The geometry of the space of measures and its applications

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Singer : 33 Children

Singer : 94 Grandchildren

Singer : 18 Great-grandchildren

Geometry of the space of probability measures

Motivation

Optimal transport

Formal Geometry of Wasserstein Space

Metric geometry of Wasserstein space

Ricci meets Wasserstein

Some more metric geometry

Generalized entropy functionals

Abstract Ricci curvature

Applications

Perelman's reduced volume

Formulas from Riemannian optimal transport

Optimal transport for Ricci flow

Monotonicity of the reduced volume

Infinite-dimensional spaces

Geometry and topology of infinite-dimensional spaces

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Today : the space of probability measures.

Questions

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How can we understand Ricci curvature?

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Perelman introduced an important monotonic quantity in Ricci flow, the reduced volume. Where does this come from?

Claim : These questions can be answered in terms of optimal transport, or the geometry of the space of probability measures.

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LV = Lott-Villani, S = Sturm

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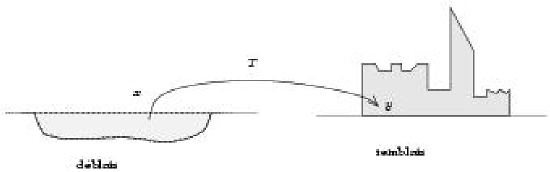
Dirtmoving

Given a before and an after dirtpile, what is the most efficient way to move the dirt from one place to the other?



Let's say that the cost to move a gram of dirt from x to y is $d(x, y)^2$.

Gaspard Monge

Mémoire sur la théorie des déblais et des remblais (1781)

Memoir on the theory of excavations and fillings (1781)

Gaspard Monge

MÉMOIRE
SUR LA
THÉORIE DES DÉBLAIS ET DES REMBLAIS.

Par M. MONGE.

LORSQU'ON doit tranſporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit tranſporter, & le nom de *Remblai* à l'eſpace qu'elles doivent occuper après le tranſport.

Le prix du tranſport d'une molécule étant, toutes choſes d'ailleurs égales, proportionnel à ſon poids & à l'eſpace qu'on lui fait parcourir, & par conſéquent le prix du tranſport total devant être proportionnel à la ſomme des produits des molécules multipliées chacune par l'eſpace parcouru, il s'enſuit que le déblai & le remblai étant donnés de figure & de poſition, il n'eſt pas indifférent que telle molécule du déblai ſoit tranſportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine diſtribution à faire des molécules du premier dans le ſecond, d'après laquelle la ſomme de ces produits ſera la moindre poſſible, & le prix du tranſport total ſera un *minimum.*

C'eſt la ſolution de cette queſtion que je me propoſe de donner ici. Je diviſerai ce Mémoire en deux parties, dans la première je ſuppoſerai que les déblais & les remblais ſont des aires contenues dans un même plan; dans le ſecond, je ſuppoſerai que ce ſont des volumes.

PREMIÈRE PARTIE.

Du tranſport des aires planes ſur des aires compriſes dans un même plan.

I.

QUELLE que ſoit la route que doive ſuivre une molécule

Wasserstein space

Let (X, d) be a compact metric space.

Notation

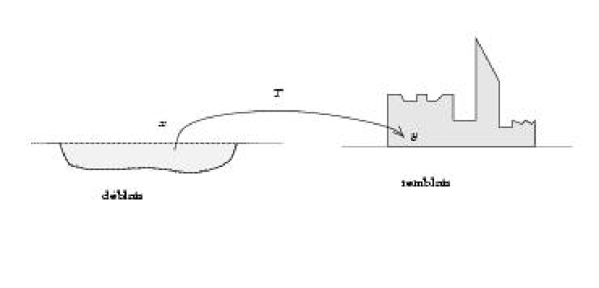
$P(X)$ is the set of Borel probability measures on X.

That is, $\mu \in P(X)$ iff μ is a nonnegative Borel measure on X with $\mu(X) = 1$.

Definition

Given $\mu_0, \mu_1 \in P(X)$, the Wasserstein distance $W_2(\mu_0, \mu_1)$ is the square root of the minimal cost to transport μ_0 to μ_1.

Wasserstein space



$$W_2(\mu_0, \mu_1)^2 = \inf \left\{ \int_{X \times X} d(x, y)^2 \, d\pi(x, y) \right\},$$

where

$$\pi \in P(X \times X), (p_0)_* \pi = \mu_0, (p_1)_* \pi = \mu_1.$$

Wasserstein space

Fact :
$(P(X), W_2)$ is a metric space, called the Wasserstein space.

The metric topology is the weak-$*$ topology, i.e. $\lim_{i\to\infty} \mu_i = \mu$ if and only if for all $f \in C(X)$, $\lim_{i\to\infty} \int_X f\, d\mu_i = \int_X f\, d\mu$.

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Note : There is an isometric embedding $X \to P(X)$ by $x \to \delta_x$.

Displacement interpolations

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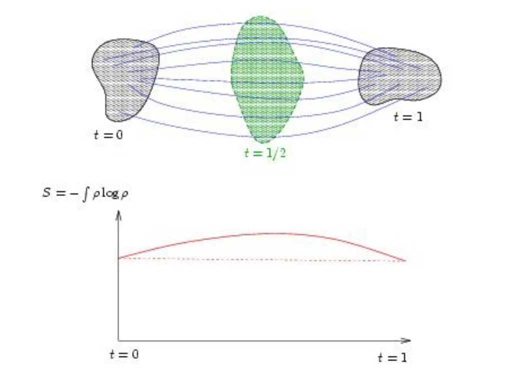

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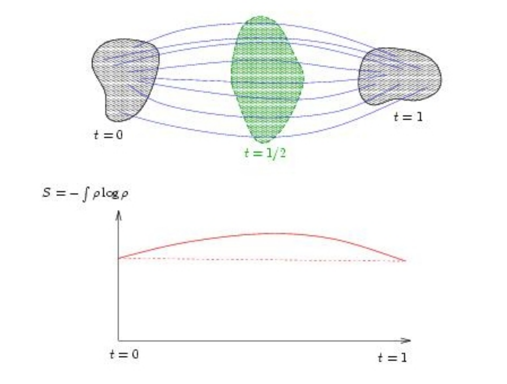

Take a snapshot at time t. We get a family of meaures $\{\mu_t\}_{t\in[0,1]}$, called a displacement interpolation. We would like to say that this is a "geodesic" in $P(X)$.

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Smooth measures

If (M, g) is a compact connected Riemannian manifold, let $P^\infty(M) \subset P(M)$ be the smooth probability measures with positive density.

$$P^\infty(M) = \{\rho \,\mathrm{dvol}_M \;:\; \rho \in C^\infty(M),\ \rho > 0,\ \textstyle\int_M \rho \,\mathrm{dvol}_M \;=\; 1\}.$$

Otto's formal Riemannian metric on $P^\infty(M)$

Given $\mu = \rho \, \mathrm{dvol}_M \in P^\infty(M)$, consider an infinitesimally nearby measure $\mu + \delta\mu$, i.e.

$$\delta\mu = (\delta\rho) \, \mathrm{dvol}_M \in T_\mu P^\infty(M).$$

Solve $\delta\rho = -\sum_i \nabla^i(\rho \nabla_i \phi)$ for $\phi \in C^\infty(M)$, unique up to an additive constant.

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Definition :

$$\langle \delta\mu, \delta\mu \rangle = \int_M |\nabla\phi|^2 \, \rho \, \mathrm{dvol}_M \, .$$

This is the H^{-1} Sobolev metric, in terms of ρ.

Corresponding energy of a curve

Say $c : [0,1] \to P^{\infty}(M)$ is a smooth curve.

Write $c(t) = \rho(t)\ \mathrm{dvol}_M$.

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Fact : We can solve

$$\frac{\partial \rho}{\partial t} = -\sum_i \nabla^i(\rho \nabla_i \phi)$$

for $\phi \equiv \phi(t) \in C^{\infty}(M)$.

Benamou-Brenier variational problem

From $\{\rho(t)\}_{t\in[0,1]}$, we got $\{\phi(t)\}_{t\in[0,1]}$.

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Definition

$$E(c) = \frac{1}{2}\int_0^1 \int_M |\nabla\phi|^2 \rho \, \mathrm{dvol}_M \, dt.$$

This is the energy of the curve c.

Geodesic distance equals Wasserstein distance

Theorem : (Otto-Westdickenberg 2005)

$$\frac{1}{2} W_2(\mu_0, \mu_1)^2 \;=\; \inf\{E(c) \;:\; c(0) = \mu_0, c(1) = \mu_1\}.$$

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That is, the geodesic distance coming from Otto's metric is the Wasserstein distance W_2, at least on $P^\infty(M)$.

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That is, the geodesic distance coming from Otto's metric is the Wasserstein distance W_2, at least on $P^\infty(M)$.

Note : the infimum may not be achieved. A minimizing c is a smooth displacement interpolation.

Euler-Lagrange equations

The Euler-Lagrange equation for the functional E is

Hamilton-Jacobi equation

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2} |\nabla \phi|^2.$$

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We also had

Conservation equation

$$\frac{\partial \rho}{\partial t} = -\sum_i \nabla^i(\rho \nabla_i \phi).$$

These are the equations for optimal transport and can be solved explicitly. (First worked out for Riemannian manifolds by Robert McCann 2001.)

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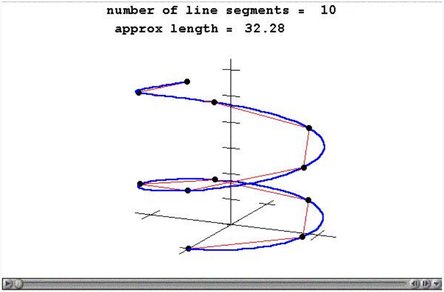
Can we extend these statements from formal results about $P^\infty(M)$ to rigorous results about $P(M)$?

Or, more generally, about $P(X)$ for a nonsmooth space X?

Use ideas from metric geometry.

Review of length spaces

Say (X, d) is a compact metric space and $\gamma : [0, 1] \to X$ is a continuous map.



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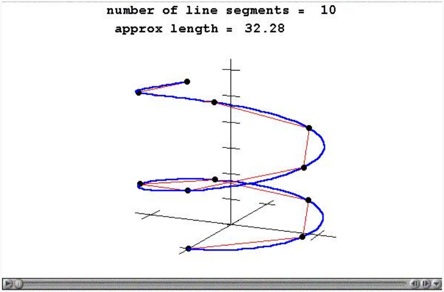

The length of γ is

$$L(\gamma) = \sup_J \sup_{0=t_0 \leq t_1 \leq \ldots \leq t_J = 1} \sum_{j=1}^{J} d(\gamma(t_{j-1}), \gamma(t_j)).$$

Length spaces

Definition

(X, d) is a length space if the distance between two points $x_0, x_1 \in X$ equals the infimum of the lengths of curves joining them, i.e.

$$d(x_0, x_1) = \inf\{L(\gamma) \; : \; \gamma(0) = x_0, \gamma(1) = x_1\}.$$

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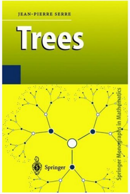
A length-minimizing curve is called a geodesic.

Length spaces

Examples of length spaces :

1. The underlying metric space of any Riemannian manifold.
2.



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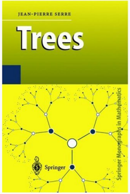
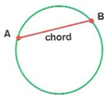


Nonexamples :

1. A finite metric space with more than one point.
2. A circle with the chordal metric.



Wasserstein space as a length space

Proposition : (LV,S)

If X is a length space then so is the Wasserstein space $P(X)$.

Hence we can talk about its (minimizing) geodesics $\{\mu_t\}_{t\in[0,1]}$, called Wasserstein geodesics.

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Proposition : (LV)

The Wasserstein geodesics are exactly the displacement interpolations $\{\mu_t\}_{t\in[0,1]}$.

Curvature of Wasserstein space

Formal calculation : (Otto 2001)

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Open question :

To what extent is $P(M)$ an infinite-dimensional Riemannian manifold?

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Back to smooth manifolds.

Ricci curvature is an averaging of sectional curvature.

Fix a unit-length vector $\mathbf{v} \in T_m M$.

Definition

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Example : $S^2 \times S^2$ has *nonnegative* sectional curvatures but has *positive* Ricci curvatures.

Regularity Issue :

To define Ric_M, we need a Riemannian metric which is C^2-regular.

Can we make sense of Ricci curvature for nonsmooth spaces?
Can we make sense at least of “nonnegative Ricci curvature”?

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The analogous question for *sectional curvature* was solved by Alexandrov in the 1950’s.

The relation of Ricci curvature to optimal transport

Say M is a compact Riemannian manifold.

Definition: The (negative) entropy functional $\mathcal{E} : P(M) \to \mathbb{R} \cup \infty$ is given by

$$\mathcal{E}(\mu) = \begin{cases} \int_M \rho \log \rho \ \mathrm{dvol}_M & \text{if } \mu = \rho \ \mathrm{dvol}_M, \\ \infty & \text{if } \mu \text{ is not a.c. w.r.t. } \mathrm{dvol}_M\,. \end{cases}$$

Otto-Villani calculation

How does the entropy function behave along geodesics in $P(M)$?

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Suppose that $c(t) = \rho(t) \ \mathrm{dvol}_M$ is a smooth Wasserstein geodesic.

We defined $\phi(t) \in C^\infty(M)$ by

$$\frac{\partial \rho}{\partial t} = - \sum_i \nabla^i (\rho \nabla_i \phi).$$

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A local calculation : (Otto-Villani 2000)
Along the geodesic c,

$$\frac{d^2}{dt^2}\mathcal{E}(c(t)) = \int_M \left[|\operatorname{Hess}(\phi)|^2 + \operatorname{Ric}_M(\nabla\phi, \nabla\phi) \right] \rho\ \mathrm{dvol}_M .$$

Convexity of entropy along Wasserstein geodesics

Corollary :
If $\mathrm{Ric}_M \geq 0$ then $\frac{d^2\mathcal{E}}{dt^2} \geq 0$, i.e. $\mathcal{E}$ is convex along any Wasserstein geodesic c.

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A new way of thinking about Ricci curvature :

Nonnegative Ricci curvature is equivalent to convexity of $\mathcal{E}$ (on $P(M)$).

We will use this property to define the notion of “nonnegative Ricci curvature” for a nonsmooth space.

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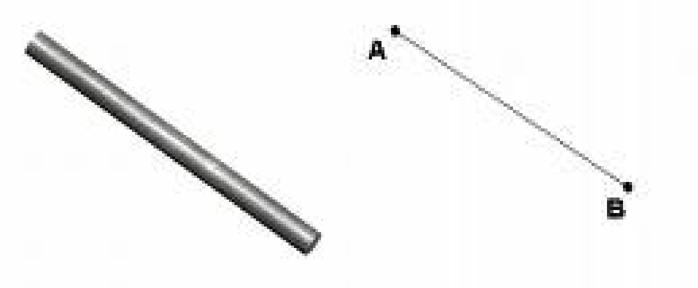
Gromov-Hausdorff topology

A topology on the set of all compact metric spaces (modulo isometry).

(X_1, d_1) and (X_2, d_2) are close in the Gromov-Hausdorff topology if somebody with bad vision has trouble telling them apart.

Example : a cylinder with a small cross-section is Gromov-Hausdorff close to a line segment.



Gromov's precompactness theorem

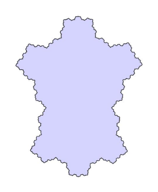
Theorem : (Gromov 1981)

Given $N \in \mathbb{Z}^+$ and $D > 0$,

$$\{(M, g) \ : \ \mathrm{dim}(M) = N, \mathrm{diam}(M) \leq D, \mathrm{Ric}_M \geq 0\}$$

is precompact in the Gromov-Hausdorff topology on $\{$compact metric spaces$\}$/isometry.

Artist's rendition of Gromov-Hausdorff space



Each point represents a compact metric space.
Each interior point is a Riemannian manifold (M, g) with $\dim(M) = N$, $\operatorname{diam}(M) \le D$ and $\operatorname{Ric}_M \ge 0$.

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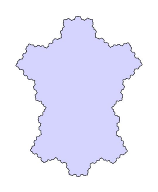

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Each interior point is a Riemannian manifold (M, g) with $\dim(M) = N$, $\mathrm{diam}(M) \leq D$ and $\mathrm{Ric}_M \geq 0$.

The boundary points are compact metric spaces (X, d) with $\dim_H X \leq N$ and $\mathrm{diam}(X) \leq D$. They are generally not manifolds.
(Example : $X = M/G$.)

In some moral sense, the boundary points are metric spaces with "nonnegative Ricci curvature".

Question :

What can we say about the Gromov-Hausdorff limits of Riemannian manifolds with nonnegative Ricci curvature?

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To answer this, it turns out to be useful to consider instead metric-measure spaces.

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Canonical Example

If M is a compact Riemannian manifold then $\left(M, d_M, \frac{\mathrm{dvol}_M}{\mathrm{vol}(M)}\right)$ is a metric-measure space.

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If M is a compact Riemannian manifold then $\left(M, d_M, \frac{\mathrm{dvol}_M}{\mathrm{vol}(M)}\right)$ is a metric-measure space.

More generally, a smooth measured length space is a compact Riemannian manifold (M, g) equipped with a smooth probability measure $d\nu = e^{-\Psi}\ \mathrm{dvol}_M$.

Measured Gromov-Hausdorff limits

An easy consequence of Gromov precompactness :

$$\left\{ \left(M, g, \frac{\mathrm{dvol}_M}{\mathrm{vol}(M)} \right) \; : \; \dim(M) = N, \mathrm{diam}(M) \leq D, \mathrm{Ric}_M \geq 0 \right\}$$

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Measured Gromov-Hausdorff limits

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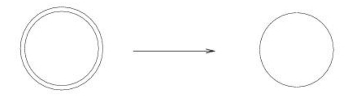
What can we say about the limit points? (Work of Cheeger-Colding 1996-2000)

What are the *smooth* limit points?

Measured Gromov-Hausdorff (MGH) topology



Definition

$\lim_{i\to\infty}(X_i, d_i, \nu_i) = (X, d, \nu)$ if there are Borel maps $f_i : X_i \to X$ and a sequence $\epsilon_i \to 0$ such that

1. (Almost isometry) For all $x_i, x_i' \in X_i$,

$$|d_X(f_i(x_i), f_i(x_i')) - d_{X_i}(x_i, x_i')| \le \epsilon_i.$$

2. (Almost surjective) For all $x \in X$ and all i, there is some $x_i \in X_i$ such that

$$d_X(f_i(x_i), x) \le \epsilon_i.$$

3. $\lim_{i\to\infty}(f_i)_*\nu_i = \nu$ in the weak-$*$ topology.

Passage to $P(X)$

To one compact metric space we have assigned another.

$$(X, d) \longrightarrow (P(X), W_2)$$

Proposition : (LV)

If $X_i \to X$ in the Gromov-Hausdorff topology then $P(X_i) \to P(X)$ in the Gromov-Hausdorff topology.

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We will use the properties of the Wasserstein space $(P(X), W_2)$ to say something about the geometry of (X, d).

In particular, we will *define* what it means for (X, ν) to have "nonnegative Ricci curvature" in terms of $P(X)$.

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Notation

X a compact Hausdorff space.

$P(X) =$ Borel probability measures on X, with weak-$*$ topology.

Fix a background measure $\nu \in P(X)$.

Effective dimension

$N \in [1, \infty]$ a new parameter (possibly infinite).

It turns out that there's not a single notion of "nonnegative Ricci curvature", but rather a 1-parameter family. That is, for each N, there's a notion of a space having "nonnegative N-Ricci curvature".

Here N is an effective dimension of the space, and must be inputted.

Entropy

Definition of the "negative entropy" function

$$\mathcal{E}_N : P(X) \to \mathbb{R} \cup \infty$$

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be the Lebesgue decomposition of μ with respect to ν.

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For $N \in [1, \infty)$, the "negative entropy" of μ with respect to ν is

$$\mathcal{E}_N(\mu) \;=\; N \;-\; N \int_X \rho^{1-\frac{1}{N}} \, d\nu.$$

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$$\mathcal{E}_N(\mu) \ = \ N \ - \ N \int_X \rho^{1-\frac{1}{N}} \, d\nu.$$

For $N = \infty$,

$$\mathcal{E}_\infty(\mu) = \begin{cases} \int_X \rho \, \log \rho \, d\nu & \text{if } \mu \text{ is a.c. w.r.t. } \nu, \\ \infty & \text{otherwise.} \end{cases}$$

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(X, d) is a compact length space.

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We want to ask whether the negative entropy function $\mathcal{E}_N$ is a convex function on $P(X)$.

That is, given $\mu_0, \mu_1 \in P(X)$, whether $\mathcal{E}_N$ restricts to a convex function along a Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$ from μ_0 to μ_1.

Nonnegative N-Ricci curvature

Definition

Given $N \in [1, \infty]$, we say that a compact measured length space (X, d, ν) has nonnegative N-Ricci curvature if :

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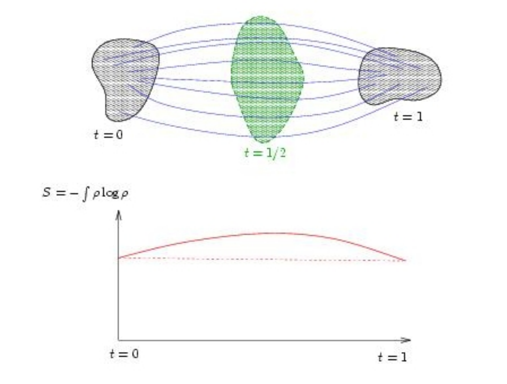
For all $\mu_0, \mu_1 \in P(X)$ with $\mathrm{supp}(\mu_0) \subset \mathrm{supp}(\nu)$ and $\mathrm{supp}(\mu_1) \subset \mathrm{supp}(\nu)$, there is *some* Wasserstein geodesic $\{\mu_t\}_{t \in [0,1]}$ from μ_0 to μ_1 so that for all $t \in [0, 1]$,

$$\mathcal{E}_N(\mu_t) \leq t\,\mathcal{E}_N(\mu_1) + (1-t)\,\mathcal{E}_N(\mu_0).$$

Nonnegative N-Ricci curvature



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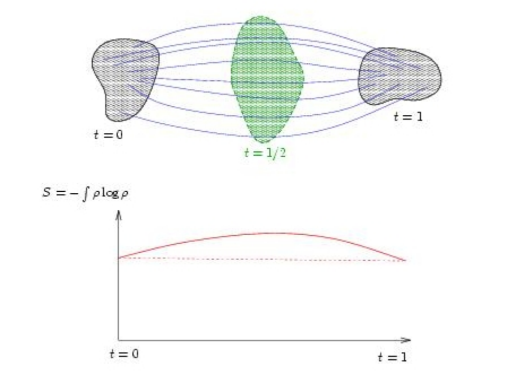
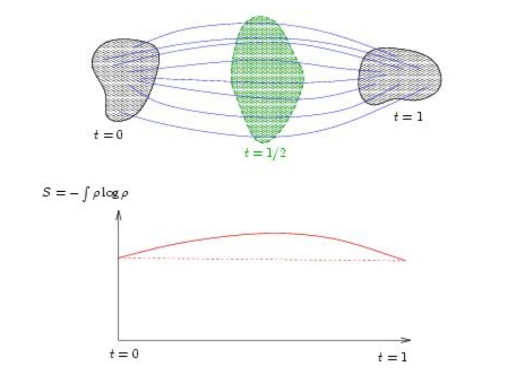


Note : We only require convexity along *some* geodesic from μ_0 to μ_1, not all geodesics.

Nonnegative *N*-Ricci curvature



Note : We only require convexity along *some* geodesic from μ_0 to μ_1, not all geodesics.

There's also a notion of "*N*-Ricci curvature bounded below by K".

Main result

Theorem : (LV,S)

Let $\{(X_i, d_i, \nu_i)\}_{i=1}^{\infty}$ be a sequence of compact measured length spaces with

$$\lim_{i\to\infty}(X_i, d_i, \nu_i) = (X, d, \nu)$$

in the measured Gromov-Hausdorff topology.

For any $N \in [1, \infty]$, if each (X_i, d_i, ν_i) has nonnegative N-Ricci curvature then (X, d, ν) has nonnegative N-Ricci curvature.

What does all this have to do with Ricci curvature?

Let (M, g) be a compact connected n-dimensional Riemannian manifold.

We could take the Riemannian measure, but let's be more general and consider any smooth measured length space.

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Let (M, g) be a compact connected n-dimensional Riemannian manifold.

We could take the Riemannian measure, but let's be more general and consider any smooth measured length space.

Say $\Psi \in C^\infty(M)$ has

$$\int_M e^{-\Psi} \, \mathrm{dvol}_M = 1.$$

Put $\nu = e^{-\Psi} \, \mathrm{dvol}_M$.

The N-Ricci tensor

For $N \in [1, \infty]$, define the N-Ricci tensor Ric_N of (M^n, g, ν) by

$$\begin{cases} \mathrm{Ric} + \mathrm{Hess}(\Psi) & \text{if } N = \infty, \\ \mathrm{Ric} + \mathrm{Hess}(\Psi) - \frac{1}{N-n} d\Psi \otimes d\Psi & \text{if } n < N < \infty, \\ \mathrm{Ric} + \mathrm{Hess}(\Psi) - \infty (d\Psi \otimes d\Psi) & \text{if } N = n, \\ -\infty & \text{if } N < n, \end{cases}$$

where by convention $\infty \cdot 0 = 0$.

Ric_N is a symmetric covariant 2-tensor field on M that depends on g and Ψ.

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Ric_N is a symmetric covariant 2-tensor field on M that depends on g and Ψ.

(If $N = n$ then Ric_N is $-\infty$ except where $d\Psi = 0$. There, $\mathrm{Ric}_N = \mathrm{Ric}$.)

$\mathrm{Ric}_\infty =$ Bakry-Emery tensor = right-hand side of Perelman's modified Ricci flow equation.

Abstract Ricci recovers classical Ricci

Recall that $\nu = e^{-\Psi} \, \mathrm{dvol}_M$.

Theorem : (LV, S)
For $N \in [1, \infty]$, the measured length space (M, g, ν) has nonnegative N-Ricci curvature if and only if $\mathrm{Ric}_N \geq 0$.

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Classical case : Ψ constant, so $\nu = \frac{\mathrm{dvol}}{\mathrm{vol}(M)}$.

Then (M^n, g, ν) has <u>abstract</u> nonnegative N-Ricci curvature if and only if it has <u>classical</u> nonnegative Ricci curvature, as soon as $N \geq n$.

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Smooth limit spaces

Had Gromov precompactness theorem. What are the limit spaces (X, d, ν)? Suppose that the limit space is a *smooth* measured length space, i.e.

$$(X, d, \nu) = (B, g_B, e^{-\Psi}\, \mathrm{dvol}_B)$$

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Note : the dimension can drop on taking limits.

Bishop-Gromov-type inequality

Theorem : (LV, S)

If (X, d, ν) has nonnegative N-Ricci curvature and $x \in \mathrm{supp}(\nu)$ then $r^{-N}\,\nu(B_r(x))$ is nonincreasing in r.

Lichnerowicz inequality

If (M, g) is a compact Riemannian manifold, let λ_1 be the smallest positive eigenvalue of the Laplacian $-\nabla^2$.

Theorem : (Lichnerowicz 1964)

If $\dim(M) = n$ and M has Ricci curvatures bounded below by $K > 0$ then

$$\lambda_1 \geq \frac{n}{n-1} K.$$

Sharp global Poincaré inequality

Theorem : (LV)

If (X, d, ν) has N-Ricci curvature bounded below by $K > 0$ and f is a Lipschitz function on X with $\int_X f \, d\nu = 0$ then

$$\int_X f^2 \, d\nu \le \frac{N-1}{N} \frac{1}{K} \int_X |\nabla f|^2 \, d\nu.$$

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(There are also log Sobolev inequalities and Sobolev inequalities.)

Ricci O'Neill Theorem

Theorem (O'Neill 1966) Sectional curvature is nondecreasing under a Riemannian submersion.

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LV : A synthetic proof of the Ricci O'Neill theorem using displacement convexity.

Some open questions

Do measured length spaces with nonnegative N-Ricci curvature admit isoperimetric inequalities?

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To what extent does the Cheeger-Gromoll splitting principle hold for measured length spaces with nonnegative N-Ricci curvature?

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Say $(M, g(t))$ is a Ricci flow solution, i.e. $\frac{dg}{dt} = -2 \operatorname{Ric}$.

Fix t_0 and put $\tau = t_0 - t$. Then $\frac{dg}{d\tau} = 2 \operatorname{Ric}$.

An important tool : monotonic quantities.

Reduced volume

Fix $p \in M$. Say $\gamma : [0, \overline{\tau}] \to M$ is a smooth curve with $\gamma(0) = p$. (The graph of γ goes "backward in time".)

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reduced volume $\widetilde{V}(\overline{\tau}) = \overline{\tau}^{-\frac{n}{2}} \int_M e^{-l(q,\overline{\tau})} \operatorname{dvol}(q)$.

Monotonicity of the reduced volume

Theorem : (Perelman 2002)

$\widetilde{V}$ is nonincreasing in $\overline{\tau}$, i.e. nondecreasing in t.

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An "entropy" functional for Ricci flow.

The only assumption : $g(t)$ satisfies the Ricci flow equation.

Main application : Perelman's "no local collapsing" theorem.

Perelman's heuristic derivation

Put $\overline{M} = M \times S^N \times \mathbb{R}^+$.

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Put
$$\overline{g} = g(\tau) + 2N\tau g_{S^N} + \left(\frac{N}{2\tau} + R\right) d\tau^2.$$

Fact : As $N \to \infty$, $\mathrm{Ric}(\overline{M}) = O\left(N^{-1}\right)$.

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Fact : As $N \to \infty$, $\mathrm{Ric}(\overline{M}) = O\left(N^{-1}\right)$.

Bishop-Gromov : $r^{-\,dim}\ \mathrm{vol}(B_r(p))$ is nonincreasing in r if $\mathrm{Ric} \geq 0$.

Apply formally to $\overline{M}$ and take $N \to \infty$. Get monotonicity of $\widetilde{V}$.

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We'll describe a (re)proof of the monotonicity of $\widetilde{V}$, using optimal transport methods.

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Say $c : [0,1] \to P^\infty(M)$ is a smooth curve.

Write $c(t) = \rho(t)\ \mathrm{dvol}_M$.

Solve

$$\frac{\partial \rho}{\partial t} = -\sum_i \nabla^i(\rho \nabla_i \phi)$$

for $\phi \equiv \phi(t) \in C^\infty(M)$.

From $\{\rho(t)\}_{t\in[0,1]}$, we got $\{\phi(t)\}_{t\in[0,1]}$. Put

$$E(c) = \frac{1}{2}\int_0^1 \int_M |\nabla \phi|^2 \rho\ \mathrm{dvol}_M\ dt.$$

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$$\frac{d^2}{dt^2}\int_M \rho \log \rho \,\mathrm{dvol}_M = \int_M \left[|\operatorname{Hess}\phi|^2 + \operatorname{Ric}_M(\nabla\phi, \nabla\phi) \right] \rho \,\mathrm{dvol}_M \,.$$

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Question

Can we do something similar for the Ricci flow?

Principle : Satisfying Ric $=0$ in the Riemannian case is like satisfying the Ricci flow equation in the spacetime case.

Optimal transport in a Ricci flow background was first considered by Peter Topping, with application to another monotonic quantity ($\mathcal{W}$-functional).

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Note : The Ricci flow equation

$$\frac{dg}{dt} = -2\ \mathrm{Ric}$$

implies

$$\frac{\mathrm{dvol}_M}{dt} = -R\ \mathrm{dvol}_M .$$

E_0 functional

Assume hereafter that $(M, g(t))$ satisfies the Ricci flow equation.

Given $c : [t_0, t_1] \to P^\infty(M)$, write $c(t) = \rho(t)\ \mathrm{dvol}_M$. Solve

$$\frac{\partial \rho}{\partial t} = -\sum_i \nabla^i(\rho \nabla_i \phi) + R\,\rho$$

for $\phi \equiv \phi(t) \in C^\infty(M)$.

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$E_0(c) = \frac{1}{2} \int_{t_0}^{t_1} \int_M \left(|\nabla \phi|^2 + R \right)\ \rho\ \mathrm{dvol}_M\ dt$

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Euler-Lagrange equation for E_0 :

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}\,|\nabla \phi|^2 + \frac{1}{2}\,R.$$

Convexity statement

Proposition

If c satisfies the Euler-Lagrange equation then

$$\frac{d^2}{dt^2}\int_M (\rho\,\ln\rho\;-\;\phi\,\rho)\;\mathrm{dvol}_M\;=\;\int_M |\,\mathrm{Ric}-\mathrm{Hess}\,\phi|^2\,\rho\;\mathrm{dvol}_M\,.$$

Convexity statement

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$$\frac{d^2}{dt^2}\int_M (\rho \ln \rho - \phi\,\rho)\,\mathrm{dvol}_M = \int_M |\,\mathrm{Ric} - \mathrm{Hess}\,\phi|^2\,\rho\,\mathrm{dvol}_M\,.$$

Corollary

If c satisfies the Euler-Lagrange equation then $\int_M (\rho \ln \rho - \phi\,\rho)\,\mathrm{dvol}_M$ is convex in t.

Corresponding optimal transport problem

Say we want to transport a measure μ_0 (at time t_0) to a measure μ_1 (at time t_1).

Take the cost to transport a unit of mass from p to q to be

$$\min\{\mathcal{L}_0(\gamma) \ : \ \gamma(t_0) = p, \gamma(t_1) = q\},$$

where

$$\mathcal{L}_0(\gamma) \ = \ \frac{1}{2} \int_{t_0}^{t_1} \left(|\dot{\gamma}|^2_{g(t)} \ + \ R(\gamma(t), t) \right) \, dt.$$

There is a corresponding notion of optimal transport, displacement interpolation, etc.

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Fix t_0 and put $\tau = t_0 - t$. The Ricci flow equation is

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Euler-Lagrange equation for E_- :

$$\frac{\partial \phi}{\partial \tau} = - \frac{1}{2} \, |\nabla \phi|^2 + \frac{1}{2} \, R - \frac{1}{2\tau} \, \phi.$$

Convexity statement

Proposition

If c satisfies the Euler-Lagrange equation then

$$\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2\left(\int_M(\rho\,\ln\rho\,+\,\phi\,\rho)\,\mathrm{dvol}_M+\frac{n}{2}\,\ln\tau\right)=$$

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Corollary

If c satisfies the Euler-Lagrange equation then $\int_M (\rho \ln\rho + \phi\,\rho)\,\mathrm{dvol}_M + \frac{n}{2}\ln\tau$ *is convex in the variable* $s = \tau^{-\frac{1}{2}}$.

Motivation
Optimal transport
Formal Geometry of Wasserstein Space
Metric geometry of Wasserstein space
Ricci meets Wasserstein
Some more metric geometry
Generalized entropy functionals
Abstract Ricci curvature
Applications
Perelman's reduced volume
Formulas from Riemannian optimal transport
Optimal transport for Ricci flow
Monotonicity of the reduced volume

The transport problem

Take $\tau_0 \to 0$, $\mu_0 = \delta_p$ and μ_1 an absolutely continuous measure.

The displacement interpolation is along $\mathcal{L}$-geodesics emanating from p.

In this case, $\phi = l$.

From convexity to monotonicity

Proposition

In this case, $\int_M (\rho \ln \rho + \phi \rho) \, \mathrm{dvol}_M + \frac{n}{2} \ln \tau$ *is nondecreasing in* τ*.*

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Proof.

We know that it is convex in $s = \tau^{-\frac{1}{2}}$. As $s \to \infty$, i.e. as $\tau \to 0$, it approaches a constant. (Almost Euclidean situation.) So it is nonincreasing in s, i.e. nondecreasing in τ. $\square$

Free energy

Trivial fact : The minimizer of

$$\int_M (\rho \, \ln \rho \; + \; \phi \, \rho) \; \mathrm{dvol}_M \; + \frac{n}{2} \, \ln \tau,$$

as $\rho \, \mathrm{dvol}_M$ ranges over absolutely continuous probability measures, is

$$- \, \ln \left(\tau^{- \, \frac{n}{2}} \int_M e^{- \, \phi} \; \mathrm{dvol}_M \right) .$$

The minimizing measure is given by

$$\rho \; = \; \frac{e^{-\phi}}{\int_M e^{- \, \phi} \; \mathrm{dvol}_M}.$$

Monotonicity of reduced volume

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$$\widetilde{V}(\tau) = \tau^{-\frac{n}{2}} \int_M e^{-l} \, \mathrm{dvol}_M$$

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Transport it to δ_p (at time zero). At the intermediate time τ' we see a measure $\mu(\tau') = \rho(\tau') \, \mathrm{dvol}_M$.

Proof

Then

$$-\ln\left((\tau')^{-\frac{n}{2}}\int_M e^{-\phi(\tau')}\,\mathrm{dvol}_M\right)$$

$$\leq \int_M \left[\rho(\tau')\ln\rho(\tau') + \phi(\tau')\,\rho(\tau')\right]\mathrm{dvol}_M + \frac{n}{2}\ln\tau'$$

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=\ & - \ln \left((\tau'')^{-\frac{n}{2}} \int_M e^{-\phi(\tau'')} \, \mathrm{dvol}_M \right) .
\end{aligned}
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End of proof

Optimal transport and heat flow

Otto, Otto-Westdickenberg Suppose that a compact Riemannian manifold has Ric ≥ 0. If $\mu_0(t)$ and $\mu_1(t)$ are two solutions of the heat flow on measures then $W_2(\mu_0(t), \mu_1(t))$ is nonincreasing in t.

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Topping Extension to a statement about the $\mathcal{L}$-transport distance between μ_0 and μ_1 at distinct but related times.