Some rigorous results about the past and future behavior of expanding vacuum spacetimes

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References

Taken from

"Kasner-like regions near crushing singularities", Class. and Quantum Gravity 38, 055005 (2021)

"On the initial geometry of a vacuum cosmological spacetime", Class. and Quantum Gravity 37, 085017 (2020)

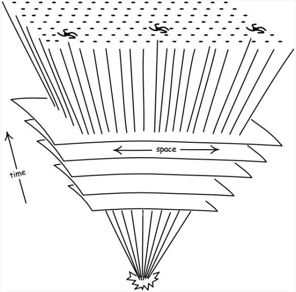
"Collapsing in the Einstein flow", Annales Henri Poincaré 19, p. 2245-2296 (2018)

Outline

The setup

Suppose that we have a *cosmological spacetime* M, i.e. it is globally hyperbolic with a compact Cauchy hypersurface.

I'm interested in *expanding* vacuum spacetimes.



By Hawking's singularity theorem, there is a past singularity. Two basic questions:

1. What is the behavior as one approaches the past singularity?
2. What is the future behavior?

Motivation

The physical relevance of looking at *vacuum* spacetimes: under some assumptions, there are heuristic arguments that for the behavior near an initial singularity, matter doesn't matter.

There are some strong results about past and future behavior if one assumes continuous symmetries (Isenberg, Moncrief,...).

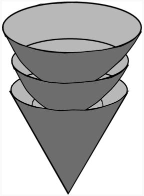
These would include spatial homogeneity, T^2-symmetry or $U(1)$-symmetry.

There are fewer results without any symmetry assumptions.

For concreteness, in this talk I will assume a 3+1 dimensional spacetime, with compact spatial slices. Some of the results work in any dimension, and for noncompact spatial slices.

Relevant example 1

A Milne spacetime is the interior of a forward lightcone in $\mathbb{R}^{3,1}$, quotiented by a discrete subgroup of $O^+(3,1)$.

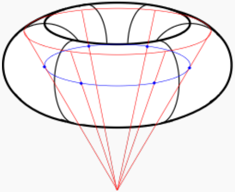


It is foliated by Riemannian 3-manifolds of constant negative curvature.

The metric is $g = -dt^2 + t^2 h_{hyp}$.

Relevant example 2

The Kasner spacetimes live on $(0, \infty) \times T^3$



with metric

$$g = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 .$$

Here

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 .$$

BKL conjectures

BKL conjectures about *generic* initial singularities (1970):

1. The evolution at different spatial points asymptotically decouples.
2. For a given spatial point, the asymptotic evolution is governed by the ODE of a spatially homogeneous vacuum spacetime of Bianchi type VIII or IX.

Are the conjectures right? Numerics indicate that there is something correct about them.

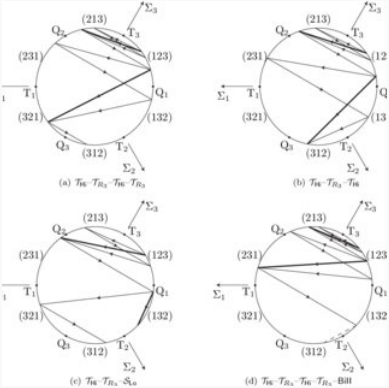
Some recent related preprints:

“Stable Big Bang formation for Einstein’s equations: The complete sub-critical regime” by Fournodavlos, Rodnianski and Speck

“On the geometry of silent and anisotropic big bang singularities” by Ringstrom

Mixmaster spacetimes

Generic spatially homogeneous vacuum spacetimes of Bianchi type VIII or IX have a chaotic behavior, with long stretches of Kasner-like geometry, punctuated by jumps between them.



Kasner-like regions

In particular, the BKL conjectures predict the existence of many regions of Kasner-like geometry near a *generic* initial singularity.

I will address a slightly different question:

What are geometric properties that characterize the existence or nonexistence of Kasner-like regions as one approaches the initial singularity?

We find that this is governed by the behavior of the spatial volume density.

Approach

The ingredients:

- A way to rescale a vacuum spacetime, that allows one to consider a blowup limit.
- A monotonicity result.
- A way to take a convergent subsequence of a sequence of vacuum spacetimes.

In combination, one can use these features to prove results about geometric asymptotics of a spacetime by contradiction.

One identifies a class of putative target spacetimes and assumes that some sequence of blowup rescalings of the given vacuum spacetime stays away from the target class.

One takes a convergent subsequence of the rescalings and uses the monotonicity result to show that the limit does in fact lie in the target class, thereby obtaining a contradiction.

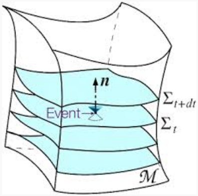
Crushing singularity

We assume that there is a *crushing singularity*, meaning that in the past there is a sequence of disjoint compact spatial Cauchy hypersurfaces whose mean curvatures approach $-\infty$ uniformly.

If there is a crushing singularity then there is a constant mean curvature (CMC) foliation in the past by compact hypersurfaces, whose mean curvatures H approach $-\infty$.

Define the Hubble time by $t = -\frac{3}{H}$. Then $t \to 0$ corresponds to approaching the singularity.

The spacetime is diffeomorphic to $(0, t_0) \times X$, where X is a compact three-dimensional manifold.



Using the foliation, one can write the metric as

$$g = -L^2 dt^2 + h(t),$$

where $L = L(t)$ is a function on X and $h(t)$ is a Riemannian metric on X. Let $K(t)$ denote the second fundamental form.

Rescaling

$$g = -L^2 dt^2 + h(t)$$

Given $s > 0$, put

$$g_s \;=\; -\, L^2(su)\, du^2 \;+\; s^{-2} h(su).$$

It is isometric to $s^{-2}g$, and so is also a vacuum solution, with Hubble time u. Hence we put

$$\begin{aligned} L_s(u) =& L(su), \\ h_s(u) =& s^{-2} h(su), \\ K_s(u) =& s^{-1} K(su). \end{aligned}$$

Taking $s \to 0$ corresponds to performing a blowup near the initial singularity.

Note: we will take blowup limits of vacuum *spacetimes*, and not just spatial slices.

Monotonicity

A remarkable monotonicity result of Fischer-Moncrief and Anderson:

$$\frac{d}{dt}\left(t^{-3}\,\mathrm{vol}(X,h(t))\right) = -\,t^{-2}\int_X L|K^0|^2\,\mathrm{dvol}_h,$$

where K^0 is the traceless second fundamental form.

In particular, $t^{-3}\,\mathrm{vol}(X,h(t))$ is nonincreasing in t. It is constant in t only for Milne spacetimes.

This gives rise to the intuition that for large time, the “noncollapsing” part of the spacetime should approach a Milne solution.

Because $t^{-3}\,\mathrm{vol}(X,h(t))$ is bounded for large t, it is useful for understanding the future behavior.

To understand the past behavior, we want a quantity that is nondecreasing in t.

Monotonicity

Claim:

$$\frac{d}{dt}\left(t^{-1}\,\mathrm{vol}(X,h(t))\right) = -\frac{1}{3}\int_X LR\ \mathrm{dvol}_h,$$

where R is the spatial scalar curvature.

Compare with

$$\frac{d}{dt}\left(t^{-3}\,\mathrm{vol}(X,h(t))\right) = -\,t^{-2}\int_X L|K^0|^2\ \mathrm{dvol}_h\,.$$

In particular, if $R \leq 0$ then $t^{-1}\,\mathrm{vol}(X,h(t))$ is *nondecreasing* in t, i.e. can only *decrease* as $t \to 0$.

If $R \leq 0$ and the three-manifold X is aspherical (i.e. has a contractible universal cover) then $t^{-1}\,\mathrm{vol}(X,h(t))$ is constant in t only for *Kasner* spacetimes.

In fact, there is a *pointwise* monotonicity statement:

If $R \leq 0$ then for any $x \in X$, the spatial volume density $t^{-1}\,\mathrm{dvol}_{h(t)}(x)$ is nondecreasing in t.

Convergent subsequences

Put $e_0 = \frac{1}{L} \frac{\partial}{\partial t}$. Let $\{e_i\}_{i=1}^3$ be an orthonormal basis at a point for $e_0^\perp$. Put

$$|\operatorname{Rm}|_T = \sqrt{\sum_{\alpha,\beta,\gamma,\delta=0}^{3} R^2_{\alpha\beta\gamma\delta}}.$$

Definition

A CMC vacuum solution is type-I if $|\operatorname{Rm}|_T \leq Ct^{-2}$ as $t \to 0$.

This is a scale-invariant condition. I don't know any crushing singularities that are not type-I.

Theorem

(Anderson,...) Let $\{g_i\}_{i=1}^\infty$ be a sequence of type-I CMC vacuum solutions (with the same C), defined on spacetimes $(0, t_0) \times X_i$, where X_i is equipped with a basepoint x_i.

Then a subsequence converges in the (pointed) weak $W^{2,p}$ and norm $C^{1,\alpha}$-topologies to a type-I CMC vacuum solution.

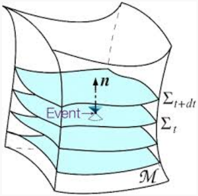
The limit is defined on $(0, t_0) \times X_\infty$ for some 3-manifold (or étale groupoid) X_∞ with a basepoint x_∞.

Milne asymptotics I

We first characterize when the past behavior is Milne-like.

Given a type-I CMC vacuum solution g defined on $(0, t_0) \times X$, choose a point $x \in X$.

Look at the spatial density $\mathrm{dvol}_{h(t)}(x)$. That is, we are looking at the spatial density along a curve toward the initial singularity that meets the hypersurfaces orthogonally.



The *fastest* that $\mathrm{dvol}_{h(t)}(x)$ can decrease as $t \to 0$ is $O(t^3)$.

Milne asymptotics II

Let $\mathcal{M}$ be the set of flat Milne spacetimes $(0, \infty) \times H^3/\Gamma$.

Theorem

(L. 2021) If $\mathrm{dvol}_{h(t)}(x) = O(t^3)$ *as* $t \to 0$ *then the blowup rescalings* g_s *approach* $\mathcal{M}$ *as* $s \to 0$.

That is, the original spacetime becomes increasing Milne-like, as measured around the point x, as one approaches the initial singularity.

Kasner asymptotics I

We now pass to Kasner asymptotics.

Choose $x \in X$. Look at the spatial density $\mathrm{dvol}_{h(t)}(x)$.

If $R \leq 0$ then the *slowest* that $\mathrm{dvol}_{h(t)}(x)$ can decrease as $t \to 0$ is $O(t)$.

Definition

The CMC vacuum solution has *asymptotically nonpositive spatial scalar curvature* if $\limsup_{t \to 0} \sup_{x \in X} t^2 R(t, x) \leq 0$.

(I don't know any crushing singularities that don't have this property.)

Kasner asymptotics II

Definition

(Kasner-like solutions) $\mathcal{K}$ is the set of type-I CMC vacuum solutions with $R = 0$, $L = \frac{1}{3}$ and $|K|^2 = H^2$.

Theorem

(L. 2021) Suppose that we have a type-I CMC vacuum solution. Suppose that it has asymptotically nonpositive spatial scalar curvature.

If $t^{-1}\,\mathrm{dvol}_{h(t)}(x)$ is (positively) bounded below as $t \to 0$ then the blowup rescalings g_s approach $\mathcal{K}$ as $s \to 0$.

Note: For a given $x \in X$, the rescalings g_s approach $\mathcal{K}$ but they may not approach a particular element of $\mathcal{K}$.

Even if they do approach a particular element of $\mathcal{K}$, that element could depend on x.

Mixmaster asymptotics

A Mixmaster solution (Bianchi VIII and Bianchi IX) is type-I with asymptotically nonpositive spatial scalar curvature.

However, it cannot have $t^{-1}\,\mathrm{dvol}_{h(t)}(x)$ bounded below as $t \to 0$.

Otherwise it would approach Kasner-like geometry as $t \to 0$, but this contradicts the existence of the "jumps" between different Kasner spacetimes.

One finds that a Mixmaster solution *almost* has $\mathrm{dvol}_{h(t)}(x) \sim t$. It decreases a bit faster, but not much.

Theorem

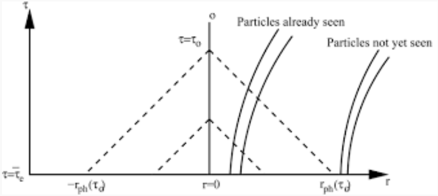
(L. 2021) Suppose that we have a type-I CMC vacuum solution. Suppose that it has asymptotically nonpositive spatial scalar curvature.

Suppose that for each $\beta > 0$, $\mathrm{dvol}_{h(t)}(x)$ fails to be $O\left(t^{1+\beta}\right)$ as $t \to 0$.

Put $\tau = \log(1/t)$, so $\tau \to \infty$ corresponds to approaching the singularity. Then as $\tau \to \infty$, the proportion of τ-time that the solution spends near $\mathcal{K}$ goes to one.

Particle horizons

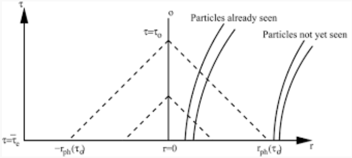
The BKL conjectures say that near generic initial singularities, particle horizons form and there is asymptotic decoupling between disjoint spatial regions as $t \to 0$.



This is not true for all crushing singularities; it is true for Kasner spacetimes but not Milne spacetimes.

It turns out that except for Milne spacetimes, approximate particle horizons always form in type-I solutions.

Particle horizons II



Let $J_-(t, x)$ denote the causal past of a spacetime point (t, x).

Theorem

Given a type-I CMC vacuum solution on $(0, t_0) \times X$, either it is a Milne solution or the following holds:

Given $x_1 \in X$, $N \in \mathbb{Z}^+$ and $\Lambda > 1$, if t is small enough then there are points $x_2, \dots, x_N \in X$ so that for each $i \neq j$, the causal pasts $J_-(t, x_i)$ and $J_-(t, x_j)$ are disjoint on the time interval $[\Lambda^{-1} t, t]$.

Note: A particle horizon corresponds to $\Lambda = \infty$. For a Milne solution, even when $N = 2$, there is a fixed upper bound on what one can take for Λ.

There is also a version of the theorem in which the points $x_2, \dots, x_N$ are localized in an arbitrary neighborhood of x_1 in X.

Some questions

1. Is every crushing singularity type-I? Are the type-I solutions generic?

2. Does every crushing singularity have asymptotically nonpositive spatial scalar curvature?

3. Can one characterize the Kasner-like solutions (type-I CMC vacuum solutions with $R = 0$, $L = \frac{1}{3}$ and $|K|^2 = H^2$), assuming complete time slices.

4. What is the most unstable direction coming from a Kasner solution? A Bianchi type-II direction?

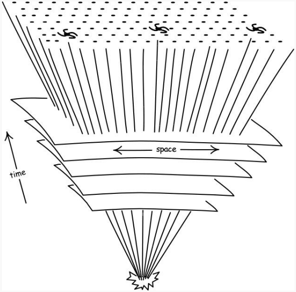
5. Can one detect the Bianchi type-II solutions that occur in the Mixmaster jumps?

6. Can one understand the self-similar vacuum solutions, i.e. those with a time-like homothetic Killing field, say with complete CMC time slices of bounded curvature?

7. Are there new monotonic quantities?

Future behavior

Consider an expanding CMC vacuum solution, again with $t = -\frac{3}{H}$, now defined for $t \in (t_0, \infty)$.

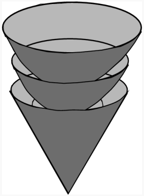


We can use similar methods to analyze its *future* asymptotics, using *blowdown* limits.

Model solutions

Here are the simply-connected expanding spatially homogeneous self-similar solutions (that admit a spatially compact quotient):

1. The Milne spacetime is the interior of a forward lightcone in $\mathbb{R}^{3,1}$. It is foliated by hyperboloids.



The metric is $g = -dt^2 + t^2 h_{hyp}$. It is scale-invariant.

2. The Bianchi-III flat spacetime is $\mathbb{R}$ times the interior of a forward lightcone in $\mathbb{R}^{2,1}$.

3. The Taub-flat spacetime is $\mathbb{R}^2$ times the interior of a forward lightcone in $\mathbb{R}^{1,1}$.

4. The Kasner spacetimes live on $(0, \infty) \times \mathbb{R}^3$, with metric

$$g = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2.$$

Here

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.$$

Definition

A CMC vacuum solution is type-III if $|\operatorname{Rm}|_T = O\left(t^{-2}\right)$ as $t \to \infty$.

Theorem

(L. 2018) Suppose that we have a type-III CMC vacuum solution on a compact aspherical three dimensional manifold X.

Suppose that the diameter of $(X, h(t))$ is $O(t)$.

Then there are arbitrarily large future time intervals where the pullback of the solution to the universal cover $\widetilde{X}$ is modelled by one of the homogeneous self-similar solutions.

That is, on any compact subset of $(0, \infty) \times \widetilde{X}$, a sequence of blowdown rescalings converges to one of the model geometries, in the weak $W^{2,p}$ and norm $C^{1,\alpha}$-topologies.

(If there is a lower volume bound $\operatorname{vol}(h(t)) \geq \text{const.}\, t^3$ then the model space is the Milne spacetime. This case is due to Mike Anderson.)

Collapsing

A new feature is that in future evolution, the spatial slices frequently collapse relative to their curvature.

In our case, this is saying that $\lim_{t \to \infty} t^{-3} \operatorname{vol}(X, h(t)) = 0$.

From work of Cheeger, Fukaya and Gromov, when a Riemannian manifold collapses with bounded curvature, it acquires Killing vector fields in the limit.

The implication is that a *collapsing limit* of blowdown rescalings g_s, as $s \to \infty$, has *continuous symmetries*.

It can then be analyzed using monotonic quantities that are special to that symmetry type.

Type-II solutions

There are expanding CMC vacuum solutions that do *not* satisfy the scale-invariant curvature condition $|\,\mathrm{Rm}\,|_T = O\left(t^{-2}\right)$ as $t \to \infty$. (Homogeneous examples are due to Hans Ringström.)

We can still do a blowdown analysis. (Rescale at points of large curvature so that the rescaled curvature tensor there has norm one.)

Theorem

(L. 2018) Suppose that we have a CMC vacuum solution on a compact three dimensional manifold X.

Suppose that the curvature is not $O(t^{-2})$ in magnitude as $t \to \infty$.

Doing a blowdown analysis at points (x_i, t_i) of spatially maximal curvature, with $t_i \to \infty$, one can extract a limit solution.

If the original solution has $|K| = O\left(t^{-1}\right)$ then the limit solution turns out to be flat.

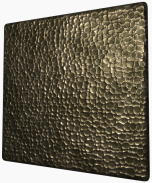
An apparent paradox

In the blowdown analysis, we rescale so that $|\operatorname{Rm}(x_i, t_i)|_T = 1$. How can the limit be flat?

The limit of the metrics exists in the *weak* $W^{2,p}$-topology, for $1 \leq p < \infty$, and in the norm $C^{1,\alpha}$-topology for $0 < \alpha < 1$.

This implies that the curvature tensors converge in the *weak* L^p-topology. The limit could well be zero.

In effect, there are increasing curvature fluctuations that average out the curvature to zero. The rescaled metrics *do* converge to a flat metric in the $C^{1,\alpha}$-topology.



3+1 decomposition

$$g = -L^2 dt^2 + h(t).$$

Constraint equations:

$$R - |K|^2 + H^2 = 0.$$

$$\nabla_i K^i_j - \nabla_j H = 0.$$

Evolution equations:

$$\frac{\partial h_{ij}}{\partial t} = -2LK_{ij}.$$

$$\frac{\partial K_{ij}}{\partial t} = LHK_{ij} - 2Lh^{kl}K_{ik}K_{lj} - L_{;ij} + LR_{ij}.$$

Suppose now that we have a CMC spacetime. From the last two equations, one obtains

$$\frac{\partial H}{\partial t} = -\triangle_h L + LH^2 + LR.$$

Maximum principle

$$\frac{\partial H}{\partial t} = -\triangle_h L + LH^2 + LR.$$

If $H = -\frac{3}{t}$, this becomes

$$\triangle_h L = \frac{9}{t^2}\left(L - \frac{1}{3}\right) + LR.$$

If $R \leq 0$ then the weak maximum principle gives $L \geq \frac{1}{3}$.

If there is a spacetime point (t, x) where $L(t, x) = \frac{1}{3}$ then the strong maximum principle says that $L \equiv \frac{1}{3}$ on the time slice and $R \equiv 0$ on the time slice.

From the constraint equations, $|K|^2 = H^2$ on the time slice.

Global monotonicity

$$\frac{\partial h_{ij}}{\partial t} = -2LK_{ij}.$$

As a pointwise statement,

$$\partial_t \operatorname{dvol}_{h(t)} = -LH \operatorname{dvol}_{h(t)}.$$

Suppose that the spatial slices X are compact.

$$\begin{aligned}
\frac{d}{dt}\left(t^{-1}\operatorname{vol}(X,h(t))\right) &= -t^{-2}\operatorname{vol}(X,h(t)) + t^{-1}\int_X (-LH)\operatorname{dvol}_{h(t)} \\
&= \int_X (-t^{-2} + 3t^{-2}L)\operatorname{dvol}_{h(t)} \\
&= 3t^{-2}\int_X \left(L - \frac{1}{3}\right)\operatorname{dvol}_{h(t)} \\
&= \frac{1}{3}\int_X (\triangle_h L - LR)\operatorname{dvol}_{h(t)} \\
&= -\frac{1}{3}\int_X LR\operatorname{dvol}_{h(t)}.
\end{aligned}$$

If $R \leq 0$ then $t^{-1}\operatorname{vol}(X,h(t))$ is nondecreasing in t.

Local monotonicity

As a pointwise statement,

$$\partial_t \, \mathrm{dvol}_{h(t)} = -LH \, \mathrm{dvol}_{h(t)} \, .$$

Then

$$\partial_t \left(t^{-1} \, \mathrm{dvol}_{h(t)} \right) = -t^{-2} \, \mathrm{dvol}_{h(t)} - t^{-1} LH \, \mathrm{dvol}_{h(t)} = \frac{3}{t^2} \left(L - \frac{1}{3} \right) \mathrm{dvol}_{h(t)} \, .$$

If $R \leq 0$ then from the weak maximum principle, $L \geq \frac{1}{3}$. Hence $t^{-1} \, \mathrm{dvol}_{h(t)}$ is nondecreasing in t.

If $t_1^{-1} \, \mathrm{dvol}_{h(t_1)}(x) = t_2^{-1} \, \mathrm{dvol}_{h(t_2)}(x)$ for some $t_1 < t_2$ then $L(t, x) = \frac{1}{3}$ for $t \in [t_1, t_2]$.

From before, using the strong maximum principle, $R = 0$, $L = \frac{1}{3}$ and $|K|^2 = H^2$ on the time interval. That is, the solution is Kasner-like on the time interval.