Long-time behavior in geometric flows

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April 1, 2019

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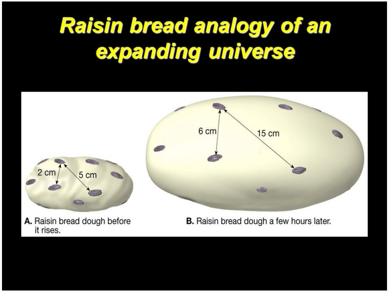
Today: Two ways to evolve a three dimensional space.

1. If a three dimensional space admits a locally homogeneous structure, can we find it with the Ricci flow?

2. For a compact universe with no matter, will gravitational dynamics make it more homogeneous?



Outline of the talk

1. Homogeneous spaces and the geometrization conjecture
2. The geometrization conjecture and Ricci flow
3. Finiteness of the number of surgeries
4. Long-time behavior of Ricci flow
5. The Einstein flow

Long-time behavior

Homogeneous spaces and the geometrization conjecture

Geometrization conjecture and Ricci flow

Finiteness of the number of surgeries

Long-time behavior of Ricci flow

Einstein flow

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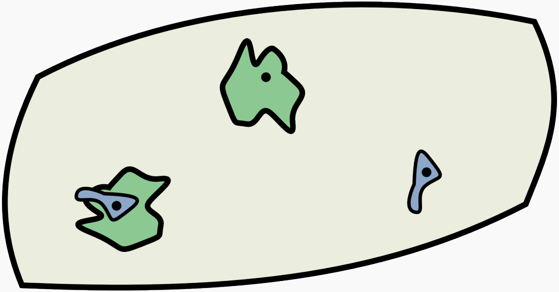
First, how do we understand three dimensional spaces?

In terms of homogeneous spaces.

Locally homogeneous metric spaces

A metric space X is *locally homogeneous* if all $x, y \in X$, there are neighbourhoods U and V of x and y and an isometric isomorphism $(U, x) \to (V, y)$.



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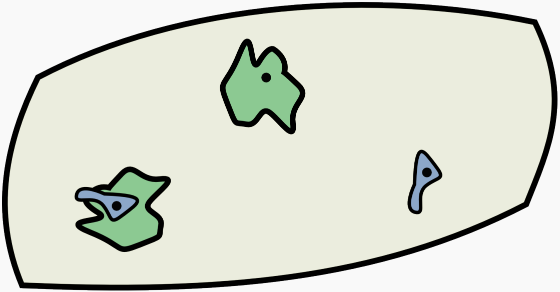
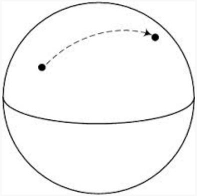


The metric space X is *globally homogeneous* if for all $x, y \in X$, there is an isometric isomorphism $\phi : X \to X$ that $\phi(x) = y$.

Locally homogeneous Riemannian manifolds

Any Riemannian manifold M gets a metric space structure.



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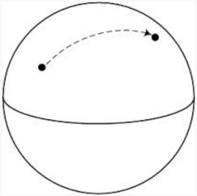

Theorem

(Singer 1960) If M is a complete, simply connected Riemannian manifold which is locally homogeneous, then M is globally homogeneous.

So passing to the universal cover turns "locally homogeneous" into "globally homogeneous".

Two-dimensional geometries

Globally homogeneous S^2,

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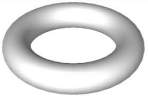
Globally homogeneous $\mathbb{R}^2$, locally homogeneous 

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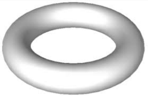
Globally homogeneous $\mathbb{R}^2$, locally homogeneous 

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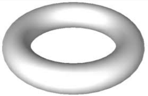
Globally homogeneous $\mathbb{R}^2$, locally homogeneous 

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Three-dimensional Thurston geometries

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Warning : Unlike in two dimensions, not every compact three-dimensional manifold admits a geometric structure, i.e. admits a locally homogeneous Riemannian metric.

Geometrization conjecture

If M is a compact orientable 3-manifold then there is a way to split M into canonical pieces, using certain embedded 2-spheres and 2-tori. (The collection of 2-spheres and 2-tori could be empty.)

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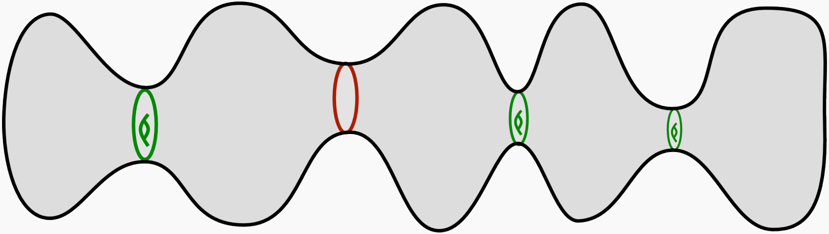

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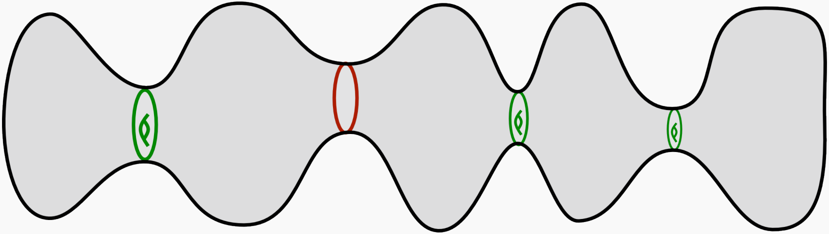

Conjecture (Thurston, 1982)

The ensuing pieces have geometric structures, i.e. admit locally homogeneous metrics

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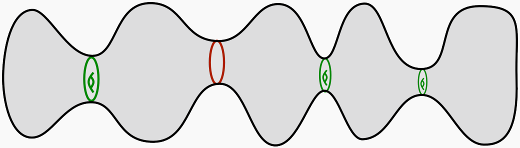

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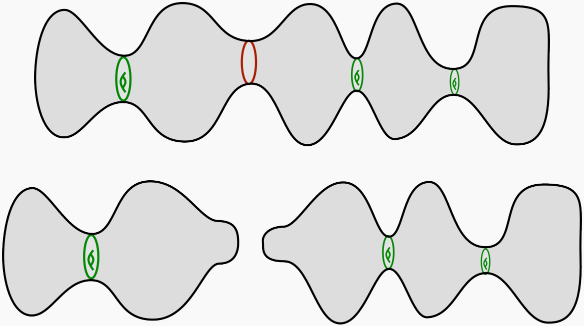

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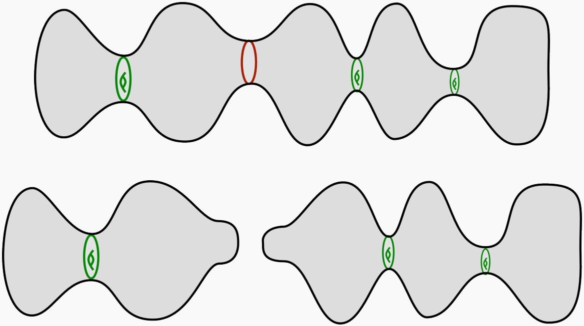

Cut along the 2-tori. The interiors of the ensuing pieces should admit complete locally homogeneous metrics.

Long-time behavior

Ricci flow approach to geometrization

Hamilton's Ricci flow equation

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Maybe the Ricci flow will evolve an initial Riemannian metric into something homogeneous.

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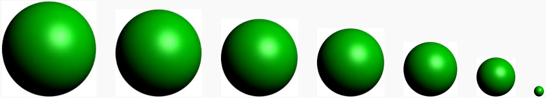

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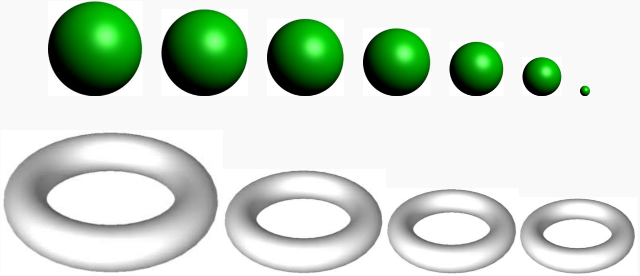

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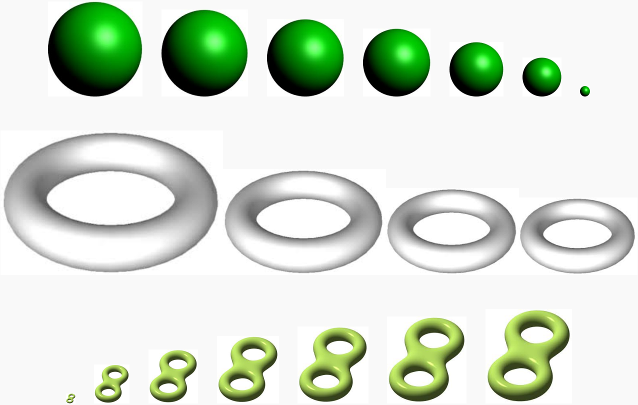
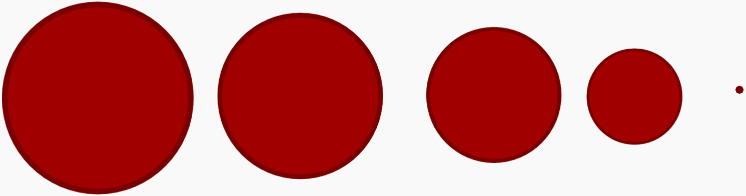


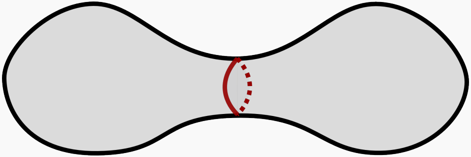
Singularities in 3D Ricci flow

Some components may disappear, e.g. a round shrinking 3-sphere.



Neckpinch

A 2-sphere pinches off. (Drawn one dimension down.)



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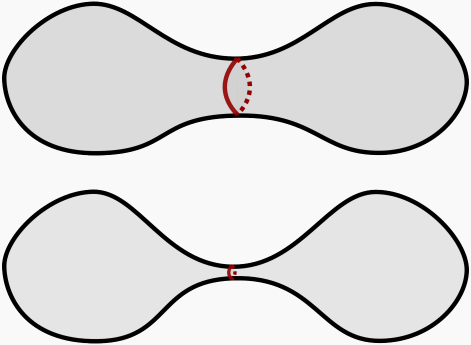

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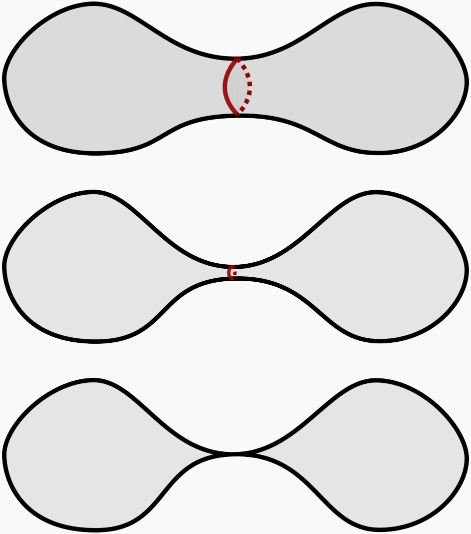
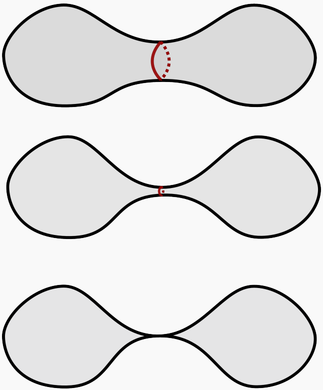


Hamilton's idea of surgery



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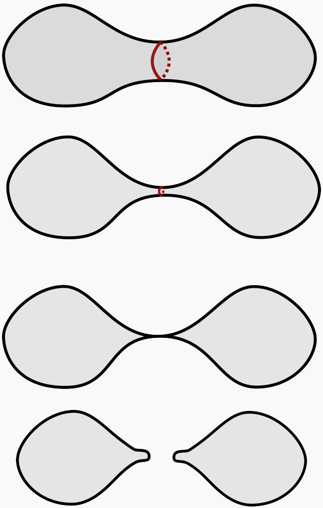

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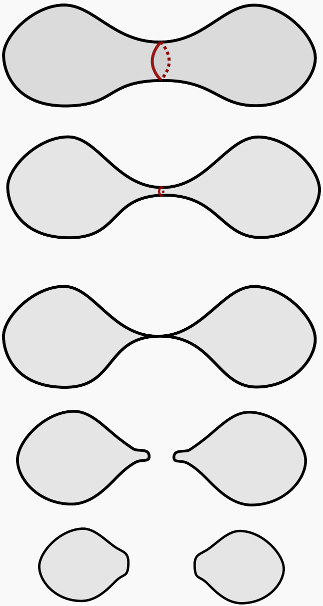

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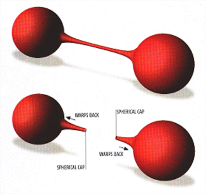

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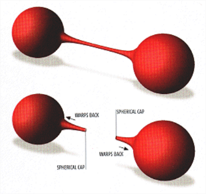

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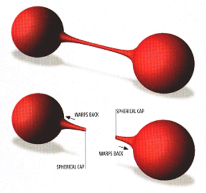

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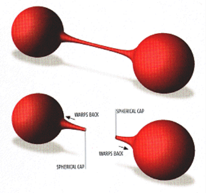

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Remark : the surgeries are done on 2-spheres, not 2-tori.

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(Relevant geometries : $\mathbb{R}^3$, H^3, $H^2 \times \mathbb{R}$, $\mathrm{S}\widetilde{\mathrm{L}(2, \mathbb{R}})$, Sol, Nil.)

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From Perelman's second Ricci flow paper : *This is a technical paper, which is a continuation of [I]. Here we verify most of the assertions, made in [I, §13]; the exceptions are ... the smoothness of the solution from some time on, which turned out to be unjustified, and, on the other hand, irrelevant for the other conclusions.*

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2. The thin part is a “graph manifold”. (This doesn’t use Ricci flow. Stated by Perelman, proofs by Shioya-Yamaguchi, Morgan-Tian, Bessières-Besson-Boileau-Maillot-Porti and Kleiner-L.)

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Graph manifolds were known to have a geometric decomposition. Along with knowledge of the topological effects of surgeries, this proved the geometrization conjecture.

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Remark : Answering these questions has no topological implication. We already know that the geometrization conjecture holds. Rather, they are *analytic* questions about the Ricci flow.

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Finiteness of the number of surgeries

Theorem

(Bamler 2018) Starting from any compact Riemannian 3-manifold, Perelman's Ricci-flow-with-surgery only encounters a finite number of surgeries.

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The statement is that if this parameter is small enough (which can always be achieved) then there is a finite number of surgeries.

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This is good because we know lots about metrics with bounded sectional curvature (Cheeger-Fukaya-Gromov).

Ingredients of the proof

Bamler's proof uses all of Perelman's work, and more. Some of the new ingredients :

1. Localizing Perelman's estimates and applying them to local covers of the manifold.

2. Use of minimal surfaces to control the geometry of the thin part.

3. Use of minimal embedded 2-complexes.

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The only case that we completely understand is when M admits *some* hyperbolic metric. Then from Perelman's work, for *any* initial metric on M, as $t \to \infty$ the rescaled Riemannian metric $\widehat{g}(t)$ approaches the metric on M of constant sectional curvature $-\frac{1}{4}$.

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Question : if M doesn't admit a hyperbolic metric, what are the candidate geometries for the long-time behavior?

Quasistatic solutions

The static solutions of the Ricci flow equation

$$\frac{dg}{dt} = -2\,\mathrm{Ric}_g$$

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Fact : On a compact 3-manifold, any self-similar solution has constant sectional curvature.

Apparent paradox : What happens to the Ricci flow if our 3-manifold doesn't admit a constant curvature metric?

Nil geometry

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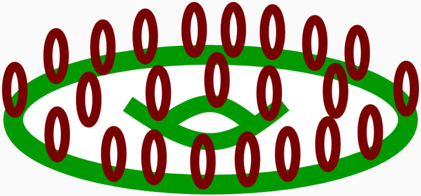

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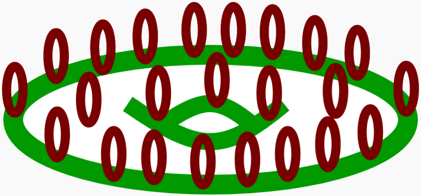

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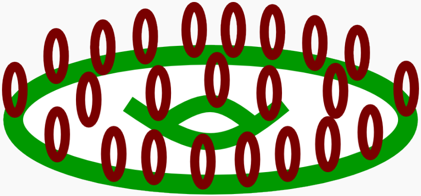

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With the rescaled metric $\widehat{g}(t) = \frac{g(t)}{t}$, $(M, \widehat{g}(t))$ shrinks to a point.

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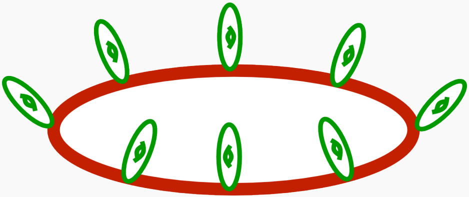
M fibers over a circle with 2-torus fibers. The monodromy is a hyperbolic element of $\mathrm{SL}(2, \mathbb{Z})$.



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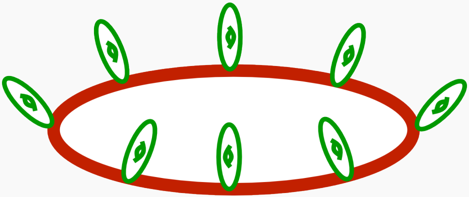

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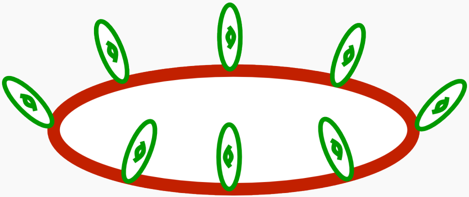

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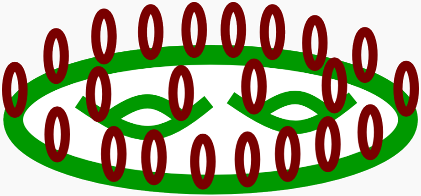
$\widetilde{SL(2,\mathbb{R})}$ geometry

Suppose that M is the unit tangent bundle of a hyperbolic surface Σ.



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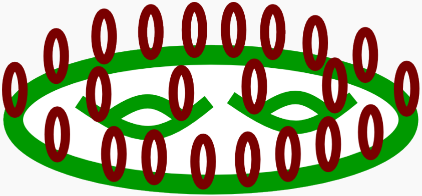

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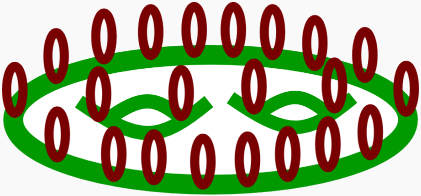

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With the rescaled metric, $(M, \widehat{g}(t))$ approaches the hyperbolic surface Σ. As the fibers shrink, the local geometry of the total space becomes more product-like.

Is there a common pattern?

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There is a common pattern, but to see it one must pass to the universal cover. That is, we are looking at the Ricci flow on a Thurston geometry of type $\mathbb{R}^3, H^3, H^2 \times \mathbb{R}, \mathrm{Sol}, \mathrm{Nil}$ or $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

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(L. 2007) For any initial globally homogeneous metric on such a Thurston geometry, there is a limiting (blowdown) Ricci flow solution, which is an expanding soliton. There is one such soliton for each homogeneity type. It is a universal attractor.

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The limiting solitons

Thurston type	Expanding soliton
H^3	$4\,t\,g_{H^3}$
$H^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}(2,\mathbb{R})}$	$2\,t\,g_{H^2} + g_{\mathbb{R}}$
Sol	$e^{-2z}\,dx^2 + e^{2z}\,dy^2 + 4\,t\,dz^2$
Nil	$\frac{1}{3t^{\frac{1}{3}}}\left(dx + \frac{1}{2}ydz - \frac{1}{2}zdy\right)^2 + t^{\frac{1}{3}}\left(dy^2 + dz^2\right)$
$\mathbb{R}^3$	$g_{\mathbb{R}^3}$

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Conjecture

For a long-time 3D Ricci flow, the diameter is $O(\sqrt{t})$ *if and only if* M *admits a locally homogeneous metric.*

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4. Local stability results for certain expanding Ricci solitons (due to Dan Knopf).

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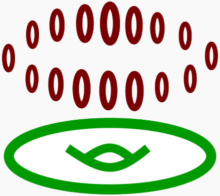

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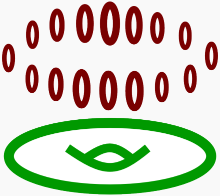

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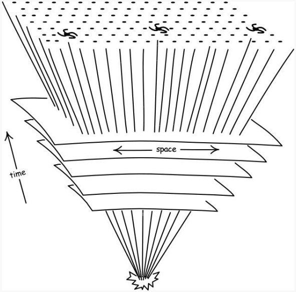
Question : is this true for *all* initial metrics on T^3?

Long-time behavior

The setup



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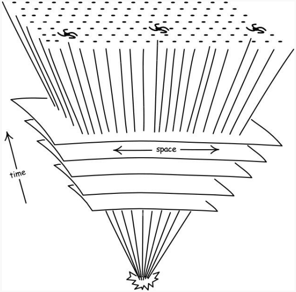

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The spacetime is diffeomorphic to $(0, \infty) \times X$, where X is a compact three-dimensional manifold.

Einstein equations

The spacetime has a Lorentzian metric g. The Einstein equation of general relativity is

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}.$$

Here $R_{\alpha\beta}$ is the Ricci tensor and $R = \sum_{\alpha,\beta} g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature function.

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i.e. the Lorentzian metric g is Ricci-flat.

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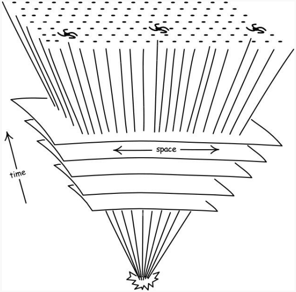

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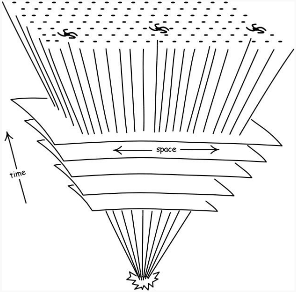

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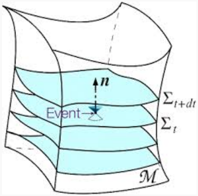
There has been lots of work on this, mostly under some symmetry assumptions for the spatial slices (e.g. locally homogeneous or T^2-symmetry). Are there more general results?

What is time?

Suppose that we have a foliation of the spacetime by compact hypersurfaces.



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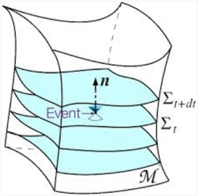

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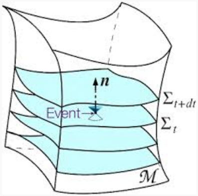

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Let's *assume* that along any given hypersurface, the expansion factor is constant. This defines a constant mean curvature (CMC) foliation.

Einstein flow

Using the foliation, the metric takes the form

$$g = -L^2 dt^2 + h(t),$$

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where $L = L(t)$ is a function on X and $h(t)$ is a Riemannian metric on X. The Ricci-flat condition on g becomes

$$\frac{\partial h_{ij}}{\partial t} = -2LK_{ij} \tag{3}$$

and

$$\frac{\partial K_{ij}}{\partial t} = LHK_{ij} - 2L \sum_{k,l} h^{kl} K_{ik} K_{lj} - L_{;ij} + LR_{ij}, \tag{4}$$

along with certain time-independent "constraint" equations. Here the mean curvature $H = \sum_{i,j} h^{ij} K_{ij}$ is spatially constant.

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The analogous statement in Ricci flow is that $t^{-\frac{3}{2}}\,\mathrm{vol}(X, h(t))$ is monotonically nonincreasing.

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On the infinitesimal level, this means that there is a future-directed vector field X with $\mathcal{L}_X g = cg$. I'll take $c = 2$.

Self-similar solutions

A Lorentzian metric g is *self-similar* if there's a one-parameter group of diffeomorphisms $\{\phi_s\}$ so that $\phi_s^* g = e^{cs} g$, for some $c \in \mathbb{R}$.

On the infinitesimal level, this means that there is a future-directed vector field X with $\mathcal{L}_X g = cg$. I'll take $c = 2$.

This is the analog of an (expanding) Ricci soliton.

Explicit solutions

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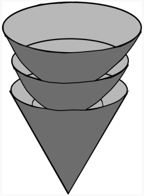

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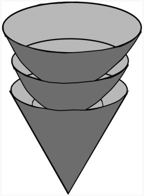

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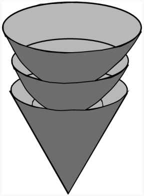

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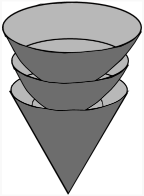

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A spatially compact quotient is called a Löbell spacetime.

2. The Bianchi-III flat spacetime is $\mathbb{R}$ times the interior of a forward lightcone in $\mathbb{R}^{2,1}$.

More explicit solutions

3. The Taub-flat spacetime is $\mathbb{R}^2$ times the interior of a forward lightcone in $\mathbb{R}^{1,1}$.

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4. The Kasner spacetimes live on $(0, \infty) \times \mathbb{R}^3$, with metric

$$g = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 .$$

Here

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 .$$

Convergence result

The scale-invariant curvature condition is that $\| \mathrm{Rm}_g \| = O(t^{-2})$ as $t \to \infty$. (This is the analog of a type-III solution in Ricci flow.)

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Theorem

(L. 2018) Suppose that $(h(t), K(t), L(t))$ is an expanding CMC Einstein flow on a compact aspherical three dimensional manifold X. Suppose that the curvature is $O(t^{-2})$ in magnitude, and the diameter of $(X, h(t))$ is $O(t)$.
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(If there is a lower volume bound $\mathrm{vol}(h(t)) \geq \mathrm{const.}\, t^3$ then the model space is the Milne spacetime. This case is due to Mike Anderson.)

Type-II solutions

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Theorem

(L. 2018) Suppose that $(h(t), K(t), L(t))$ is an expanding CMC Einstein flow on a compact three dimensional manifold X. Suppose that the curvature is not $O(t^{-2})$ in magnitude. Doing a blowdown analysis at points (x_i, t_i) of spatially maximal curvature, with $t_i \to \infty$, one can extract a limit flow.

It turns out to be flat.

An apparent paradox

In the blowdown analysis, we rescale so that $\| \operatorname{Rm}_g(x_i, t_i)\| = 1$. How can the limit be flat?

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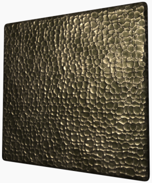
In effect, there are increasing curvature fluctuations that average out the curvature to zero. The rescaled metrics *do* converge to a flat metric in the $C^{1,\alpha}$-topology.



Some questions

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2. By Hawking's singularity theorem, if we look backward in time, there is geodesic incompleteness, and often curvature blowup. (Big bang.)

Can one understand the geometric asymptotics as one approaches the singularity? (BKL conjectures.)