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J. Math. Pures Appl. 88 (2007) 219–229

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

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Hamilton–Jacobi semigroup on length spaces and applications

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Received 16 February 2007

Available online 3 July 2007

Abstract

We define a Hamilton–Jacobi semigroup acting on continuous functions on a compact length space. Following a strategy of Bobkov, Gentil and Ledoux, we use some basic properties of the semigroup to study geometric inequalities related to concentration of measure. Our main results are that (1) a Talagrand inequality on a measured length space implies a global Poincaré inequality and (2) if the space satisfies a doubling condition, a local Poincaré inequality and a log-Sobolev inequality then it also satisfies a Talagrand inequality.

Résumé

Nous définissons un semi-groupe de Hamilton–Jacobi agissant sur les fonctions continues définies sur un espace de longueurs compact. Nous utilisons les propriétés de ce semi-groupe pour étudier certaines inégalités géométriques liées au phénomène de concentration de la mesure, selon une stratégie initiée par Bobkov, Gentil et Ledoux. Nos principaux résultats stipulent que (1) une inégalité de Talagrand sur un espace de longueurs mesuré implique une inégalité de Poincaré globale, et (2) si l'espace vérifie en outre une condition de doublement, une inégalité de Poincaré locale et une inégalité de Sobolev logarithmique, alors il admet aussi une inégalité de Talagrand.

Keywords: Metric-measure spaces; Hamilton–Jacobi semigroup; Talagrand inequality; Logarithmic Sobolev inequality; Poincaré inequality; Ricci curvature

Links between concentration of measure, log-Sobolev inequalities, Talagrand inequalities and Poincaré inequalities have been studied in the setting of Riemannian manifolds [1–3,8,9,12]. The main result in the paper of Otto and Villani [12] can be informally stated as follows: on a Riemannian manifold, a log-Sobolev inequality implies a Talagrand inequality, which in turn implies a Poincaré (or spectral gap) inequality, all of this being without any degradation of the constants.

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[1] The research of this author was supported by NSF grant DMS-0604829.

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2007.06.003

On the other hand, there has been intense recent activity to develop a theory of Ricci curvature bounds, log-Sobolev inequalities and related inequalities in the more general setting of metric-measure length spaces satisfying minimal regularity assumptions [10,11,14–16].

The goal of the present paper is to extend the main results of [12] to this generalized framework, which can be considered to be a natural degree of regularity for the problem. To do so, we adapt the strategy of Bobkov–Gentil–Ledoux [2], based on the Hamilton–Jacobi semigroup. We also establish the basic properties of the Hamilton–Jacobi semigroup for general length spaces, which is of independent interest.

1. Main results

Basic information on length spaces is in [4, Chapter 2]. For the sake of simplicity we work with compact length spaces, but the results remain valid for locally compact complete separable length spaces.

Throughout this paper, X will denote a compact length space, equipped with a metric d and a Borel reference probability measure ν. We use the following conventions:

- $\mathrm{Lip}(X)$ denotes the set of real-valued Lipschitz functions on X.
- Given $f \in C(X)$, we define the *gradient norm* of f at a point $x \in X$ by:

$$|\nabla f|(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x, y)}. \tag{1.1}$$

If $f \in \mathrm{Lip}(X)$ then $|\nabla f| \in L^\infty(X)$.

- We further define the *subgradient norm* of f at x by:

$$|\nabla^- f|(x) = \limsup_{y \to x} \frac{[f(y) - f(x)]_-}{d(x, y)} = \limsup_{y \to x} \frac{[f(x) - f(y)]_+}{d(x, y)}. \tag{1.2}$$

Here $a_+ = \max(a, 0)$ and $a_- = \max(-a, 0)$. Clearly $|\nabla^- f|(x) \leqslant |\nabla f|(x)$, so the subgradient norm is a (slightly) finer notion than the gradient norm. Note that $|\nabla^- f|(x)$ is automatically zero if f has a local minimum at x. In a sense, $|\nabla^- f|(x)$ measures the downward pointing component of f near x.

- Given two probability measures μ_0 and μ_1 on X, the Wasserstein distance (of order 2) $W_2(\mu_0, \mu_1)$ between μ_0 and μ_1 is the square root of the optimal transport cost between μ_0 and μ_1, when the infinitesimal cost is the square of the distance; see for instance [17, Theorem 7.3].
- The metric-measure space (X, d, ν) satisfies a *doubling condition* if the measure ν is doubling in the sense of [6, Eq. (0.1)].
- The metric-measure space (X, d, ν) satisfies a *local Poincaré inequality* if the measure ν satisfies the weak Poincaré inequality of type (1,1) as in [6, Eq. (4.3)].
- The metric-measure space (X, d, ν) is *nonbranching* if any two constant-speed geodesics $[0, 1] \to X$ that coincide on an interval $(t_0, t_1) \subset [0, 1]$ are equal.

We will focus on the following three functional inequalities:

- If $K > 0$, we say that (X, d, ν) satisfies a *log-Sobolev inequality* with constant K, $\mathrm{LSI}(K)$, if for any $f \in \mathrm{Lip}(X)$ with $\int_X f^2 \, d\nu = 1$, we have:

$$\int_X f^2 \log(f^2) \, d\nu \leqslant \frac{2}{K} \int_X |\nabla^- f|^2 \, d\nu. \tag{1.3}$$

- We say that (X, d, ν) satisfies a *Talagrand inequality* with constant K, $T(K)$, if for any $F \in L^2(X, \nu)$ with $\int_X F^2 \, \mathrm{d}\nu = 1$, we have:

$$W_2(F^2\nu, \nu) \leqslant \sqrt{\frac{2\int_X F^2 \log(F^2)\, \mathrm{d}\nu}{K}}. \tag{1.4}$$

- We say that (X, d, ν) satisfies a (global) *Poincaré inequality* with constant K, $P(K)$, if for any $h \in \mathrm{Lip}(X)$ with $\int_X h \, \mathrm{d}\nu = 0$, we have:

$$\int_X h^2 \, \mathrm{d}\nu \leqslant \frac{1}{K} \int_X |\nabla^- h|^2 \, \mathrm{d}\nu. \tag{1.5}$$

Remark 1.6. In (1.3) and (1.5) we use the subgradient norm defined in (1.2), instead of the gradient norm defined in (1.1). Accordingly, our log-Sobolev and Poincaré inequalities are slightly stronger statements than those discussed by many other authors.

Inequalities (1.3), (1.4) and (1.5) are associated with concentration of measure [1–3,8,9,18]. For example, $T(K)$ implies a Gaussian-type concentration of measure.

The following chain of implications, none of which is an equivalence, is well-known in the context of smooth Riemannian manifolds:

$$[\mathrm{Ric} \geqslant K] \Longrightarrow \mathrm{LSI}(K) \Longrightarrow T(K) \Longrightarrow P(K). \tag{1.7}$$

A complete proof of (1.7) is available for instance in [18, Chapters 21 and 22].

Our main result is as follows:

Theorem 1.8. *Let (X, d, ν) be a compact measured length space.*

(i) *If (X, d, ν) satisfies $T(K)$, for some $K > 0$, then it also satisfies $P(K)$.*
(ii) *Suppose that (X, d, ν) satisfies a doubling condition on the measure and a local Poincaré inequality. If (X, d, ν) satisfies $\mathrm{LSI}(K)$ for some $K > 0$, then it also satisfies $T(K)$.*

It is standard that $\mathrm{LSI}(K)$ implies $P(K)$; see, for example, [10, Theorem 6.18].

The assumptions of Theorem 1.8(ii) are satisfied if (X, d, ν) is nonbranching and has Ricci curvature bounded below in the sense of Lott–Villani and Sturm [11,14,16]. They are also satisfied if (X, d) is a length space with Alexandrov curvature bounded below and Hausdorff dimension $n < \infty$, and ν is the n-dimensional Hausdorff measure on X; the doubling property follows from the Bishop–Gromov inequality [4, Theorem 10.6.6] and the local Poincaré inequality was proven in [7, Theorem 7.2].

Theorem 1.8 will be proven in Section 3. An important technical tool in the proof is the quadratic Hamilton–Jacobi semigroup, which will be introduced and studied in Section 2. We thank Juha Heinonen for some helpful comments.

2. Hamilton–Jacobi semigroup

First, we recall the Hamilton–Jacobi semigroup in the case of Riemannian manifolds. If M is a compact Riemannian manifold, then the quadratic Hamilton–Jacobi equation on M is:

$$\frac{\partial F}{\partial t} + \frac{|\nabla F|^2}{2} = 0. \tag{2.1}$$

Given an initial condition $f \in C(M)$, the viscosity solution to the Hamilton–Jacobi equation is given by the Hopf–Lax formula,

$$F(t, x) = \inf_{y \in X} \left[f(y) + \frac{d(x, y)^2}{2t} \right], \tag{2.2}$$

where d is the geodesic distance on M. The map that sends f to $F(t,\cdot)$ defines a semigroup action of $\mathbb{R}_+$ on $C(M)$, called the Hamilton–Jacobi semigroup.

Eq. (2.2) does not require any smoothness assumption, so the following definition makes sense.

Definition 2.3. Let (X,d) be a compact metric space. Given $f \in C(X)$ and $t \geqslant 0$, we define a map $Q_t : X \to \mathbb{R}$ by:

$$(Q_t f)(x) = \inf_{y\in X}\left[f(y) + \frac{d(x,y)^2}{2t}\right], \tag{2.4}$$

with the convention that $Q_0 f = f$.

If X is a length space, then the map Q_t defines a semigroup action of $\mathbb{R}^+$ on $C(X)$; see part (i) of Theorem 2.5 below. We may then speak of the "Hamilton–Jacobi semigroup". The next theorem establishes some of its basic properties.

Theorem 2.5.

(i) *For any $s,t \geqslant 0$, $Q_t Q_s f = Q_{t+s} f$.*
(ii) *For any $x \in X$, $\inf f \leqslant (Q_t f)(x) \leqslant f(x)$.*
(iii) *For any $t > 0$, $Q_t f \in \mathrm{Lip}(X)$.*
(iv) *For any $x \in X$, $(Q_t f)(x)$ is a nonincreasing function of t, that converges monotonically to $f(x)$ as $t \to 0$. In particular, $\lim_{t\to 0} Q_t f = f$ in $C(X)$.*
(v) *For any $t \geqslant 0$, $s > 0$ and $x \in X$,*

$$\frac{|Q_{t+s} f(x) - Q_t f(x)|}{s} \leqslant \frac{\|Q_t f\|_{\mathrm{Lip}}^2}{2}. \tag{2.6}$$

(vi) *For any $x \in X$ and $t \geqslant 0$,*

$$\liminf_{s\to 0^+} \frac{(Q_{t+s} f)(x) - (Q_t f)(x)}{s} \geqslant -\frac{|\nabla^- Q_t f|(x)^2}{2}. \tag{2.7}$$

(vii) *If (X,d,ν) satisfies a doubling condition on the measure and a local Poincaré inequality, then for $t > 0$ and ν-almost any $x \in X$,*

$$\lim_{s\to 0^+} \frac{(Q_{t+s} f)(x) - (Q_t f)(x)}{s} = -\frac{|\nabla^- Q_t f|^2(x)}{2}. \tag{2.8}$$

(viii) *If (X,d) is a finite-dimensional space with Alexandrov curvature bounded below then for any $t > 0$ and any $x \in X$,*

$$\lim_{s\to 0^+} \frac{(Q_{t+s} f)(x) - (Q_t f)(x)}{s} = -\frac{|\nabla^- Q_t f|^2(x)}{2}. \tag{2.9}$$

Remark 2.10. Part (vii) of Theorem 2.5 will be used in the proof of Theorem 1.8. Part (viii) is not needed for the proof of Theorem 1.8, but may be of independent interest. Parts (vii) and (viii) show that for $t > 0$, the function $F(t,x) = Q_t f(x)$ satisfies the Hamilton–Jacobi equation,

$$\frac{\partial F}{\partial t} + \frac{|\nabla^- F|^2}{2} = 0, \tag{2.11}$$

almost everywhere in the case of (vii) and everywhere in the case of (viii).

Remark 2.12. Theorem 2.5 is reminiscent of known properties of Hamilton–Jacobi equations in a smooth setting; see e.g. [5]. However, even in the context of Riemannian manifolds, we have been unable to find exactly this statement in the literature. On the one hand, the vast majority of works are only concerned with Euclidean or Hilbert spaces. On the other hand, the use of the subgradient norm is a bit nonstandard.

Remark 2.13. More general Hamilton–Jacobi semigroups will be considered in [18, Appendix of Chapter 22], of the form:

$$Q_t f(x) = \inf_{y\in X}\left[f(y) + tL\left(\frac{d(x,y)}{t}\right)\right],$$

where $L:\mathbb{R}_+ \to \mathbb{R}_+$ is increasing, convex and locally semiconcave, with $L(0)=0$. Theorem 2.5 can be extended mutatis mutandis to this more general situation (apart maybe from (viii)). In this generalization, a few minor complications arise if $L'(\infty) < +\infty$. (For simplicity, only Riemannian manifolds are considered in [18, Chapter 22], but nonsmooth spaces can be treated as in the present paper.) At the level of geometric applications, an interesting case in relation to Poincaré inequalities (as opposed to log Sobolev inequalities) is when $L(s)$ is asymptotic to s^2 for small s, and to s for large s.

Proof of Theorem 2.5. To prove (i), we first claim that for all $x, y \in X$ and $s, t > 0$,

$$\frac{d(x,y)^2}{t+s} = \inf_{z\in X}\left[\frac{d(x,z)^2}{t} + \frac{d(z,y)^2}{s}\right]. \tag{2.14}$$

The triangle inequality implies that the left-hand side of (2.14) is less than or equal to the right-hand side. The equality in (2.14) comes from choosing a minimal geodesic between x and y, and a point z on this geodesic with $d(x,z) = \frac{t}{s+t} d(x,y)$.

From (2.14), we obtain:

$$(Q_{t+s}f)(x) = \inf_{y\in X}\left[f(y) + \frac{d(x,y)^2}{2(t+s)}\right] = \inf_{y\in X}\inf_{z\in X}\left[f(y) + \frac{d(x,z)^2}{2t} + \frac{d(z,y)^2}{2s}\right] = (Q_t Q_s f)(x), \tag{2.15}$$

which proves (i).

For part (ii), the inequality on the left is obvious, while the inequality on the right follows from the choice $y = x$ in the definition of $(Q_t f)(x)$.

Part (iii) follows from

$$\begin{aligned}(Q_t f)(x) - (Q_t f)(x') &\leqslant \frac{1}{2t}\sup_{y\in X}\left[d(x,y)^2 - d(x',y)^2\right] \\ &\leqslant \left(\frac{1}{2t}\sup_{y\in X}\left[d(x,y) + d(x',y)\right]\right) d(x,x') \\ &\leqslant \frac{\operatorname{diam}(X)}{t} d(x,x').\end{aligned} \tag{2.16}$$

In view of (i) and (ii), for any $s, t > 0$ and $x \in X$,

$$(Q_{t+s}f)(x) \leqslant (Q_t f)(x), \tag{2.17}$$

so $(Q_t f)(x)$ is indeed a nonincreasing function of t. Given $f \in C(X)$, put $C = C(f) = 2(\sup f - \inf f)$. If y is such that $d(x,y) \geqslant \sqrt{Ct}$, then

$$f(y) + \frac{d(x,y)^2}{2t} \geqslant (\inf f) + \frac{C}{2} = \sup f \geqslant f(x). \tag{2.18}$$

We conclude that

$$(Q_t f)(x) = \inf_{y\in B_{\sqrt{Ct}}(x)}\left[f(y) + \frac{d(x,y)^2}{2t}\right]. \tag{2.19}$$

Given $x \in X$ and $\varepsilon > 0$, choose $\delta > 0$ so that

$$d(x,y) < \delta \Longrightarrow \left|f(x) - f(y)\right| < \varepsilon. \tag{2.20}$$

If $t \leqslant \delta^2/C$ then $\sqrt{Ct} \leqslant \delta$, so

$$(Q_t f)(x) \geqslant \inf_{y\in B_\delta(x)}\left[f(y) + \frac{d(x,y)^2}{2t}\right] \geqslant f(x) - \varepsilon. \tag{2.21}$$

This shows that $\lim_{t\to 0}(Q_t f)(x) = f(x)$. Since the convergence is monotone and X is compact, the convergence is uniform. This proves part (iv) of the theorem.

Next, for $g \in C(X)$, we write (with $C = C(g)$ and the convention that $0 \cdot \infty = 0$),

$$\begin{aligned}\frac{g(x) - (Q_s g)(x)}{s} &= \frac{1}{s} \sup_{y \in B_{\sqrt{Cs}}(x)} \left[g(x) - g(y) - \frac{d(x,y)^2}{2s} \right] \\ &\leqslant \sup_{y \in B_{\sqrt{Cs}}(x)} \left(\frac{[g(x) - g(y)]_+}{d(x,y)} \frac{d(x,y)}{s} - \frac{d(x,y)^2}{2s^2} \right) \\ &\leqslant \sup_{y \in B_{\sqrt{Cs}}(x)} \frac{1}{2} \left(\frac{[g(x) - g(y)]_+}{d(x,y)} \right)^2.\end{aligned}$$

If $g = Q_t f$, in view of (i) and (ii) this becomes:

$$0 \leqslant \frac{Q_t f(x) - (Q_{t+s} f)(x)}{s} \leqslant \sup_{y \in B_{\sqrt{Cs}}(x)} \frac{1}{2} \left(\frac{[Q_t f(x) - Q_t f(y)]_+}{d(x,y)} \right)^2. \tag{2.22}$$

Then statement (v) follows immediately. If now we let $s \to 0^+$ then the definition of $|\nabla^- Q_t f|$ implies that

$$\limsup_{s \to 0^+} \sup_{y \in B_{\sqrt{Cs}}(x)} \frac{1}{2} \left(\frac{[Q_t f(x) - Q_t f(y)]_+}{d(x,y)} \right)^2 \leqslant \frac{|\nabla^- Q_t f|^2(x)}{2}, \tag{2.23}$$

and (vi) is also true.

We now turn to (vii) and (viii), which are the most delicate parts of the theorem. Again with $g = Q_t f$, we want to prove that

$$\liminf_{s \to 0^+} \left[\frac{g(x) - (Q_s g)(x)}{s} \right] \geqslant \frac{|\nabla^- g|^2(x)}{2}. \tag{2.24}$$

The case when $|\nabla^- g|(x) = 0$ is obvious, since $(Q_t g)(x)$ is a nonincreasing function of t. So in what follows we assume that $|\nabla^- g|(x) > 0$.

We write

$$\begin{aligned}\frac{g(x) - (Q_s g)(x)}{s} &= \frac{1}{s} \sup_{y \in X} \left[g(x) - g(y) - \frac{d(x,y)^2}{2s} \right] \\ &\geqslant \sup_{y \in S_{s|\nabla^- g|(x)}(x)} \left(\left[\frac{g(x) - g(y)}{d(x,y)} \right] |\nabla^- g|(x) - \frac{|\nabla^- g|^2(x)}{2} \right).\end{aligned} \tag{2.25}$$

Put

$$\psi(r) = \sup_{y \in S_r(x)} \frac{g(x) - g(y)}{d(x,y)}. \tag{2.26}$$

As $\limsup_{r \to 0^+} \psi(r) = |\nabla^- g|(x) > 0$, if we can show that $\liminf_{r \to 0^+} \psi(r) = |\nabla^- g|(x)$ then Eq. (2.25) will imply (2.24).

For (vii), we use results from [6]. By [6, Theorem 10.2], the Lipschitz function g admits generalized linear derivatives $g_{0,x}$ at x for ν-almost all $x \in X$. For such an x, suppose that there is a sequence $r_i \to 0$ such that $\lim_{i \to 0} \psi(r_i) = |\nabla^- g|(x) - \varepsilon$ for some $\varepsilon > 0$. After passing to a further subsequence, we can assume that the rescaled measured length spaces $(X, x, r_i^{-1/2} d, \nu)$ converge to a measured tangent cone $(X_x, x_\infty, d_\infty, \nu_\infty)$ and the rescaled functions $g_{r_i^{1/2}, x} = (g - g(x))/r_i^{1/2}$ converge to a generalized linear function $g_{0,x}$ on X_∞; see [6, Theorem 10.2]. (Note that we rescale by $r_i^{1/2}$ and not r_i; any rate $s_i \to 0$ such that $r_i = o(s_i)$ would do.) Then $|\nabla^- g_{0,x}|(x_\infty) = |\nabla^- g|(x) - \varepsilon$. From [6, Theorem 8.10], there is a unit-speed line γ in X_∞ through x_∞ which is an integral curve for $g_{0,x}$. That is, $\gamma(0) = x_\infty$ and $\frac{\mathrm{d}}{\mathrm{d}t} g_{0,x}(\gamma(t)) = |\nabla g_{0,x}|$. It follows that $|\nabla^- g_{0,x}|(x_\infty) \geqslant |\nabla g_{0,x}|(x_\infty)$. However, from [6, Theorem 10.2], one has $|\nabla g_{0,x}|(x_\infty) = |\nabla g|(x)$. Thus $|\nabla g|(x) \leqslant |\nabla^- g|(x) - \varepsilon$, which is a contradiction. This proves statement (vii).

Remark 2.27. This reasoning shows actually shows that $|\nabla^- g|(x) \geqslant |\nabla g|(x)$, so $|\nabla^- g|(x) = |\nabla g|(x)$ (for ν-almost all x).

Statement (viii), with convergence for all $x \in X$, requires additional regularity for X. We will use the notion of *quasigeodesics* in Alexandrov spaces, as studied in [13]. The following properties will be useful: (a) squared distance functions, when restricted to quasigeodesics, satisfy the same curvature-dependent differential inequalities as when restricted to geodesics (inequality (2.30) below); (b) quasigeodesics can be extended to all positive times; (c) uniform limits of quasigeodesics are quasigeodesics, and this statement goes through for quasigeodesics defined on a Gromov–Hausdorff converging sequence of Alexandrov spaces. We recall that by definition, nontrivial quasi-geodesics are parametrized by arc-length.

Lemma 2.28. *Let X be a finite-dimensional compact length space with Alexandrov curvature bounded below. Fix $x \in X$. Then*

(i) *There is some $\delta > 0$ so that each complete quasigeodesic $\gamma : [0, \infty) \to X$ starting from x intersects $S_\delta(x)$.*
(ii) *There is a function $\sigma : (0, \delta) \to \mathbb{R}_+$ with $\lim_{r\to 0^+} \sigma(r) = 0$ so that if $\gamma : [0, L] \to X$ is a quasigeodesic segment starting from x with $\gamma(L) \in S_r(x)$ and $\gamma([0, L]) \subset \overline{B_r(x)}$ then $|L/r - 1| \leqslant \sigma(r)$.*

Remark 2.29. Of course, if X is a Riemannian manifold then this lemma holds true for geodesics. (Take δ to be the injectivity radius at x and take $\sigma = 0$.)

Proof of Lemma 2.28. Suppose that (i) is not true. Then for each $i \in \mathbb{Z}^+$, there is a quasigeodesic $\gamma_i : [0, \infty) \to X$ starting from x that remains in $B_{1/i}(x)$. Taking a convergent subsequence of $\{\gamma_i\}_{i=1}^\infty$ [13, §2] gives a quasigeodesic $\gamma_\infty : [0, \infty) \to X$ whose image is $\{x\}$. This contradicts the fact that a quasigeodesic has unit speed.

Suppose that (ii) is not true. Then there is an $\varepsilon > 0$ along with a sequence $\{r_i\}_{i=1}^\infty$ converging to zero and a sequence of quasigeodesic segments $\gamma_i : [0, L_i] \to X$ starting from x so that for all $i \in \mathbb{Z}^+$, we have that $\gamma_i(L_i) \in S_{r_i}(x)$, $\gamma_i([0, L_i]) \subset \overline{B_{r_i}(x)}$ and $L_i/r_i \geqslant 1 + \varepsilon$. Rescaling the pointed Alexandrov space (X, x) by $1/r_i$ and taking a convergent subsequence of $\{\gamma_i\}_{i=1}^\infty$ [13, Theorem 2.2], we obtain a quasigeodesic segment $\gamma_\infty : [0, L_\infty] \to C_x X$ starting at the vertex o of the tangent cone $C_x X$ so that $\gamma_\infty(L_\infty) \in S_1(o)$ (if $L_\infty < \infty$), $\gamma_\infty([0, L_\infty]) \subset \overline{B_1(o)}$ and $L_\infty \geqslant 1 + \varepsilon$. However, one can check that a quasigeodesic in $C_x X$ starting at o must be a radial geodesic, which is a contradiction. □

Let us go back to the proof of Theorem 2.5, part (viii). As X is compact with Alexandrov curvature bounded below, there is a $K \geqslant 0$ so that for all quasigeodesic segments $\gamma : [0, u] \to X$ starting from x, all $u' \in [0, u]$ and all $z \in X$,

$$d\big(\gamma(u'), z\big)^2 - \frac{u'}{u} d\big(\gamma(u), z\big)^2 - \frac{u - u'}{u} d(x, z)^2 \geqslant -K u'(u - u'). \tag{2.30}$$

Then

$$\frac{1}{2t} \inf_{z\in X} \left(d\big(\gamma(u'), z\big)^2 - \frac{u'}{u} d\big(\gamma(u), z\big)^2 - \frac{u - u'}{u} d(x, z)^2 \right) \geqslant -\frac{1}{2t} K u'(u - u'). \tag{2.31}$$

As

$$\begin{aligned} g\big(\gamma(u')\big) - \frac{u'}{u} g\big(\gamma(u)\big) - \frac{u - u'}{u} g(x) = \inf_{z'\in X} \sup_{z,w\in X} \bigg[& f(z') - \frac{u'}{u} f(z) - \frac{u - u'}{u} f(w) \\ & + \frac{d(\gamma(u'), z')^2}{2t} - \frac{u'}{u} \frac{d(\gamma(u), z)^2}{2t} - \frac{u - u'}{u} \frac{d(x, w)^2}{2t} \bigg], \end{aligned} \tag{2.32}$$

by considering the case when $z' = z = w$, we obtain:

$$g\big(\gamma(u')\big) - \frac{u'}{u} g\big(\gamma(u)\big) - \frac{u - u'}{u} g(x) \geqslant -\frac{1}{2t} K u'(u - u'). \tag{2.33}$$

Equivalently,

$$\frac{g(x) - g(\gamma(u))}{u} + \frac{1}{2t} K u \geqslant \frac{g(x) - g(\gamma(u'))}{u'} + \frac{1}{2t} K u'. \tag{2.34}$$

In order to prove that $\liminf_{r\to 0^+} \psi(r) = |\nabla^- g|(x)$, suppose that $\liminf_{r\to 0^+} \psi(r) = |\nabla^- g|(x) - \varepsilon$ for some $\varepsilon > 0$. Then there are sequences $\{u'_i\}_{i=1}^{\infty}$ and $\{v_j\}_{j=1}^{\infty}$ converging to zero with

$$\lim_{i\to\infty} \psi(u'_i) = |\nabla^- g|(x), \tag{2.35}$$

and

$$\lim_{i\to\infty} \psi(v_i) = |\nabla^- g|(x) - \varepsilon. \tag{2.36}$$

We may assume that $u'_i < v_i < \delta$, where δ is from Lemma 2.28(i).

In particular, there are points $y'_i \in S_{u'_i}(x)$ so that

$$\lim_{i\to\infty} \frac{g(x) - g(y'_i)}{u_i} = |\nabla^- g|(x). \tag{2.37}$$

Choose a minimizing geodesic γ_i from x to y'_i. Extend it to a complete quasigeodesic $\gamma_i : [0,\infty) \to X$. Put

$$u_i = \inf\bigl\{w_i\colon\ \gamma_i(w_i) \in S_{v_i}(x)\bigr\}. \tag{2.38}$$

From Lemma 2.28(i), u_i exists. As γ_i is parametrized by arclength, $u_i \geqslant v_i > u'_i$. From (2.34),

$$\frac{v_i}{u_i} \frac{g(x) - g(\gamma(u_i))}{v_i} + \frac{1}{2t} K u_i \geqslant \frac{g(x) - g(y'_i)}{u'_i} + \frac{1}{2t} K u'_i. \tag{2.39}$$

In particular,

$$\frac{v_i}{u_i} \psi(v_i) + \frac{1}{2t} K u_i \geqslant \frac{g(x) - g(y'_i)}{u'_i} + \frac{1}{2t} K u'_i. \tag{2.40}$$

From Lemma 2.28(ii),

$$\lim_{i\to\infty} \frac{v_i}{u_i} = 1. \tag{2.41}$$

Taking $i \to \infty$ in (2.40), we get a contradiction to (2.36) and (2.37). □

3. Proof of Theorem 1.8

Armed with Theorem 2.5, we can now use the strategy of [2] to prove Theorem 1.8.

Proof of Theorem 1.8, part (i). Let $h \in \mathrm{Lip}(X)$ satisfy $\int_X h\,\mathrm{d}\nu = 0$. Introduce:

$$\psi(t) = \int_X \mathrm{e}^{KtQ_t h}\,\mathrm{d}\nu. \tag{3.1}$$

From Talagrand's inequality in its dual formulation (see [3, p. 16], [17, Exercise 9.15] or [18, Chapter 22]), we know that $\psi(t) \leqslant \exp(Kt \int_X h\,\mathrm{d}\nu) = 1$. Hence ψ has a maximum at $t = 0$. Combining this with $\int h\,\mathrm{d}\nu = 0$, we find:

$$0 \leqslant \limsup_{t\to 0^+} \left(\frac{1-\psi(t)}{Kt^2}\right) = \limsup_{t\to 0^+} \int_X \left(\frac{1 + Kth - \mathrm{e}^{KtQ_t h}}{Kt^2}\right) \mathrm{d}\nu. \tag{3.2}$$

By the boundedness of $Q_t h$ and Theorem 2.5(iv),

$$\begin{aligned} \mathrm{e}^{KtQ_t h} &= 1 + KtQ_t h + \frac{K^2 t^2}{2} (Q_t h)^2 + \mathrm{O}(t^3) \\ &= 1 + KtQ_t h + \frac{K^2 t^2}{2} h^2 + \mathrm{o}(t^2). \end{aligned} \tag{3.3}$$

So the right-hand side of (3.2) equals:

$$\limsup_{t\to 0^+}\int_X\left(\frac{h-Q_th}{t}\right)\mathrm{d}\nu-\frac{K}{2}\int_X h^2\,\mathrm{d}\nu. \tag{3.4}$$

By Theorem 2.5(v), $(h-Q_th)/t$ is bounded, which allows us to apply Fatou's lemma in the form:

$$\limsup_{t\to 0^+}\int_X\left(\frac{h-Q_th}{t}\right)\mathrm{d}\nu\leqslant\int_X\limsup_{t\to 0^+}\left(\frac{h-Q_th}{t}\right)\mathrm{d}\nu. \tag{3.5}$$

Then Theorem 2.5(vi) implies that

$$\int_X\limsup_{t\to 0^+}\left(\frac{h-Q_th}{t}\right)\mathrm{d}\nu\leqslant\int_X\frac{|\nabla^-h|^2}{2}\,\mathrm{d}\nu. \tag{3.6}$$

All in all, the right-hand side of (3.2) can be bounded above by:

$$\frac{1}{2}\int_X|\nabla^-h|^2\,\mathrm{d}\nu-\frac{K}{2}\int_X h^2\,\mathrm{d}\nu, \tag{3.7}$$

so this expression is nonnegative. This concludes the proof. □

Proof of Theorem 1.8, part (ii). From Talagrand's inequality in its dual formulation, it is sufficient to show that for all $g\in C(X)$,

$$\int_X \mathrm{e}^{K\inf_y[g(y)+\frac{d(x,y)^2}{2}]}\,\mathrm{d}\nu(x)\leqslant\mathrm{e}^{K\int_X g\,\mathrm{d}\nu}. \tag{3.8}$$

Put

$$\phi(t)=\frac{1}{Kt}\log\left(\int_X\mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu\right). \tag{3.9}$$

Since g is bounded, Theorem 2.5(ii) implies that Q_tg is bounded, uniformly in t. Thus

$$\int_X\mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu=1+Kt\int_X Q_tg\,\mathrm{d}\nu+\mathrm{O}(t^2) \tag{3.10}$$

and

$$\phi(t)=\int_X Q_tg\,\mathrm{d}\nu+\mathrm{O}(t). \tag{3.11}$$

By Theorem 2.5(iv), Q_tg converges uniformly to g as $t\to 0^+$, and so

$$\lim_{t\to 0^+}\phi(t)=\int_X g\,\mathrm{d}\nu. \tag{3.12}$$

Therefore, our goal will be achieved if we can show that $\phi(1)\leqslant\lim_{t\to 0^+}\phi(t)$. For this, it suffices to show that $\phi(t)$ is nonincreasing in t.

Let $t\in(0,1]$ be given. For $s>0$, we have:

$$\begin{aligned}\frac{\phi(t+s)-\phi(t)}{s}&=\frac{1}{s}\left(\frac{1}{K(t+s)}-\frac{1}{Kt}\right)\log\int_X\mathrm{e}^{K(t+s)Q_{t+s}g}\,\mathrm{d}\nu\\&\quad+\frac{1}{Kts}\left(\log\int_X\mathrm{e}^{K(t+s)Q_{t+s}g}\,\mathrm{d}\nu-\log\int_X\mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu\right).\end{aligned} \tag{3.13}$$

As $s \to 0^+$, $\mathrm{e}^{K(t+s)Q_{t+s}g}$ converges uniformly to e^{KtQ_tg}. Thus the limit of the first term in the right-hand side above, as $s \to 0^+$, is

$$-\frac{1}{Kt^2}\log\left(\int_X \mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu\right), \tag{3.14}$$

while the limit of the second term is:

$$\frac{1}{Kt\int \mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu}\lim_{s\to 0^+}\left[\frac{1}{s}\left(\int_X \mathrm{e}^{K(t+s)Q_{t+s}g}\,\mathrm{d}\nu - \int_X \mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu\right)\right], \tag{3.15}$$

provided that the latter limit exists. We rewrite the expression inside the square brackets as

$$\int_X \left(\frac{\mathrm{e}^{K(t+s)Q_{t+s}g} - \mathrm{e}^{KtQ_{t+s}g}}{s}\right)\mathrm{d}\nu + \int_X \left(\frac{\mathrm{e}^{KtQ_{t+s}g} - \mathrm{e}^{KtQ_tg}}{s}\right)\mathrm{d}\nu. \tag{3.16}$$

The integrand of the first term in (3.16) can be rewritten as $(\mathrm{e}^{KtQ_{t+s}g})(\mathrm{e}^{KsQ_{t+s}g}-1)/s$, which converges uniformly to $(\mathrm{e}^{KtQ_tg})KQ_tg$ as $s \to 0^+$. So the first integral in (3.16) converges to $\int_X (KQ_tg)\mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu$.

We now turn to the second term of (3.16). By Theorem 2.5(vii), for ν-almost all $x \in X$, we have:

$$Q_{t+s}g(x) = Q_tg(x) - s\left(\frac{|\nabla^- Q_tg(x)|^2}{2} + \mathrm{o}(1)\right), \tag{3.17}$$

and therefore

$$\lim_{s\to 0^+}\frac{\mathrm{e}^{KtQ_{t+s}g(x)} - \mathrm{e}^{KtQ_tg(x)}}{s} = -Kt\mathrm{e}^{KtQ_tg}\frac{|\nabla^- Q_tg(x)|^2}{2}. \tag{3.18}$$

On the other hand, parts (iv) and (v) of Theorem 2.5 imply that

$$Q_{t+s}g = Q_tg + \mathrm{O}(s). \tag{3.19}$$

Since $Q_tg(x)$ is uniformly bounded in t and x, we deduce that

$$\frac{\mathrm{e}^{KtQ_{t+s}g} - \mathrm{e}^{KtQ_tg}}{s} = \mathrm{O}(1) \tag{3.20}$$

as $s \to 0^+$. The combination of (3.18) and (3.20) makes it possible to pass to the limit by dominated convergence, to obtain:

$$\lim_{s\to 0^+}\int_X \left(\frac{\mathrm{e}^{KtQ_{t+s}g} - \mathrm{e}^{KtQ_tg}}{s}\right)\mathrm{d}\nu = -Kt\int_X \frac{|\nabla^- Q_tg|^2}{2}\mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu. \tag{3.21}$$

In summary,

$$\begin{aligned}\lim_{s\to 0^+}\left[\frac{\phi(t+s)-\phi(t)}{s}\right] = \frac{1}{Kt^2\int_X \mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu}\Bigg[&-\left(\int_X \mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu\right)\log\left(\int_X \mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu\right)\\ &+\int_X (KtQ_tg)\mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu - \frac{1}{2K}\int_X \big(Kt|\nabla^- Q_tg|\big)^2\mathrm{e}^{KtQ_tg}\,\mathrm{d}\nu\Bigg].\end{aligned} \tag{3.22}$$

Inequality LSI(K) implies that this quantity is nonpositive, which concludes the proof. □

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