition 11.5 and Theorem 12.8]. As each M_i is orientable, X_∞ has isolated singular points. For large i there is a pointed $\delta_{4,i}$-approximation $\pi_i : M_i \to X_\infty$, which is a contradiction.

Then $\pi^{-1}(B(x_0, r))$ is a solid torus. As γ is a short loop in $\pi^{-1}(B(x_0, r))$ that is not contractible in $B(m_0, 2r)$, it is homotopic to a nonzero multiple of a circle fiber of the fibration. Then from (3.4), the loop γ is not contractible in $B(m_0, 2L)$, which is a contradiction. Hence (M, m_0) isn't δ_4-close to an element of A_{K_0,δ_1}. In particular, if δ_4 is small enough relative to δ_1 then (M, m_0) is δ_1-close to a pointed line or ray (X, x_0).

If (X, x_0) is a line and δ_1^{-1} is large compared to L then $B(m_0, 4L)$ is homeomorphic to a product of an interval with a 2-torus, with $B(m_0, L/10)$ and $B(m_0, 2L)$ homeomorphic to a products of subintervals with the 2-torus. This contradicts the assumption that γ is contractible in $B(m_0, 2L)$ but not in $B(m_0, L/10)$. Hence (X, x_0) is a ray, to which (M, m_0) is δ_1-close. If we take $\delta_1 \ll \delta_3$ then the conclusion of the lemma holds. $\square$

The next lemma is a statement about the lengths of curves γ in Lemma 3.5.

Lemma 3.6 *Given $\vec{K}$ and $L < \infty$, there are $\delta_5 = \delta_5(\vec{K}, L) > 0$ and $\mathcal{L} = \mathcal{L}(\vec{K}, L) > 0$ with the following property. Suppose that (M, m_0) is a complete pointed orientable $\vec{K}$-regular Riemannian 3-manifold so that (M, m_0) has pointed Gromov-Hausdorff distance at most δ_5 from a ray (X, x_0). Suppose that there is a loop γ in $B(m_0, L/100)$, going through m_0, that can be contracted in $B(m_0, 2L)$ but not in $B(m_0, L/10)$. Then the length of γ is at least $\mathcal{L}$.*

Proof We can assume $K_0 > 0$. For δ_5 small, we first describe the local geometry of M. In what follows, it will be sufficient to have estimates with respect to a metric that is e^ϵ-biLipschitz to the metric g on M, for some $\epsilon > 0$. Fixing ϵ, we can approximate the metric on M by a model (ρ, k)-round metric $\widehat{g}$ in the sense of [4]. In doing so we may increase the curvature bound by some factor depending on ϵ. Although it's not essential for us, we can assume that the sectional curvatures of $\widehat{g}$ are bounded above by $2K_0$ in magnitude [32, Theorem 2.1].

In the region of M corresponding to the tip x_1 of X, the relevant nilpotent group is $N = \mathbb{R} \times U(1)$ and the relevant covering group is $\Lambda = \mathbb{Z}$, a discrete subgroup of N. From [4, Theorem 1.3], there is an open subset V of M whose Λ-cover $\widetilde{V}$ carries a model N-invariant metric. The fixed-point set $\widetilde{c}$ of $U(1)$ will be an infinite geodesic in $\widetilde{V}$ and we can assume that $\widetilde{V}$ is the $\rho(2K_0)^{-1/2}$-neighborhood of $\widetilde{c}$, topologically $\mathbb{R} \times B^2$. A point on $\widetilde{c}$ has injectivity radius at least $\rho(2K_0)^{-1/2}$. The image c of $\widetilde{c}$, in V, is a closed geodesic that can be considered to lie over x_1 in a singular fibration $V \to B(x_1, \rho(2K_0)^{-1/2})$, given by the distance from c, with the other fibers being 2-tori. A generator of Λ acts by an element $\left(\tau, e^{i\phi}\right)$ of N. For V to be Gromov-Hausdorff close to a ray, ϕ has to be a small nonzero number and τ has to be small relative to ϕ. Note that $V = \widetilde{V}/\Lambda$ is diffeomorphic to a solid torus.

Given $\beta > 0$, let $\widetilde{W}$ be the $\beta K_0^{-1/2}$-neighborhood of $\widetilde{c}$ in $\widetilde{V}$. We claim that β can be chosen small enough, independent of the other parameters, so that any noncontractible loop on $\partial \widetilde{W}$ (in the sense that it cannot be contracted in $\partial \widetilde{W}$) has length at least $\pi \beta K_0^{-1/2}$. To see this, if we rescale $\widetilde{W}$ so that it is the unit distance neighborhood of $\widetilde{c}$ then as $\beta \to 0$, the result approaches the flat isometric product $\mathbb{R} \times B^2$ uniformly in the $C^{1,\alpha}$-topology. For the latter, the shortest noncontractible loop on the boundary of the unit distance neighborhood of $\mathbb{R} \times \{0\}$ has length 2π, from which the claim follows. We will take $\beta \ll K_0^{1/2} L/100$.

Put $W = \widetilde{W}/\Lambda$. If σ is a noncontractible loop on ∂W that contracts in W then σ lifts to a noncontractible loop on $\partial \widetilde{W}$, and hence has length at least $\pi \beta K_0^{-1/2}$.

There is an open set $U \subset M$ with $B(m_0, 3.9L) \subset U \subset B(m_0, 4L)$ so that on $U - W$, the model geometry has a Riemannian submersion $F : (U - W) \to [a, b)$ whose fibers are flat 2-tori with diameter at most const. δ_5 and second fundamental form at most const. $K_0^{1/2}$ in norm (where const. depends on β) [4, Theorem 2.6]. Using the (integrable) horizontal distribution to trivialize the fibration as $G : (U - W) \to [a, b) \times T^2$, we can write the model metric as $\widehat{g} = dt^2 + h_t$, where $t \in [a, b)$ and h_t is a flat metric on T^2. The shape operator of a fiber is $\frac{1}{2} h_t^{-1} \partial_t h_t$.

We can assume that this torus fibration matches up with the torus fibration on a neighborhood of ∂W in W [4, Theorem 1.7]. Suppose that γ is a loop in $B(m_0, L/100)$, going through m_0, that can be contracted in $B(m_0, 2L)$ but not in $B(m_0, L/10)$. If the distance from x_0 to the tip x_1 of the ray is less than $L/20$ then $B(m_0, 2L) - B(m_0, L/10)$ lies in the torus-fibered region $U - W$ and $B(m_0, 2L)$ can be retracted to $B(m_0, L/10)$, which contradicts the assumption about γ. Hence $d(x_0, x_1) \geq L/20$. For similar reasons, $d(x_0, x_1) \leq 5L$. Hence $|b - a| \leq 10L$.

As γ cannot be contracted in $B(m_0, L/10)$, and lies in $B(m_0, L/100) \subset U - W$, it is homotopic to a noncontractible loop in $F^{-1}(m_0)$. We now sweep γ to ∂W. That is, writing $G(\gamma(t)) = (\gamma_1(t), \gamma_2(t))$, we put $\widehat{\gamma}(t) = G^{-1}(a, \gamma_2(t))$. The loop $\widehat{\gamma}$ is noncontractible in ∂W. By the bound on the shape operators of the fibers of F, there is a bound $l(\widehat{\gamma}) \leq e^{\text{const.}\, K_0^{1/2} L} l(\gamma)$. As γ contracts in $B(m_0, 2L)$, the loop $\widehat{\gamma}$ contracts in W. Hence $l(\gamma) \geq \pi \beta K_0^{-1/2} e^{-\text{const.}\, K_0^{1/2} L}$, as measured with $\widehat{g}$. Taking into account that g and $\widehat{g}$ are e^{ϵ}-biLipschitz, this proves the lemma. ◻

We now prove Proposition 3.1. Fix a time parameter $u > 0$. From Corollary 2.14, there is a sequence $\{K_i\}_{i=0}^{\infty}$ so that for all $s \geq 1$, the metric $g_s(u)$ is $\vec{K}$-regular.

We first describe how we choose parameters. Put $L = 1$. Let δ_5 be the parameter of Lemma 3.6. Choose $\delta_3 < \delta_5$ with $\delta_3 \ll 1$. Let $\delta_4 = \delta_4(\vec{K}, 1, \delta_3)$ be the parameter from Lemma 3.5. In reference to the parameters $\overline{\delta_1}$ and $\mathrm{const.}_1$ of Lemma 3.2, and $\mathcal{L}$ of Lemma 3.6, choose $\delta_1 < \min(\overline{\delta_1}, \delta_4)$ so that $\mathrm{const.}_1\, \delta_1 < \min(\delta_4, \mathcal{L})$. Given this value of δ_1 and in reference to the parameters $\overline{\delta_2}$ and $\mathrm{const.}_2$ of Lemma 3.3, choose $\delta_2 < \min(\overline{\delta_2}, \delta_4)$ so that $\mathrm{const.}_2\, \delta_2 < \min(\delta_4, \mathcal{L})$.

Suppose that the proposition is not true. Then there is a positive nonincreasing function ϵ_s with $\lim_{s\to\infty} \epsilon_s = 0$ so that the metric space $(M, \widehat{d}_{s,u}, m_0)$ has pointed Gromov-Hausdorff distance at most ϵ_s from a one or two dimensional complete pointed Alexandrov space with Alexandrov curvature bounded below by $-2K_0$. Take s_0 so that $\epsilon_{s_0} \ll \min(\delta_1, \delta_2)$. If $(M, \widehat{d}_{s_0,u}, m_0)$ is δ_1-close to a one dimensional complete pointed Alexandrov space then we can apply Lemma 3.2. If not, we can apply Lemma 3.3. In either case we get a loop γ through m_0, with length at most $\min(\delta_4, \mathcal{L})$, that is not contractible in $B_{\widehat{d}_{s_0,u}}(m_0, 1)$.

As M is diffeomorphic to $\mathbb{R}^3$, there is some $\Delta > 1$ so that γ can be contracted in $B_{\widehat{d}_{s_0,u}}(m_0, \Delta)$. Let $\mathcal{C}$ be the set of $s \geq s_0$ so that γ can be contracted in $B_{\widehat{d}_{s,u}}(m_0, 1)$. By the right-hand inclusion of (2.20), large values of s are in $\mathcal{C}$, and if $s \in \mathcal{C}$ then $s' \in \mathcal{C}$ for all $s' > s$. By (2.20), $\mathcal{C}$ is open in $\mathbb{R}$. Hence it is a half-open interval (s_1, ∞) for some $s_1 \geq s_0$.

In particular, γ cannot be contracted in $B_{\widehat{d}_{s_1,u}}(m_0, 1/10)$. On the other hand, if s is slightly greater than s_1 then γ can be contracted in $B_{\widehat{d}_{s,u}}(m_0, 1)$. Then by the left-hand inclusion of (2.20), γ can be contracted in $B_{\widehat{d}_{s_1,u}}(m_0, 2)$. By (2.19), the length of γ with respect to $g_{s_1}(u)$ is still less than $\min(\delta_4, \mathcal{L})$.

Lemma 3.5 implies that (M, m_0) has pointed Gromov-Hausdorff distance at most $\delta_3 < \delta_5$ from some ray (X, x_0). Lemma 3.6 now implies that the length of γ, with respect to $g_{s_1}(u)$, is at least $\mathcal{L}$. This is a contradiction.

4 Cubic volume growth

Proposition 4.1 *Under the hypotheses of Proposition 1.6, and with reference to Proposition 3.1, both (M, g_0) and $(M_\infty, g_\infty(u))$ have cubic volume growth. In addition, each tangent cone at infinity of $(M_\infty, g_\infty(u))$ is isometric to the tangent cone at infinity $T_\infty M = \lim_{i\to\infty} \left(M, m_0, s_i^{-\frac{1}{2}} d_0\right)$ of M.*

Proof We know that the pointed limit $\lim_{i\to\infty}(M, g_{s_i}(\cdot), m_0)$ exists as a Ricci flow $(M_\infty, g_\infty(\cdot), m_\infty)$ on a pointed 3-manifold (M_∞, m_∞). We claim first that (M, g_0) has cubic volume growth. Fix $u > 0$. Given $R > 0$, put $U_i = B_{\widehat{d}_{s_i,u}}(m_0, R)$ and $C_R = \mathrm{vol}(B(m_\infty, R), g_\infty(u))$. Then for large i, using (2.17) we have

$$s_i^{-\frac{3}{2}} \mathrm{vol}(U_i, d_0) = \mathrm{vol}\left(U_i, s_i^{-\frac{1}{2}} d_0\right) \geq \mathrm{vol}\left(U_i, \widehat{d}_{s_i,u}\right) \geq \frac{1}{2} C_R, \tag{4.2}$$

where vol denotes the 3-dimensional Hausdorff mass computed with the given metric. Also from (2.18), we have $U_i \subset B_{d_0}\left(m_0, s_i^{\frac{1}{2}}(R + C'\sqrt{u})\right)$. Hence

$$\operatorname{vol}\left(B_{d_0}\left(m_0, s_i^{\frac{1}{2}}(R + C'\sqrt{u})\right)\right) \geq \frac{1}{2} C_R s_i^{\frac{3}{2}}. \tag{4.3}$$

Since $r^{-3}\operatorname{vol}(B(m_0, r), g_0)$ is nonincreasing in r, it follows that there is some $v_0 > 0$ so that for all $r > 0$, we have $\operatorname{vol}(B(m_0, r), d_0) \geq v_0 r^3$.

Let d_∞ denote the metric on $T_\infty M$. Let $\widehat{d}_{\infty,u}$ denote the metric on $(M_\infty, g_\infty(u))$. From (2.17), we have

$$d_\infty - C'\sqrt{u} \leq \widehat{d}_{\infty,u} \leq d_\infty \tag{4.4}$$

on $T_\infty M - B_{d_\infty}(\star_\infty, C'\sqrt{u})$. Hence the tangent cone at infinity of $(M_\infty, g_\infty(u))$ is unique and is isometric to $(T_\infty M, d_\infty)$. □

Proposition 4.5 *If $\{s_i\}_{i=1}^\infty$ is any sequence tending to infinity then after passing to a subsequence, there is a pointed limit $\lim_{i\to\infty}(M, g_{s_i}(\cdot), m_0)$ as a Ricci flow on a pointed 3-manifold, defined for times $u \in (0, \infty)$.*

Proof Put $v_\infty = \lim_{r\to\infty} r^{-3}\operatorname{vol}(B_{d_0}(m_0, r), g_0) > 0$, the asymptotic volume ratio of (M, g_0). Fix $u > 0$. For any $s > 1$, from (2.17) a tangent cone at infinity of $(M, g_s(u))$ is isometric to a tangent cone at infinity of (M, g_0). Hence the asymptotic volume ration of $(M, g_s(u))$ is v_∞. Given $R > 0$, the Bishop-Gromov inequality implies that $\operatorname{vol}(B_{\widehat{d}_{s,u}}(m_\infty, R), \widehat{d}_{s,u}) \geq v_\infty R^3$. As $|\operatorname{Rm}(g_s(u))| \leq \frac{C}{u}$, the claim follows from the Hamilton compactness theorem. □

The next lemma will be used in Sect. 5.

Lemma 4.6 *A three dimensional complete gradient expanding soliton (M, g) with bounded sectional curvature, c-pinched nonnegative Ricci curvature, and cubic volume growth, must be isometric to flat $\mathbb{R}^3$.*

Proof If (M, g) is flat then because of the cubic volume growth, it must be isometric to $\mathbb{R}^3$. Hence we can assume that $\operatorname{Ric}(M, g) > 0$. From [28, Proposition 3.1], (M, g) has exponential curvature decay. Fix a basepoint m_0. We can find a sequence $\alpha_i \to \infty$ so that $\{(M, \alpha_i^{-2} g, m_0)\}_{i=1}^\infty$ converges in the pointed Gromov-Hausdorff topology to a three dimensional tangent cone at infinity (X_∞, x_∞) of (M, g). In particular, (X_∞, x_∞) is a cone over a connected compact surface. Because of the quadratic curvature decay, after passing to a further subsequence we can assume that there is a $W^{2,p}$-regular Riemannian metric on $X_\infty - x_\infty$, along with convergence of metrics in the weak $W^{2,p}_{loc}$-topology; c.f. [19] and [30, Sections 4 and 5]. From the weak $W^{2,p}_{loc}$-convergence and the exponential curvature decay of (M, g), the Riemannian metric on $X_\infty - x_\infty$ is flat. Hence X_∞ is a cone over the round S^2 or its $\mathbb{Z}_2$-quotient $\mathbb{R}P^2$. As M was orientable, the second possibility cannot occur, so X_∞ is the flat $\mathbb{R}^3$. Then by [10, Theorem 0.3], (M, g) is flat, which is a contradiction. □

Remark 4.7 Under the additional assumption of nonnegative sectional curvature, Lemma 4.6 was proven in [8].

5 Proof of Theorem 1.3

Proposition 5.1 *If* (M, g_0) *has nonnegative sectional curvature then Conjecture* 1.1 *holds.*

Proof It is enough to prove that Conjecture 1.2 holds, so we will assume that $\mathrm{Ric}_M > 0$, with M noncompact, and derive a contradiction. Using Proposition 4.1 and [35, Theorem 1.2], there is a blowdown limit $(M_\infty, g_\infty(\cdot), m_\infty)$ that is an gradient expanding soliton. From Lemma 4.6, it must be isometric to $\mathbb{R}^3$. Hence $T_\infty M$ is isometric to $\mathbb{R}^3$. By [10, Theorem 0.3], (M, g_0) is isometric to $\mathbb{R}^3$, which contradicts our assumption that $\mathrm{Ric}_M > 0$. □

Remark 5.2 To clarify a technical point, in [8] use is made of [17, Theorem 16.5] to say that $A = \limsup_{t\to\infty} t \| \mathrm{Rm}(g(t))\|_\infty$ is positive. The proof of [17, Theorem 16.5] is based on [17, Theorem 16.4], which has a similar conclusion without an assumption of positivity of curvature, but whose proof is only valid in the compact case (since it invokes the diameter). In fact, there are noncompact counterexamples to [17, Theorem 16.4]. With nonnegative curvature operator, the trace Harnack inequality directly implies that $A > 0$ for nonflat solutions.

Lemma 5.3 *If* (M, g) *has c-pinched nonnegative Ricci curvature then for any tangent vector* v*, we have*

$$Rg(v, v) \le \left(1 + \frac{2}{c}\right) \mathrm{Ric}(v, v). \tag{5.4}$$

Proof Working in a tangent space $T_m M$, we can restrict to unit vectors v. Lagrange multipliers show that the maximum of $Rg(v, v) - \left(1 + \frac{2}{c}\right) \mathrm{Ric}(v, v)$, over unit vectors in $T_m M$, is realized at a unit eigenvector of the Ricci operator. Let $r_1 \le r_2 \le r_3$ be the eigenvalues of the Ricci operator. As $r_2 \le r_3 \le \frac{1}{c} r_1$, we have

$$R = r_1 + r_2 + r_3 \le \left(1 + \frac{2}{c}\right) r_1 \le \left(1 + \frac{2}{c}\right) r_i \tag{5.5}$$

for each $i \in \{1, 2, 3\}$. This proves the lemma. □

Proposition 5.6 *If* (M, g) *has quadratic curvature decay then Conjecture* 1.1 *holds.*

Proof We will assume that $\mathrm{Ric}_M > 0$, with M noncompact, and derive a contradiction. Given $\alpha > 1$, consider the rescaled metric $\alpha^{-2} g$. Using Proposition 4.1, there is a sequence $\{\alpha_i\}_{i=1}^\infty$ tending to infinity so that $\{\left(M, \alpha_i^{-2} g, m_0\right)\}_{i=1}^\infty$ has a limit $T_\infty M = \lim_{i\to\infty} \left(M, \alpha_i^{-2} g_0, m_0\right)$ in the pointed Gromov-Hausdorff topology, where $T_\infty M$ is a three dimensional metric cone with a connected link [3, Theorem 7.6]. Furthermore, from the quadratic curvature decay, $T_\infty M$ is a smooth manifold away from the vertex, where it has a $W^{2,p}_{loc}$, $p < \infty$, or $C^{1,\alpha}_{loc}$, $\alpha \in (0, 1)$, metric g_∞, and the convergence to g_∞ is in the weak $W^{2,p}_{loc}$-topology. The inequality (5.4) will pass to such a limit. Since $T_\infty M$ is a cone, if ∂_r denotes the radial vector field then from the cone structure, $\mathrm{Ric}_{g_\infty}(\partial_r, \partial_r) = 0$. Then by (5.4), we conclude that $R_{g_\infty} = 0$. This means that Ric_{g_∞} vanishes, so g_∞ is smooth and flat. Hence $T_\infty M$ is a cone over the round S^2 or $\mathbb{R}P^2$. Since M is orientable, $T_\infty M$ must be a cone over the round S^2, and hence is isometric to $\mathbb{R}^3$. By [10, Theorem 0.3], (M, g) is isometric to $\mathbb{R}^3$, which contradicts our assumption that $\mathrm{Ric}_M > 0$. □

6 Proof of Theorem 1.4

To prove Theorem 1.4 we will use a rescaling argument as in the proof of Proposition 5.6. The rescalings no longer have uniform local double sided bounds on their curvatures, so we need a different convergence result. This will come from [23], which provides a weak convergence of curvature operators. It turns out that this is enough to obtain a contradiction.

We recall some results from [23]. Given an n-dimensional Riemannian manifold (M, g), let Riem be the curvature operator of M and let $\star_M : \Lambda^{n-2}(TM) \to \Lambda^2(TM)$ be Hodge duality. Given C^1-functions $\{f_j\}_{j=1}^{n-2}$ on M, put

$$\sigma = \star_M(\nabla f_1 \wedge \nabla f_2 \wedge \cdots \wedge \nabla f_{n-2}) \tag{6.1}$$

and define

$$r_M(f_1, \ldots, f_{n-2}) = \langle \sigma, \mathrm{Riem}(\sigma)\rangle \ \mathrm{dvol}_M, \tag{6.2}$$

a measure on M.

Suppose that $\{(M_i, g_i)\}_{i=1}^\infty$ is a sequence of n-dimensional pointed complete Riemannian manifolds with sectional curvatures uniformly bounded below, that converges to an n-dimensional pointed Alexandrov space X_∞ in the Gromov-Hausdorff topology. There is a notion of a test function f_∞ on X_∞. Given C^1-functions $\{f_i\}_{i=1}^\infty$ on the M_i's, there is a notion of the sequence C^1_δ-converging to f_∞.

The main result of [23] is the following. Suppose that for each i, $\{f_{i,j}\}_{1\le j\le n-2}$ is a collection of C^1-functions on M_i. Suppose that for each j, there is a C^1_δ-limit $\lim_{i\to\infty} f_{i,j} = f_{\infty,j}$, where $f_{\infty,j}$ is a test function on X_∞. Then there is a weak limit

$$\lim_{i\to\infty} r_{M_i}(f_{i,1}, \ldots, f_{i,n-2}) = r_{X_\infty}(f_{\infty,1}, \ldots, f_{\infty,n-2}). \tag{6.3}$$

Furthermore, the measure $r_{X_\infty}(f_{\infty,1}, \ldots, f_{\infty,n-2})$ is intrinsic to X_∞. It vanishes on the strata of X_∞ with codimension greater than two, and has descriptions on the codimension-two stratum and the set of regular points. Similarly, there is a measure R_{X_∞} on X_∞ to which the scalar curvature measures converge, i.e. $\lim_{i\to\infty} R_{M_i}\,\mathrm{dvol}_{M_i} = R_{X_\infty}$ in the weak topology.

The preceding constructions can also be carried out locally.

Lemma 6.4 *Suppose that M is a 3-manifold with c-pinched nonnegative Ricci curvature. Given $f \in C^1(M)$, put $V = \nabla f$. Then*

$$Rg(V, V)\, \mathrm{dvol}_M \leq \left(1 + \frac{2}{c}\right) \left(\frac{1}{2} R\, \mathrm{dvol}_M\, g(V, V) - r_M(f)\right). \tag{6.5}$$

Proof The proof is similar to that of Lemma 5.3. Fixing $m \in M$ and restricting to unit vectors $V_m \in T_m M$, using Lagrange multipliers one sees that it is enough to check the inequality when V_m is an eigenvector associated to the quadratic form coming from the difference of the two sides of (6.5). One finds that these eigenvectors are the eigenvectors of the Ricci operator, in which case (6.5) reduces to (5.4). □

Proposition 6.6 *If there is some $A < \infty$ so that the sectional curvatures of (M, g) satisfy $K(m) \geq -\frac{A}{d(m,m_0)^2}$, where m_0 is some basepoint, then Conjecture 1.1 holds.*

Proof We will assume that $\mathrm{Ric}_M > 0$, with M noncompact, and derive a contradiction. From Proposition 4.1, there is a sequence $\{\alpha_i\}_{i=1}^\infty$ tending to infinity so that putting $g_i = \alpha_i^{-2} g$ and $M_i = (M, g_i)$, the sequence $\{(M_i, m_0)\}_{i=1}^\infty$ converges to a three-dimensional metric cone (X_∞, x_∞) in the pointed Gromov-Hausdorff topology. From the curvature assumption, the cone X_∞ has curvature bounded below by the function $-\frac{A}{d(x,x_\infty)^2}$ in the Alexandrov sense. As a locally Alexandrov space, the cone will have no boundary points, i.e. no codimension-one stratum. Let Σ_∞ denote the link of the cone, so that $X_\infty = \mathrm{cone}(\Sigma_\infty)$. Then Σ_∞ is a connected Alexandrov surface with curvature bounded below by $-A$. The underlying topological space of Σ_∞ is a 2-manifold Y without boundary, which hence admits a smooth structure. Let ω_Y denote the curvature measure on Y, in the sense of [31]. (If Y is a smooth Riemannian 2-manifold then $\omega_Y = K\, \mathrm{dvol}_Y$, where K is the Gaussian curvature.)

Lemma 6.7 *Let ∂_r denote the radial vector field on X_∞. Then*

$$r_{X_\infty}(f) = (\partial_r f)^2 dr \wedge (d\omega_Y - \mathrm{dvol}_Y), \tag{6.8}$$

where $d\omega_Y$ is the curvature measure of the Alexandrov surface Y and dvol_Y *is the two-dimensional Hausdorff measure of Y. Also,*

$$R_{X_\infty} = 2dr \wedge (d\omega_Y - \mathrm{dvol}_Y). \tag{6.9}$$

Proof From [33, Section 1 and Appendix A], there is a 1-parameter family of smooth Riemannian metrics $\{h_s\}_{s\in(0,\epsilon)}$ on Y so that $\lim_{s\to 0}(Y, h_s) = \Sigma_\infty$ in the Gromov-Hausdorff topology, and the curvature of (Y, h_s) is bounded below by $-A$. For $s \in (0, \epsilon)$, let Y_s denote Y with the Riemannian metric h_s. We first compute $r_{\mathrm{cone}(Y_s)}$. Writing $\mathrm{cone}(Y_s) - \star = (0, \infty) \times Y_s$, if V is a vector field on Y_s then we can also consider it to be a vector field on $\mathrm{cone}(Y_s) - \star$. We have $\mathrm{Riem}(\partial_r \wedge V) = 0$. If V

and W are vector fields on Y_s then $\mathrm{Riem}(V \wedge W) = \frac{K_s - 1}{r^2} V \wedge W$, where K_s is the Gaussian curvature of Y_s. Hence if f is the radial function on $\mathrm{cone}(Y_s)$ then

$$\begin{aligned} r_{\mathrm{cone}(Y_s)}(f) =& \langle \star_{\mathrm{cone}(Y_s)} \partial_r, \mathrm{Riem}(\star_{\mathrm{cone}(Y_s)} \partial_r) \rangle \ \mathrm{dvol}_{\mathrm{cone}(Y_s)} \\ =& \frac{K_s - 1}{r^2} \langle \star_{\mathrm{cone}(Y_s)} \partial_r, \star_{\mathrm{cone}(Y_s)} \partial_r \rangle \ r^2 dr \wedge \mathrm{dvol}_{Y_s} \\ =& (K_s - 1) dr \wedge \mathrm{dvol}_{Y_s} = dr \wedge (K_s \, \mathrm{dvol}_{Y_s} - \mathrm{dvol}_{Y_s}). \end{aligned} \tag{6.10}$$

Then in general,

$$r_{\mathrm{cone}(Y_s)}(f) = (\partial_r f)^2 dr \wedge (K_s \, \mathrm{dvol}_{Y_s} - \mathrm{dvol}_{Y_s}). \tag{6.11}$$

As $s \to 0$, we have pointed Gromov-Hausdorff convergence $\lim_{s \to 0} \mathrm{cone}(Y_s) = X_\infty$. Working locally on X_∞, say on an annular region $a \leq r \leq A$, there is a weak limit $\lim_{s \to \infty} r_{\mathrm{cone}(Y_s)} = r_{X_\infty}$, which gives (6.8).

As

$$R_{\mathrm{cone}(Y_s)} \, \mathrm{dvol}_{\mathrm{cone}(Y_s)} = \frac{2(K_s - 1)}{r^2} r^2 dr \wedge \mathrm{dvol}_{Y_s} = 2(K_s - 1) dr \wedge \mathrm{dvol}_{Y_s} \tag{6.12}$$

equation (6.9) follows. □

If f is the radial function r on X_∞ then from (6.8) and (6.9), we have

$$R_{X_\infty} = 2 r_{X_\infty}(f). \tag{6.13}$$

Returning to the sequence $\{M_i\}_{i=1}^\infty$, the inequality (6.5) will pass to the weak limit, so (6.13) implies that $R_{X_\infty} = 0$. Equation (6.9) gives

$$d\omega_Y = \mathrm{dvol}_Y \, . \tag{6.14}$$

Integrating (6.14) over Y shows that the Euler characteristic of Y is positive. By Perelman stability, $\mathrm{cone}(X_\infty) - \star$ is orientable, so Y is a 2-sphere. As an Alexandrov surface, the Alexandrov geometry on Y comes from a Riemannian metric of the form $e^{2\phi} g_{S^2}$ which is subharmonic in the sense of [31, Section 7]. Equation (6.14) becomes

$$\triangle_{S^2} \phi - 1 = -e^{2\phi}, \tag{6.15}$$

where $\triangle_{S^2} \phi$ is *a priori* a measure on S^2 and $e^{2\phi}$ is an L^1-function. From (6.14) the curvature measure $d\omega_Y$ is absolutely continuous with respect to the Riemannian density on the round S^2, so [1, Proposition 2.8] implies that $e^{2\phi}$ is L^p-regular on S^2 for all $p < \infty$. Then (6.15) implies that ϕ is $W^{2,p}$-regular, hence $C^{1,\alpha}$-regular for all $\alpha \in (0, 1)$. We can now bootstrap (6.15) to conclude that ϕ is smooth on S^2. Then (6.15) becomes the statement that $e^{2\phi} g_{S^2}$ has constant curvature 1. Thus Y is isometric to the round S^2. Hence X_∞ is the flat $\mathbb{R}^3$. By [10, Theorem 0.3], (M, g) is flat, which is a contradiction. □

References

1. Ambrosio, L., Bertrand, J.: On the regularity of Alexandrov surfaces with curvature bounded below. Anal. Geom. Metr. Spaces **2016**, 282–287 (2016)
2. Brendle, S., Huisken, G., Sinestrari, C.: Ancient solutions to the Ricci flow with pinched curvature. Duke Math. J. **158**, 537–551 (2011)
3. Cheeger, J., Colding, T.: Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. Math. **144**, 189–237 (1996)
4. Cheeger, J., Fukaya, K., Gromov, M.: Nilpotent structures and invariant metrics on collapsed manifolds. J. Am. Math. Soc. **5**, 327–372 (1992)
5. Cheeger, J., Gromov, M.: On the characteristic numbers of complete manifolds of bounded curvature and finite volume. In: Differential Geometry and Complex Analysis, pp. 115–154. Springer, Berlin (1985)
6. Cheeger, J., Gromov, M.: Collapsing Riemannian manifolds while keeping their curvature bounded II. J. Differ. Geom. **32**, 269–298 (1990)
7. Cheeger, J., Tian, G.: Curvature and injectivity radius estimates for Einstein 4-manifolds. J. Am. Math. Soc. **19**, 487–525 (2005)
8. Chen, B.-L., Zhu, X.-P.: Complete Riemannian manifolds with pointwise pinched curvature. Inv. Math. **140**, 423–452 (2000)
9. Chow, B., Lu, P., Ni, L.: Hamilton's Ricci flow. Am. Math. Soc. **2006**, 58 (2006)
10. Colding, T.: Ricci curvature and volume convergence. Ann. Math. **145**, 477–501 (1997)
11. Deruelle, A.: Smoothing out positively curved metric cones by Ricci expanders. Geom. Funct. Anal. **26**, 188–249 (2016)
12. Deruelle, A., Schulze, F., Simon, M.: Initial stability estimates for Ricci flow and three dimensional Ricci-pinched manifolds. arXiv:2203.15313 (2022)
13. Fukaya, K.: Hausdorff convergence of Riemannian manifolds and its applications. In: Recent Topics in Differential and Analytic Geometry, ed. T. Ochiai, Math. Soc. of Japan, Tokyo, pp. 143–238 (1990)
14. Grove, K., Karcher, H.: How to conjugate C1-close group actions. Math. Z. **132**, 11–20 (1973)
15. Hamilton, R.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. **17**, 255–306 (1982)
16. Hamilton, R.: Convex hypersurfaces with pinched second fundamental form. Comm. Anal. Geom. **2**, 167–172 (1994)
17. Hamilton, R.: The formation of singularities in the Ricci flow. In: Surveys in Differential Geometry vol. 2, pp. 7–136. International Press, Boston (1995)
18. Hilaire, C.: Ricci flow on Riemannian groupoids. arXiv:1411.6058 (2014)
19. Kasue, A.: A convergence theorem for Riemannian manifolds and some applications. Nagoya Math. J. **114**, 21–51 (1989)
20. Kleiner, B., Lott, J.: Notes on Perelman's papers. Geom. Top. **12**, 2587–2855 (2008)
21. Kleiner, B., Lott, J.: Locally collapsed 3-manifolds. Astérisque **365**, 7–99 (2014)
22. Kleiner, B., Lott, J.: Geometrization of three-dimensional orbifolds via Ricci flow. Asterisque **365**, 101–177 (2014)
23. Lebedeva, N., Petrunin, A.: Curvature tensor of smoothable Alexandrov spaces. arXiv:2202.13420 (2022)
24. Lee, M.-C., Topping, P.: Three-manifolds with non-negatively pinched Ricci curvature. arXiv:2204.00504 (2022)
25. Lott, J.: On the long-time behavior of type-III Ricci flow solutions. Math. Ann. **339**, 627–666 (2007)
26. Lott, J., Zhang, Z.: Ricci flow on quasiprojective manifolds II. J. Eur. Math. Soc. **18**, 1813–1854 (2016)
27. Lu, P.: A local curvature bound in Ricci flow. Geom. Top. **14**, 1095–1110 (2010)
28. Ni, L.: Ancient solutions to Kähler-Ricci flow. Math. Res. Lett. **12**, 633–654 (2005)
29. Ni, L., Wu, B.: Complete manifolds with nonnegative curvature operator. Proc. Am. Math. Soc. **135**(2007), 3021–3028 (2007)
30. Petersen, P.: Convergence theorems in Riemannian geometry. In: Comparison geometry, MSRI Publ. 30, pp. 167–202. Cambridge University Press, Cambridge(1997)
31. Reshetnyak, Y.: Two-dimensional manifolds of bounded curvature. In: Geometry IV. Springer, New York (1993)
32. Rong, X.: On the fundamental groups of manifolds of positive sectional curvature. Ann. Math. **1996**, 397–411 (1996)
33. Richard, T.: Canonical smoothing of compact Alexandrov surfaces. Ann. Sci. ENS **51**, 263–279 (2018)

34. Schoen, R., Yau, S.-T.: Complete three dimensional manifolds with positive Ricci curvature and scalar curvature. In: Seminar on Differential Geometry, ed. S.-T. Yau, Annals of Math. Studies 102. Princeton University Press, Princeton (1982)
35. Schulze, F., Simon, M.: Expanding solitons with non-negative curvature operator coming out of cones. Math. Z. **275**, 625–639 (2013)
36. Scott, P.: The geometries of 3-manifolds. Bull. Lond. Math. Soc. **15**, 401–487 (1983)
37. Vogt, E.: A foliation of $\mathbb{R}^3$ and other punctured 3-manifolds by circles. Publ. Math. IHES **69**, 215–232 (1989)