SEMINAR 1

ETA AND TORSION

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1. Introduction

We discuss two spectral invariants of Riemannian manifolds. The first, the eta-invariant η, was introduced by Atiyah, Patodi and Singer in order to prove an index theorem for manifolds with boundary [1]. The second, the analytic torsion T, was introduced by Ray and Singer as an analytic analog of the Reidemeister torsion [15]. In the first part of this paper we define the invariants and give examples of how they arise.

The two invariants η and T, especially T, have been somewhat mysterious in nature. They have the flavor of being "secondary" invariants. In recent years much progress has been made in making this precise, in showing how η and T arise via transgression from "primary" invariants. To do this one must look at the invariants not of a single manifold, but of a family of manifolds. In the second part of this paper we explain the work of Bismut and Cheeger on eta-forms [6] and the work of Bismut and Lott on analytic torsion forms [8].

2. Eta-Invariant

Let Z be an odd-dimensional closed (= boundaryless compact) Riemannian spin manifold. Let E be a Hermitian vector bundle on Z and let ∇^E be a connection on E which is compatible with the Hermitian metric. If S denotes the spinor bundle on Z, there is an essentially self-adjoint Dirac-type operator D acting on smooth sections of $S \otimes E$.

Example 1 : Take $Z = S^1$, parametrized by $\theta \in [0, 2\pi)$. Give Z the spin structure so that spinors on Z are periodic functions. Take $E = S^1 \times \mathbf{C}$ with the connection $\nabla^E = d\theta(\partial_\theta + ia)$, where $a \in \mathbf{R}$. Then $D = -i(\partial_\theta + ia)$. The eigenfunctions of D are $\{e^{ik\theta}\}_{k \in \mathbf{Z}}$ and the eigenvalues are $\{k + a\}_{k \in \mathbf{Z}}$.
End of Example 1

Question : Does D have more positive or negative eigenvalues?

To give meaning to this question, first let T be a self-adjoint invertible $N \times N$ matrix with eigenvalues $\{\lambda_k\}_{k=1}^N$. Then the answer to the question for T is given by

$$\sum_{k=1}^{N} \mathrm{sign}(\lambda_k) = \mathrm{Tr}(\mathrm{sign}(T)) = \mathrm{Tr}(T(T^2)^{-1/2}) = \mathrm{Tr}(T \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2 T^2} ds).$$

Returning to the Dirac-type operator, this motivates the following definition.

Definition 1 $\eta(D) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}(De^{-s^2 D^2}) ds.$

It is a nontrivial fact that the above integral converges [7, 4]. The above definition of $\eta(D)$ is equivalent to that of Atiyah-Patodi-Singer[1].

Return to Example 1 : Using the Poisson summation formula, one finds

$$\begin{aligned} \eta(D) &= \frac{2}{\sqrt{\pi}} \int_0^\infty \sum_{k \in \mathbf{Z}} (k+a) e^{-s^2(k+a)^2} ds = \sum_{m \neq 0} -\frac{i}{\pi m} e^{2\pi i a m} \\ &= \begin{cases} 0 & \text{if } a \in \mathbf{Z} \\ 1 - 2a & \text{if } a \in (0,1). \end{cases} \end{aligned} \tag{2.1}$$

In particular, $\eta(D)$ is 1-periodic in a. This is a reflection of the fact that the connection $d\theta(\partial_\theta + i(a+1))$ is gauge-equivalent to $d\theta(\partial_\theta + ia)$, and so the spectrum of D is 1-periodic in a.

We also see that as a goes from slightly negative to slightly positive, $\eta(D)$ jumps by 2. This is because as a passes through zero, an eigenvalue of D goes from negative to positive. Recalling that $\eta(D)$ is formally the number of + eigenvalues minus the number of − eigenvalues, such a jump is expected. **End of Return to Example 1**

In general, if $D(\epsilon)$ is a smooth 1-parameter family of Dirac-type operators then $\eta(D(\epsilon))$ varies smoothly in ϵ, except for jumps coming from eigenvalues of $D(\epsilon)$ crossing zero.

Example 2 : Think of Z as the space in a static spacetime M which is isometrically $\mathbf{R} \times Z$. Let ψ be a spinor field on M. Its electric current is $J^\mu = e\overline{\psi}\gamma^\mu\psi$, where we take $e < 0$. The spinor Hamiltonian is of the form $H = -i\gamma^0\gamma^j\nabla_j$, plus possible mass terms. We now second-quantize ψ and ask for the electric charge of the vacuum state $\langle Q \rangle = \int_Z e\langle\overline{\psi}(x)\gamma^0\psi(x)\rangle dvol(x)$.

Claim : $\langle Q \rangle = -\frac{1}{2}e\eta(H)$.
"Pf." : If we think of filling the Dirac sea, the charge of the vacuum will be

$$\begin{aligned}\langle Q \rangle &= e\,(\#\text{ of - energy states of }H)\\ &= \frac{1}{2}e\,[[(\#\text{ of + energy states of }H) + (\#\text{ of - energy states of }H)]\\ &\quad - [(\#\text{ of + energy states of }H) - (\#\text{ of - energy states of }H)]]\,.\end{aligned}$$

After regularization, one can show that this formal expression becomes $\frac{1}{2}e\,[0 - \eta(H)]$, giving the claim. One can also give a rigorous proof of the claim [12]. **End of Example 2**

We now give the motivation of Atiyah-Patodi-Singer to define the eta-invariant. Let us first recall the statement of the Atiyah-Singer index theorem. Let X be an even-dimensional closed Riemannian spin manifold and let E be a Hermitian vector bundle on X. If ∇^E is a compatible connection on E with curvature F^E, its Chern form is

$$\mathrm{Ch}\left(\nabla^E\right) = \mathrm{Tr}\left(e^{-\frac{F^E}{2\pi i}}\right) \in \Omega^{\mathrm{even}}(X). \tag{2.2}$$

The Chern class of E is $\mathrm{Ch}(E) = [\mathrm{Ch}\left(\nabla^E\right)] \in \mathrm{H}^{\mathrm{even}}(X;\mathbf{R})$. We denote the A-hat form of ∇^{TX} by $\widehat{A}(\nabla^{TX}) \in \Omega^{4*}(X)$ and we let $\widehat{A}(TX) \in H^{4*}(X;\mathbf{R})$ be its cohomology class [4].

Theorem 1 *[2]*

$$\mathrm{Index}(D_X) = \int_X \widehat{A}(\nabla^{TX}) \wedge \mathrm{Ch}\left(\nabla^E\right) = \left(\widehat{A}(TX) \cup \mathrm{Ch}(E)\right)[X]. \tag{2.3}$$

Suppose now that X has a boundary Z, that X is isometrically a product near Z and that ∇^E has a product structure near Z. The integral in (2.3) still makes sense but has no topological meaning. Atiyah-Patodi-Singer found that one must add the eta-invariant of Z in order to obtain an index theorem. Put $\xi = \frac{1}{2}\left[\eta(D_Z) + \dim(\mathrm{Ker}(D_Z))\right]$.

Theorem 2 *[1]*

$$\mathrm{Index}(D_X) = \int_X \widehat{A}(\nabla^{TX}) \wedge \mathrm{Ch}\left(\nabla^E\right) - \xi. \tag{2.4}$$

Here D_X is defined using certain boundary conditions.

Example 3 : Take X to be topologically $[0,1] \times Z$. Let $\{g(\epsilon)\}_{\epsilon \in [0,1]}$ be a 1-parameter family of metrics on Z with $g'(0) = g'(1) = 0$. Give X the metric $d\epsilon^2 + g(\epsilon)$. Let $\mathcal{E}$ be a Hermitian vector bundle on Z and let $\{\nabla(\epsilon)\}_{\epsilon \in [0,1]}$ be a 1-parameter family of compatible connections on $\mathcal{E}$ with $\nabla'(0) = \nabla'(1) = 0$. Let E be the pulled-back Hermitian vector bundle on X, with the connection $\nabla^E = d\epsilon\, \partial_\epsilon + \nabla(\epsilon)$. Then Theorem 2 gives

$$\mathrm{Index}(D_X) = \int_{[0,1]} \int_Z \widehat{A}(\nabla^{TX}) \wedge \mathrm{Ch}\left(\nabla^E\right) - (\xi_1 - \xi_0). \tag{2.5}$$

As the index is an integer, by varying $g(1)$ and $\nabla(1)$, we obtain a variation formula for ξ (mod $\mathbf{Z}$). In the case that $\dim(\mathrm{Ker}(D_Z(\epsilon)))$ is constant in ϵ, we obtain an equation in $\Omega^1([0,1])$:

$$d\epsilon\, \partial_\epsilon \xi = \int_Z \widehat{A}(\nabla^{TX}) \wedge \mathrm{Ch}\left(\nabla^E\right). \tag{2.6}$$

End of Example 3

3. Analytic Torsion

Let Z be a closed manifold, let E be a complex vector bundle on Z and let ∇^E be a flat connection on E. If $\Omega^*(Z;E)$ denotes the forms on Z with value in E, we have the exterior derivative $d : \Omega^*(Z;E) \to \Omega^{*+1}(Z;E)$. Let us now add a Riemannian metric g^{TZ} to Z and a Hermitian metric h^E to E (we do not assume that ∇^E is compatible with h^E!) Then there is an inner product on $\Omega^*(Z;E)$:

$$\langle \omega_1, \omega_2 \rangle = \int_Z \langle \omega_1 \wedge * \omega_2 \rangle_{h^E} \tag{3.1}$$

and we obtain the adjoint operator $d^* : \Omega^{*+1}(Z;E) \to \Omega^*(Z;E)$. The Laplacian on p-forms $\triangle_p$ is the restriction of $d^*d + dd^*$ to $\Omega^p(Z;E)$. Recall that by Hodge theory, $\mathrm{Ker}(\triangle_p) \cong \mathrm{H}^p(Z;E)$. This gives an inner product to $\mathrm{H}^p(Z;E)$ and hence also a volume form $\mathrm{vol}_{\mathrm{H}^p}$. Let $\triangle'_p$ be the projection of $\triangle_p$ to $\Omega^p(Z;E)/\mathrm{Ker}(\triangle_p)$.

Recall the zeta-function definition of the determinant :

$$\ln \det \triangle'_p = - \frac{d}{ds}\Big|_{s=0} \mathrm{Tr}\ (\triangle'_p)^{-s}. \tag{3.2}$$

We can now define the analytic torsion.

Definition 2 *[15]* $T = \sum_{p=0}^{\infty} (-1)^p\, p\, \ln \det \triangle'_p$.

It turns out that T is almost topological, meaning almost independent of g^{TZ} and h^E. This is achieved by the funny factors in the definition. More precisely, let $g^{TZ}(\epsilon)$ and $h^E(\epsilon)$ be smooth 1-parameter families in ϵ.

Theorem 3 *[15, 14, 9]*

$$\frac{dT}{d\epsilon} = - \sum_{p=0}^{\dim(Z)} (-1)^p\ \frac{d\mathrm{vol}_{\mathrm{H}^p}}{d\epsilon} / \mathrm{vol}_{\mathrm{H}^p} + \int_Z (\text{local stuff on } Z). \tag{3.3}$$

One can give a precise description of (local stuff on Z) [9]; this will also follow from Theorem 8. If $\dim(Z)$ is odd then the local stuff vanishes.

Corollary 1 *If* $\dim(Z)$ *is odd and* $\mathrm{H}^*(Z;E) = 0$ *then* T *is topological.*

If, in addition, E has a covariantly-constant volume form, then T equals the classical Reidemeister torsion, a topological invariant of simplicial complexes [10, 13, 14].

Example 4 : Taking Z, E and ∇^E as in Example 1, one has that $\mathrm{H}^*(Z;E) = 0$ iff $a \notin \mathbf{Z}$. Then $T = -\ln|1 - e^{2\pi i a}|^2$.
End of Example 4

Example 5 : Suppose that $\dim(Z) = 3$ and ∇^E is compatible with h^E. Given $B \in \Omega^1(Z;E)$, we define the Chern-Simons action by $I(B) = \frac{1}{2}\int_Z \langle B \wedge dB \rangle_{h^E}$. Suppose that $\mathrm{H}^*(Z;E) = 0$. Then one can show that the path integral $\int e^{iI(B)} \mathcal{D}B$ equals $e^{-T/4}$ [16]. (We are neglecting a possible phase). Note that the path integral is formally independent of any choice of Riemannian metric g^{TZ}. This gives some explanation of the topological nature of T. Of course, to really define the path integral one must choose a Riemannian metric. **End of Example 5**

4. Eta-Forms

We now explain how the eta-invariant arises from the Atiyah-Singer families index theorem. Let us first recall what this theorem says. Let Z be an even-dimensional closed spin manifold. Let $\pi : M \to B$ be a smooth fiber bundle with fiber Z. We can think of M as being a family of copies of Z, parameterized by B. Let TZ be the bundle of tangents to the fibers, a real vector bundle on M of dimension $\dim(Z)$ with $\widehat{A}(TZ) \in \mathrm{H}^{4*}(M; \mathbf{R})$.

Let E be a Hermitian vector bundle on M with compatible connection ∇^E and Chern class $\mathrm{Ch}(E) \in \mathrm{H}^{\mathrm{even}}(M; \mathbf{R})$. We assume that the fibers $Z_b = \pi^{-1}(b)$ have Riemannian metrics g^{TZ}. For each point $b \in B$, we define $D(b)$, a Dirac-type operator on Z_b, using $E|_{Z_b}$. Its kernel $\mathrm{Ker}(D(b))$ is a $\mathbf{Z}_2$-graded vector space. We assume that these vector spaces vary smoothly in b, so as to form a $\mathbf{Z}_2$-graded vector bundle $\mathrm{Ker}(D)$ on B. Then there is the following identity in $\mathrm{H}^{\mathrm{even}}(B; R)$:

Theorem 4 *[3]* $\mathrm{Ch}(\mathrm{Ker}(D)_+) - \mathrm{Ch}(\mathrm{Ker}(D)_-) = \int_Z \widehat{A}(TZ) \cup \mathrm{Ch}(E)$.

If B is a point then we recover Theorem 1.

Suppose that we want a differential-form version of Theorem 4. We first need connections on TZ and $\mathrm{Ker}(D)$. It turns out that these come from choosing a horizontal distribution $T^H M$, meaning a subbundle of TM so that $TM = TZ \oplus T^H M$ [5, 4]. It follows immediately from Theorem 4 that

$$\mathrm{Ch}\left(\nabla^{\mathrm{Ker}(D)_+}\right) - \mathrm{Ch}\left(\nabla^{\mathrm{Ker}(D)_-}\right) \equiv \int_Z \widehat{A}\left(\nabla^{TZ}\right) \wedge \mathrm{Ch}\left(\nabla^E\right) \mod \mathrm{Im}(d)$$

and the question is whether one can do better. One can, and in [6, 11] a differential form $\widetilde{\eta} \in \Omega^{\mathrm{odd}}(B)$ is analytically constructed so that one has the following identity in $\Omega^{\mathrm{even}}(B)$:

Theorem 5 *[6, 11]*

$$d\widetilde{\eta} = \int_Z \widehat{A}\left(\nabla^{TZ}\right) \wedge \mathrm{Ch}\left(\nabla^E\right) - \left[\mathrm{Ch}\left(\nabla^{\mathrm{Ker}(D)_+}\right) - \mathrm{Ch}\left(\nabla^{\mathrm{Ker}(D)_-}\right)\right]. \tag{4.1}$$

The degree-2 part of Theorem 5 is related to the nonabelian chiral anomaly.

If instead $\dim(Z)$ is odd, we assume that $\mathrm{Ker}(D)$ is an (ungraded) vector bundle on B. Then there is an analytically constructed differential form $\widetilde{\eta} \in \Omega^{\mathrm{even}}(B)$ such that

Theorem 6 *[6, 11]* $d\widetilde{\eta} = \int_Z \widehat{A}\left(\nabla^{TZ}\right) \wedge \mathrm{Ch}\left(\nabla^E\right)$.

The degree-0 part $\widetilde{\eta}_{[0]} \in \Omega^0(B)$ of $\widetilde{\eta}$ is one half of the function on B which assigns to b the eta-invariant $\eta(D(b))$ of the fiber Z_b. One sees that (2.6) is equivalent to Theorem 6 in the case when $B = \mathbf{R}$.

The moral of the story is that the eta-invariant arises from writing a topological theorem, Theorem 4, in terms of explicit differential form representatives.

5. Analytic Torsion Forms

We now wish to give a similar interpretation for the analytic torsion, showing that it arises from writing a topological theorem in terms of differential forms. The topological theorem in question is an index theorem for flat vector bundles. We must first define certain characteristic classes of flat vector bundles.

Let E be a complex vector bundle over a smooth manifold M. Let ∇^E be a flat connection on E. Choose an arbitrary Hermitian metric h^E on E. Let $\{e_i\}_{i=1}^{\dim(E)}$ be a local covariantly-constant basis of E. Then locally, we can write h^E as a matrix-valued function h^E_{ij} on M. Put $\omega = (h^E)^{-1} dh^E$. Then ω is a globally-defined element of $\Omega^1(M; \mathrm{End}(E))$.

Definition 3 *For k odd, $k > 0$, define $c_k(E, h^E) \in \Omega^k(M)$ by*

$$c_k(E, h^E) = (2\pi i)^{-(k-1)/2}\, 2^{-k}\, \mathrm{Tr}\left(\omega^k\right). \tag{5.1}$$

Lemma 1 *The form $c_k(E, h^E)$ is closed. Its de Rham cohomology class $c_k(E) \in \mathrm{H}^k(M; \mathbf{R})$ is independent of h^E.*

The classes $c_k(E)$ are the characteristic classes of flat vector bundles that we need. In algebraic K-theory, they are known as the Borel classes. One can think of $c_k(E, h^E)$ as a Chern-Weil-type representative of $c_k(E)$.

Example 6 : If $k = 1$ then $c_1(E, h^E) = \frac{1}{2}\mathrm{Tr}\left((h^E)^{-1} dh^E\right)$ is locally $\frac{1}{2} d\left(\ln \det h^E\right)$. So $c_1(E)$ vanishes iff E admits a covariantly-constant volume form. The higher c_k-classes do not have such a simple characterization, but they all represent obstructions to E admitting a covariantly-constant Hermitian metric. **End of Example 6**

Now let Z be a closed manifold. Let $\pi : M \to B$ be a smooth fiber bundle with fiber Z. Let $o(TZ)$ be the orientation bundle of TZ, a flat real line bundle on M. Let $e(TZ) \in \mathrm{H}^{\dim(Z)}(M; o(TZ))$ be the Euler class of TZ [4].

Let E be a complex vector bundle on M with a flat connection ∇^E. For each $p \in [0, \dim(Z)]$ and $b \in B$, we have a complex vector space $\mathrm{H}^p(Z_b; E|_{Z_b})$. As b varies in B, these vector spaces fit together to form a complex vector bundle H^p on B with a flat connection ∇^{H^p}; this last fact uses the global flatness of E on M.

Theorem 7 *[8] For k odd, $k > 0$, we have an identity in* $\mathrm{H}^k(B; \mathbf{R})$ *:*

$$\sum_{p=0}^{\dim(Z)} (-1)^p \, c_k(H^p) = \int_Z e(TZ) \cup c_k(E). \tag{5.2}$$

Theorem 7 has similarities to Theorem 4. However, there is the important difference that Theorem 4 arises from a more general statement in topological K-theory, whereas Theorem 7 is related to algebraic K-theory.

We now wish to give a differential form version of Theorem 7. Let us choose Riemannian metrics g^{TZ} on the fibers, a horizontal distribution $T^H M$ on M and a Hermitian metric h^E on E. Using the fiberwise isomorphism $\mathrm{Ker}(\triangle_p) \cong \mathrm{H}^p(Z_b; E|_{Z_b})$, there is an induced L^2 Hermitian metric h^{H^p} on H^p. One can analytically construct a differential form $T_{k-1} \in \Omega^{k-1}(B)$ such that the following identity holds in $\Omega^k(B)$:

Theorem 8 *[8]*

$$dT_{k-1} = \int_Z e\left(\nabla^{TZ}\right) \wedge c_k\left(E, h^E\right) - \sum_{p=0}^{\dim(Z)} (-1)^p \, c_k\left(H^p, h^{H^p}\right). \tag{5.3}$$

In the case $k = 1$, one finds that $T_0 \in \Omega^0(B)$ is one half of the function on B which assigns to b the analytic torsion T of the fiber Z_b, defined using $E|_{Z_b}$. The forms $\{T_{k-1}\}_{k \text{ odd}}$ can be called analytic torsion forms.

Corollary 2 *If* $\dim(Z)$ *is odd and* $\mathrm{H}^*(Z; E|_Z) = 0$ *then* T_{k-1} *is closed.*

Theorem 9 *[8] In this case, the de Rham cohomology class* $[T_{k-1}] \in \mathrm{H}^{k-1}(B; \mathbf{R})$ *is independent of* g^{TZ}, $T^H M$ *and* h^E.

Thus in this case, we have defined topological invariants $\{[T_{k-1}]\}_{k \text{ odd}}$ of the smooth fiber bundle.

Example 7 : To see the relationship with the lectures of Loday, let X be a closed manifold and let $P(X)$ be the space of pseudo-isotopies of

X. For k odd, $k \geq 3$, represent an element $\alpha \in \pi_{k-2}(P(X))$ by a map $A : S^{k-2} \rightarrow P(X)$. Let D^{k-1} be a hemisphere in S^{k-1} and glue two copies of $D^{k-1} \times [0,1] \times X$ along the common boundary using A, to obtain a fiber bundle $\pi : M \rightarrow S^{k-1}$ with fiber $[0,1] \times X$. Although the fiber now has boundary, this is not a problem. (In the analysis, use differential forms on the fibers which have absolute boundary conditions on $\{0\} \times X$ and relative boundary conditions on $\{1\} \times X$.) Taking E to be the trivial complex line bundle on M, we obtain an invariant $\int_{S^{k-1}} T_{k-1} \in \mathbf{R}$ of α. There are reasons to believe that this invariant detects nontrivial rational information of the homotopy type of $P(X)$.
End of Example 7

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