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Collapsing and Dirac-Type Operators

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Abstract. We analyze the limit of the spectrum of a geometric Dirac-type operator under a collapse with bounded diameter and bounded sectional curvature. In the case of a smooth limit space B, we show that the limit of the spectrum is given by the spectrum of a certain first-order differential operator on B, which can be constructed using superconnections. In the case of a general limit space X, we express the limit operator in terms of a transversally elliptic operator on a G-manifold $\check{X}$ with $X = \check{X}/G$. As an application, we give a characterization of manifolds which do not admit uniform upper bounds, in terms of diameter and sectional curvature, on the k-th eigenvalue of the square of a Dirac-type operator. We also give a formula for the essential spectrum of a Dirac-type operator on a finite-volume manifold with pinched negative sectional curvature.

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1. Introduction

In previous papers we analyzed the limit of the spectrum of the differential form Laplacian on a manifold, under a collapse with bounded diameter and bounded sectional curvature [17, 22]. In the present paper, we extend the analysis of [17, 22] to geometric Dirac-type operators. As the present paper is a sequel to [17, 22], we refer to the introduction of [17] for background information about collapsing with bounded curvature and its relation to analytic questions.

Let M be a connected closed oriented Riemannian manifold of dimension $n > 0$. If M is spin then we put $G = \mathrm{Spin}(n)$ and if M is not spin then we put $G = \mathrm{SO}(n)$. The spinor-type fields that we consider are sections of a vector bundle E^M associated to a G-Clifford module V, the latter being in the sense of Definition 2 of Section 2. The ensuing Dirac-type operator D^M acts on sections of E^M. We will think of the spectrum $\sigma(D^M)$ of D^M as a set of real numbers with multiplicities, corresponding to possible multiple eigenvalues. For simplicity, in this introduction we will sometimes refer to the Dirac-type operators as acting on spinors, even though the results are more general.

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We first consider a collapse in which the limit space is a smooth Riemannian manifold. The model case is that of a Riemannian affine fiber bundle.

DEFINITION 1. An affine fiber bundle is a smooth fiber bundle $\pi\colon M \to B$ whose fiber Z is an infranilmanifold and whose structure group can be reduced from $\mathrm{Diff}(Z)$ to $\mathrm{Aff}(Z)$. A Riemannian affine fiber bundle is an affine fiber bundle along with

- A horizontal distribution $T^H M$ whose holonomy lies in $\mathrm{Aff}(Z)$,
- A family of vertical Riemannian metrics g^{TZ} which are parallel with respect to the flat affine connections on the fibers Z_b and
- A Riemannian metric g^{TB} on B.

Given a Riemannian affine fiber bundle $\pi\colon M \to B$, there is a Riemannian metric g^{TM} on M constructed from $T^H M$, g^{TZ} and g^{TB}. Let R^M denote the Riemann curvature tensor of (M, g^{TM}), let Π denote the second fundamental forms of the fibers $\{Z_b\}_{b\in B}$ and let $T \in \Omega^2(M; TZ)$ be the curvature of $T^H M$. Given $b \in B$, there is a natural flat connection on $E^M\big|_{Z_b}$ which is constructed using the affine structure of Z_b. We define a Clifford bundle E^B on B whose fiber over $b \in B$ consists of the parallel sections of $E^M\big|_{Z_b}$. The operator D^M restricts to a first-order differential operator D^B on $C^\infty(B; E^B)$. If V happens to be the spinor module then we show that D^B is the 'quantization' of a certain superconnection on B. For general V, there is an additional zeroth-order term in D^B which depends on Π and T.

We show that the spectrum of D^M coincides with that of D^B up to a high level, which depends on the maximum diameter $\mathrm{diam}(Z)$ of the fibers $\{Z_b\}_{b\in B}$.

THEOREM 1. *There are positive constants A, A' and C which only depend on n and V such that if $\| R^Z \|_\infty \ \mathrm{diam}\,(Z)^2 \leqslant A'$ then the intersection of $\sigma(D^M)$ with the interval*

$$\begin{aligned}&[-(A \,\mathrm{diam}\,(Z)^{-2} - C(\| R^M \|_\infty + \| \Pi \|_\infty^2 + \| T \|_\infty^2))^{1/2},\\ &\qquad (A \,\mathrm{diam}\,(Z)^{-2} - C(\| R^M \|_\infty + \| \Pi \|_\infty^2 + \| T \|_\infty^2))^{1/2}]\end{aligned} \tag{1.1}$$

equals the intersection of $\sigma(D^B)$ with (1.1).

If $Z = S^1$, $\Pi = 0$ and V is the spinor module then we recover some results of [1, Section 4]; see also [12, Theorem 1.5]. The proof of Theorem 1 follows the same strategy as the proof of the analogous [17, Theorem 1]. Consequently, in the proof of Theorem 1, we only indicate the changes that need to be made in the proof of [17, Theorem 1] and refer to [17] for details.

Given B, Cheeger, Fukaya and Gromov showed that under some curvature bounds, any Riemannian manifold M which is sufficiently Gromov–Hausdorff close to B can be well approximated by a Riemannian affine fiber bundle [11]. Using this fact, we show that the spectrum of D^M can be uniformly approximated by that of a certain first-order differential operator D^B on B, at least up to a high level which depends on the Gromov-Hausdorff distance between M and B.

Given $\varepsilon > 0$ and two collections of real numbers $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$, we say that $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are ε-close if there is a bijection $\alpha: I \to J$ such that for all $i \in I$, $|b_{\alpha(i)} - a_i| \leqslant \varepsilon$.

THEOREM 2. *Let B be a fixed smooth connected closed Riemannian manifold. Given $n \in \mathbb{Z}^+$, take $G \in \{\mathrm{SO}(n), \mathrm{Spin}(n)\}$ and let V be a G-Clifford module. Then for any $\varepsilon > 0$ and $K > 0$, there are positive constants $A(B, n, V, \varepsilon, K)$, $A'(B, n, V, \varepsilon, K)$, and $C(B, n, V, \varepsilon, K)$ so that the following holds. Let M be an n-dimensional connected closed oriented Riemannian manifold with a G-structure such that $\| R^M \|_\infty \leqslant K$ and $d_{GH}(M, B) \leqslant A'$. Then there are a Clifford module E^B on B and a certain first-order differential operator D^B on $C^\infty(B; E^B)$ such that*

(1) $\{\sinh^{-1}(\lambda/\sqrt{2K}) \colon \lambda \in \sigma(D^M)$ *and* $\lambda^2 \leqslant \mathrm{A} d_{GH}(M, B)^{-2} - C\}$ *is ε-close to a subset of* $\{\sinh^{-1}(\lambda/\sqrt{2K}) \colon \lambda \in \sigma(D^B)\}$, *and*

(2) $\{\sinh^{-1}(\lambda/\sqrt{2K}) \colon \lambda \in \sigma(D^B)$ *and* $\lambda^2 \leqslant \mathrm{A} d_{GH}(M, B)^{-2} - C\}$ *is ε-close to a subset of* $\{\sinh^{-1}(\lambda/\sqrt{2K}) \colon \lambda \in \sigma(D^M)\}$.

The other results in this paper concern collapsing to a possibly-singular space. Let X be a limit space of a sequence $\{M_i\}_{i=1}^\infty$ of n-dimensional connected closed oriented Riemannian manifolds with uniformly bounded diameter and uniformly bounded sectional curvature. In general, X is not homeomorphic to a manifold. However, Fukaya showed that X is homeomorphic to $\check{X}/G$, where $\check{X}$ is a manifold and G is a compact Lie group which acts on $\check{X}$ [15]. This comes from writing $M_i = P_i/G$, where $G = \mathrm{SO}(n)$ and P_i is the oriented orthonormal frame bundle of M_i. There is a canonical Riemannian metric on P_i. Then $\{P_i\}_{i=1}^\infty$ has a subsequence which Gromov–Hausdorff converges to a manifold $\check{X}$. As the convergence argument can be done G-equivariantly, the corresponding subsequence of $\{M_i\}_{i=1}^\infty$ converges to $X = \check{X}/G$. In general, $\check{X}$ is a smooth manifold with a metric which is $C^{1,\alpha}$ regular for all $\alpha \in (0, 1)$.

In [22] we dealt with the limit of the spectra of the differential form Laplacians $\{\triangle^{M_i}\}_{i=1}^\infty$ on the manifolds $\{M_i\}_{i=1}^\infty$. We defined a limit operator $\triangle^X$ which acts on the 'differential forms' on X, coupled to a superconnection. In order to make this precise, we defined the 'differential forms' on X to be the G-basic differential forms on $\check{X}$. We constructed the corresponding differential form Laplacian $\triangle^X$ and showed that its spectrum described the limit of the spectra of $\{\triangle^{M_i}\}_{i=1}^\infty$. We refer to [22] for the precise statements.

In the case of geometric Dirac-type operators D^{M_i}, there is a fundamental problem in extending this approach. Namely, if $\check{X}$ is a spin manifold on which a compact Lie group G acts isometrically and preserving the spin structure then there does not seem to be a notion of G-basic spinors on $\check{X}$. In order to get around this problem, we take a different approach. For a given n-dimensional Riemannian spin manifold

M, put $G = \mathrm{Spin}(n)$, let P be the principal $\mathrm{Spin}(n)$-bundle of M and let V be the spinor module. One can identify the spinor fields on M with $(C^\infty(P) \otimes V)^G$, the G-invariant subspace of $C^\infty(P) \otimes V$. There are canonical horizontal vector fields $\{\mathfrak{Y}_j\}_{j=1}^n$ on P and the Dirac operator takes the form $D^M = -i \sum_{j=1}^n \gamma^j \mathfrak{Y}_j$. Furthermore, $(D^M)^2$ can be written in a particularly simple form. As in equation (4.2) below, when acting on $(C^\infty(P) \otimes V)^G$, $(D^M)^2$ becomes the scalar Laplacian on P (acting on V-valued functions) plus a zeroth-order term.

Following this viewpoint, it makes sense to define the limiting 'spinor fields' on X to be the elements of $(C^\infty(\check{X}) \otimes V)^G$. We can then extend Theorem 1 to the setting of G-equivariant Riemannian affine fiber bundles. Namely, the limit operator D^X turns out to be a G-invariant first-order differential operator on $C^\infty(\check{X}) \otimes V$, transversally elliptic in the sense of Atiyah [2], which one then restricts to the G-invariant subspace $(C^\infty(\check{X}) \otimes V)^G$. In Theorem 6 below, we show that the analog of Theorem 1 holds, in which D^B is replaced by D^X.

Theorem 6 refers to a given G-equivariant Riemannian affine fiber bundle. In order to deal with arbitrary collapsing sequences, we use the aforementioned representation of $(D^M)^2$ as a Laplace-type operator on P. If $\{M_i\}_{i=1}^\infty$ is a sequence of n-dimensional Riemannian manifolds with uniformly bounded diameter and uniformly bounded sectional curvature then we show that after taking a subsequence, the spectra of $\{(D^{M_i})^2\}_{i=1}^\infty$ converge to the spectrum of a Laplace-type operator on a limit space. Let $\{\lambda_k(|D^M|)\}_{k=1}^\infty$ denote the eigenvalues of $|D^M|$, counted with multiplicity.

THEOREM 3. *Given $n \in \mathbb{Z}^+$ and $G \in \{\mathrm{SO}(n), \mathrm{Spin(n)}\}$, let $\{M_i\}_{i=1}^\infty$ be a sequence of connected closed oriented n-dimensional Riemannian manifolds with a G-structure. Let V be a G-Clifford module. Suppose that for some $D, K > 0$ and for each $i \in \mathbb{Z}^+$, we have $\mathrm{diam}\,(M_i) \leqslant D$ and $\| R^{M_i} \|_\infty \leqslant K$. Then there are*

(1) *A subsequence of $\{M_i\}_{i=1}^\infty$, which we relabel as $\{M_i\}_{i=1}^\infty$,*
(2) *A smooth closed G-manifold $\check{X}$ with a G-invariant Riemannian metric $g^{T\check{X}}$ which is $C^{1,\alpha}$-regular for all $\alpha \in (0, 1)$,*
(3) *A positive G-invariant function $\chi \in C(\check{X})$ with $\int_{\check{X}} \chi dvol = 1$ and*
(4) *A G-invariant function $\mathcal{V} \in L^\infty(\check{X}) \otimes \mathrm{End(V)}$ such that if $\triangle^{\check{X}}$ denotes the Laplacian on $L^2(\check{X}, \chi d\mathrm{vol}) \otimes \mathrm{V}$ [14, (0.8)] and $|D^X|$ denotes the operator $\sqrt{\triangle^{\check{X}} + \mathcal{V}}$ acting on $(L^2(\check{X}, \chi d\mathrm{vol}) \otimes \mathrm{V})^{\mathrm{G}}$ then for all $k \in \mathbb{Z}^+$,*

$$\lim_{i \to \infty} \lambda_k\big(|D^{M_i}|\big) = \lambda_k\big(|D^X|\big). \tag{1.2}$$

In the special case of the signature operator, the proof of Theorem 3 is somewhat simpler than that of the analogous [22, Proposition 3], in that we essentially only have to deal with scalar Laplacians. However, [22, Proposition 3] gives more detailed information. In particular, it expresses the limit operator in terms of a basic flat degree-1 superconnection on $\check{X}$. This seems to be necessary in order to prove the results of [22] concerning small eigenvalues. Of course, one does not expect to have analogous results concerning the small eigenvalues of general geometric Dirac-type operators, as their zero-eigenvalues have no topological meaning.

As an application of Theorem 3, we give a characterization of manifolds which do not have a uniform upper bound on the k-th eigenvalue of $|D^M|$, in terms of diameter and sectional curvature.

THEOREM 4. *Let M be a connected closed oriented manifold with a G-structure. Let V be a G-Clifford module. Suppose that for some $K > 0$ and $k \in \mathbb{Z}^+$, there is no uniform upper bound on $\lambda_k(|D^M|)$ among Riemannian metrics on M with* $\operatorname{diam}(M) = 1$ *and* $\| R^M \|_\infty \leqslant K$. *Then M is the total space of a possibly-singular affine fiber bundle $M \to X$ whose generic fiber is an infranilmanifold Z such that the restriction of E^M to Z does not have any nonzero affine-parallel sections.*

As a partial converse, let M be the total space of a smooth affine fiber bundle whose fiber is Z and whose base B has positive dimension. If the restriction of E^M to Z does not have any nonzero affine-parallel sections then there is some $K > 0$ such that for any $k \in \mathbb{Z}^+$, there is no uniform upper bound on $\lambda_k(|D^M|)$ among Riemannian metrics on M with $\operatorname{diam}(M) = 1$ *and* $\| R^M \|_\infty \leqslant K$.

More precisely, the possibly-singular affine fiber bundle $M \to X$ of Theorem 4 is the G-quotient of a G-equivariant affine fiber bundle $P \to \check{X}$. Theorem 4 is an analog of [22, Theorem 1.2]. A simple example of Theorem 4 comes from considering spinors on $M = S^1 \times N$, where N is a spin manifold and the spin structure on S^1 is the one that does not admit a harmonic spinor. Upon shrinking the S^1-fiber, the eigenvalues of D_M go off to $\pm\infty$.

Finally, we give a result about the essential spectrum of a geometric Dirac-type operator on a finite-volume manifold of pinched negative curvature, which is an analog of [19, Theorem 2]. Let M be a complete connected oriented n-dimensional Riemannian manifold with a G-structure. Suppose that M has finite volume and its sectional curvatures satisfy $-b^2 \leqslant K \leqslant -a^2$, with $0 < a \leqslant b$. Let V be a G-Clifford module. Label the ends of M by $I \in \{1, \ldots, N\}$. An end of M has a neighborhood U_I whose closure is homeomorphic to $[0, \infty) \times Z_I$, where the first coordinate is the Busemann function corresponding to a ray exiting the end, and Z_I is an infranilmanifold. Let E^M be the vector bundle on M associated to the pair (G, V) and let D^M be the corresponding Dirac-type operator. If U_I lies far enough out the end then for each $s \in [0, \infty)$, $C^\infty(\{s\} \times Z_I; E^M|_{\{s\}\times Z_I})$ decomposes as the direct sum of a finite-dimensional space $E^B_{I,s}$, consisting of 'bounded energy' sections, and its orthogonal complement, consisting of 'high energy' sections. The vector spaces $\{E^B_{I,s}\}_{s\in[0,\infty)}$ fit together to form a vector bundle E^B_I on $[0, \infty)$. Let P_0 be orthogonal projection from $\bigoplus_{I=1}^N C^\infty(\overline{U_I}; E^M|_{\overline{U_I}})$ to $\bigoplus_{I=1}^N C^\infty([0, \infty); E^B_I)$. Let D^M_{end} be the restriction of D^M to $\bigoplus_{I=1}^N C^\infty(\overline{U_I}; E^M|_{\overline{U_I}})$, say with Atiyah-Patodi-Singer boundary conditions. Then $P_0 D^M_{\mathrm{end}} P_0$ is a first-order ordinary differential operator on $\bigoplus_{I=1}^N C^\infty\big([0, \infty); E^B_I\big)$.

THEOREM 5. *The essential spectrum of* D^M *is the same as that of* $P_0 D^M_{\mathrm{end}} P_0$.

There is some intersection between Theorem 5 and the results of [4, Theorem 0.1], concerning the essential spectrum of D^M when $n = 2$ and under an additional curvature assumption, and [5, Theorem 1], concerning the essential spectrum of D^M when M is hyperbolic and V is the spinor module.

2. Dirac-type Operators and Infranilmanifolds

Given $n \in \mathbb{Z}^+$, let G be either $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$.

DEFINITION 2. A G-Clifford module consists of a finite-dimensional Hermitian G-vector space V and a G-equivariant linear map $\gamma : \mathbb{R}^n \to \mathrm{End}(V)$ such that $\gamma(v)^2 = |v|^2 \mathrm{Id}.$ and $\gamma(v)^* = \gamma(v)$.

Let M be a connected closed oriented smooth n-dimensional Riemannian manifold. Put $G = \mathrm{Spin}(n)$ or $G = \mathrm{SO}(n)$, according as to whether or not M is spin. If M is spin, fix a spin structure. Let P be the corresponding principal G-bundle, covering the oriented orthonormal frame bundle. Its topological isomorphism class is independent of the choice of Riemannian metric. Given the Riemannian metric, there is a canonical $\mathbb{R}^n$-valued 1-form θ on P, the soldering form.

With respect to the standard basis $\{e_j\}_{j=1}^n$ of $\mathbb{R}^n$, we write $\gamma^j = \gamma(e_j)$. We also take generators $\{\sigma^{ab}\}_{a,b=1}^n$ for the representation of the Lie algebra $\mathfrak{g}$ on V, so that $\sigma^{ba} = -\sigma^{ab}$, $\left(\sigma^{ab}\right)^* = -\sigma^{ab}$ and

$$[\sigma^{ab}, \sigma^{cd}] = \delta^{ad}\sigma^{bc} - \delta^{ac}\sigma^{bd} + \delta^{bc}\sigma^{ad} - \delta^{bd}\sigma^{ac}. \tag{2.1}$$

The G-equivariance of γ implies

$$[\gamma^a, \sigma^{bc}] = \delta^{ab}\gamma^c - \delta^{ac}\gamma^b. \tag{2.2}$$

EXAMPLES. (1) If $G = \mathrm{Spin}(n)$ and V is the spinor representation of G then $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$.

(2) If $G = \mathrm{SO}(n)$ and $V = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$, let E^j and I^j denote exterior and interior multiplication by e^j, respectively. Put $\gamma^j = i\left(E^j - I^j\right)$ and $\widehat{\gamma}^j = E^j + I^j$. Then $\sigma^{ab} = \frac{1}{4}\left([\gamma^a, \gamma^b] + [\widehat{\gamma}^a, \widehat{\gamma}^b]\right)$.

Put $E^M = P \times_G V$. The Dirac-type operator D^M acts on the space $C^\infty(M; E^M)$. As the topological vector space $C^\infty(M; E^M)$ is independent of any choice of Riemannian metric on M, it makes sense to compare Dirac-type operators for different Riemannian metrics on M; see [18, Section 2] for further discussion.

Let g^{TM} be the Riemannian metric on M. Let ω be the Levi-Civita connection on P. Let $\{e_j\}_{j=1}^n$ be a local oriented orthonormal basis of TM, with dual basis

$\{\tau^j\}_{j=1}^n$. Then we can write ω locally as a matrix-valued 1-form $\omega_b^a = \sum_{j=1}^n \omega_{bj}^a \tau^j$, and

$$D^M = -i\sum_{j=1}^n \gamma^j \nabla_{e_j} = -i\sum_{j=1}^n \gamma^j \left(e_j + \frac{1}{2}\sum_{a,b=1}^n \omega_{abj}\sigma^{ab}\right). \tag{2.3}$$

We have the Bochner-type equation

$$(D^M)^2 = \nabla^*\nabla - \frac{1}{8}\sum_{a,b,i,j=1}^n R^M_{abij}(\gamma^i\gamma^j - \gamma^j\gamma^i)\sigma^{ab}. \tag{2.4}$$

As the set of Riemannian metrics on M is an open convex subset of a Fréchet space, it makes sense to talk about an analytic 1-parameter family $\{c(t)\}_{t\in[0,1]}$ of metrics. Then for $t \in [0, 1]$, $\dot{c}(t)$ is a symmetric 2-tensor on M. Let $\| \dot{c}(t) \|_{c(t)}$ denote the norm of $\dot{c}(t)$ with respect to $c(t)$, i.e.

$$\| \dot{c}(t) \|_{c(t)} = \sup_{v\in TM-0} \frac{|\dot{c}(t)(v, v)|}{c(t)(v, v)}. \tag{2.5}$$

Put $l(c) = \int_0^1 \| \dot{c}(t) \|_{c(t)}\, dt$. We extend the definition of $l(c)$ to piecewise-analytic families of metrics in the obvious way. Given $K > 0$, let $\mathcal{M}(M, K)$ be the set of Riemannian metrics on M with $\| R^M \|_\infty \leqslant K$. Let d be the corresponding length metric on $\mathcal{M}(M, K)$, computed using piecewise-analytic paths in $\mathcal{M}(M, K)$. Let $\sigma(D^M, g^{TM})$ denote the spectrum of D^M as computed with g^{TM}, a discrete subset of $\mathbb{R}$ which is counted with multiplicity.

PROPOSITION 1. *There is a constant $C = C(n, V) > 0$ such that for all $K > 0$ and $g_1^{TM}, g_2^{TM} \in \mathcal{M}(M, K)$,*

$$\left\{\sinh^{-1}\left(\frac{\lambda}{\sqrt{K}}\right): \lambda \in \sigma(D^M, g_1^{TM})\right\} \tag{2.6}$$

and

$$\left\{\sinh^{-1}\left(\frac{\lambda}{\sqrt{K}}\right): \lambda \in \sigma(D^M, g_2^{TM})\right\} \tag{2.7}$$

are $Cd(g_1^{TM}, g_2^{TM})$-close.

Proof. It is enough to show that there is a number C such that if $\{c(t)\}_{t\in[0,1]}$ is an analytic 1-parameter family of metrics contained in $\mathcal{M}(M, K)$ then

$$\left\{\sinh^{-1}\left(\frac{\lambda}{\sqrt{K}}\right): \lambda \in \sigma(D^M, c(0))\right\}$$

and

$$\left\{\sinh^{-1}\left(\frac{\lambda}{\sqrt{K}}\right): \lambda \in \sigma(D^M, c(1))\right\}$$

are $Cd(c(0), c(1))$-close. By eigenvalue perturbation theory [20, Chapter XII], the subset $\bigcup_{t\in[0,1]}\{t\} \times \sigma(D^M, c(t))$ of $\mathbb{R}^2$ is the union of the graphs of functions $\{\lambda_j(t)\}_{j\in\mathbb{Z}}$ which are analytic in t. Thus it is enough to show that for each $j \in \mathbb{Z}$,

$$\left|\sinh^{-1}\left(\frac{\lambda_j(1)}{\sqrt{K}}\right) - \sinh^{-1}\left(\frac{\lambda_j(0)}{\sqrt{K}}\right)\right| \leqslant Cl(c). \tag{2.8}$$

Let $D(t)$ denote the Dirac-type operator constructed with the metric $c(t)$. It is self-adjoint when acting on $L^2(E^M, d\mathrm{vol}(t))$. In order to have all of the operators $\{D(t)\}_{t\in[0,1]}$ acting on the same Hilbert space, define $f(t) \in C^\infty(M)$ by $f(t) = d\mathrm{vol}(t)/d\mathrm{vol}(0)$. Then the spectrum of $D(t)$, acting on $L^2(E^M, d\mathrm{vol}(t))$, is the same as the spectrum of the self-adjoint operator $f(t)^{1/2}D(t)f(t)^{-1/2}$ acting on $L^2(E^M, d\mathrm{vol}(0))$. One can now compute $d\lambda_j/dt$ using eigenvalue perturbation theory, as in [20, Chapter XII]. Let $\psi_j(t)$ be a smoothly-varying unit eigenvector whose eigenvalue is $\lambda_j(t)$. Define a quadratic form $T(t)$ on TM by

$$\begin{aligned} T(t)(X, Y) = {} & \langle\psi_j, -i(\gamma(X)\nabla_Y\psi_j + \gamma(Y)\nabla_X\psi_j)\rangle + \\ & + \langle -i(\gamma(X)\nabla_Y\psi_j + \gamma(Y)\nabla_X\psi_j), \psi_j\rangle. \end{aligned} \tag{2.9}$$

Using the metric $c(t)$ to convert the symmetric tensors $\dot{c}(t)$ and $T(t)$ to self-adjoint sections of $End(TM)$, one finds

$$\frac{\mathrm{d}\lambda_{\mathrm{j}}}{\mathrm{d}t} = -\frac{1}{8}\int_M \mathrm{Tr}(\dot{c}(t)T(t))d\mathrm{vol}(t). \tag{2.10}$$

(This equation was shown for the pure Dirac operator, by different means, in [10].) Then

$$\left|\frac{\mathrm{d}\lambda_j}{\mathrm{d}t}\right| \leqslant \mathrm{const.} \parallel \dot{c}(t) \parallel_{c(t)} \int_M \mathrm{Tr}(|T(t)|)d\mathrm{vol}(t). \tag{2.11}$$

Letting $\{x_i\}_{i=1}^n$ be an orthonormal basis of eigenvectors of $T(t)$ at a point $m \in M$, we have $\mathrm{Tr}(|T(t)|) = \sum_{i=1}^n |T(t)(x_i, x_i)|$. Then from (2.9), we obtain

$$\int_M \mathrm{Tr}(|T(t)|)d\mathrm{vol}(t) \leqslant \mathrm{const.}\left(\int_M |\nabla\psi_j|^2 d\mathrm{vol}(t)\right)^{1/2}. \tag{2.12}$$

From (2.4),

$$\int_M |\nabla\psi_j|^2 d\mathrm{vol}(t) \leqslant \lambda_j^2 + \mathrm{const.}\ K. \tag{2.13}$$

In summary, from (2.11), (2.12) and (2.13), there is a positive constant C such that

$$\left|\frac{\mathrm{d}\lambda_j}{\mathrm{d}t}\right| \leqslant C \parallel \dot{c}(t) \parallel_{c(t)} \left(\lambda_j^2 + K\right)^{1/2}. \tag{2.14}$$

Integration gives Equation (2.8). The proposition follows. □

For some basic facts about infranilmanifolds, we refer to [17, Section 3]. Let N be a simply-connected connected nilpotent Lie group. Let Γ be a discrete subgroup of $\mathrm{Aff}(N)$ which acts freely and cocompactly on N, with $\Gamma \cap N$ of finite index in Γ. Put $Z = \Gamma\backslash N$, an infranilmanifold. There is a canonical flat linear connection ∇^{aff} on TZ. Put $\widehat{\Gamma} = \Gamma \cap N$, a cocompact subgroup of N. There is a short exact sequence

$$1 \longrightarrow \widehat{\Gamma} \longrightarrow \Gamma \longrightarrow F \longrightarrow 1, \tag{2.15}$$

with F a finite group. Put $\widehat{Z} = \widehat{\Gamma}\backslash N$, a nilmanifold which finitely covers Z with covering group F.

Let g^{TZ} be a Riemannian metric on Z which is parallel with respect to ∇^{aff}. Let us discuss the condition for Z to be spin. Suppose first that Z is spin. Choose a spin structure on Z. Fix the basepoint $z_0 = \Gamma e \in Z$. As ∇^{aff} preserves g^{TZ}, its holonomy lies in $\mathrm{SO}(n)$. Hence ∇^{aff} lifts to a flat connection on the principal $\mathrm{Spin}(n)$-bundle, which we also denote by ∇^{aff}. There is a corresponding holonomy representation $\Gamma \to \mathrm{Spin}(n)$.

Conversely, suppose that we do not know *a priori* if Z is spin. Suppose that the affine holonomy $\Gamma \to F \to \mathrm{SO}(n)$ lifts to a homomorphism $\Gamma \to \mathrm{Spin}(n)$. Naturally, the existence of this lifting is independent of the particular choice of g^{TZ}. Then there is a corresponding spin structure on Z with principal bundle $\Gamma\backslash(N \times \mathrm{Spin}(n))$. The different spin structures on Z correspond to different lifts of $\Gamma \to \mathrm{SO}(n)$ to $\Gamma \to \mathrm{Spin}(n)$. These are labelled by $H^1(\Gamma; \mathbb{Z}_2) \cong H^1(Z; \mathbb{Z}_2)$. Note that there are examples of nonspin flat manifolds [3]. Also, even if Z is spin and has a fixed spin structure, the action of $\mathrm{Aff}(Z)$ on Z generally does not lift to the principal $\mathrm{Spin}(n)$-bundle, as can be seen for the $SL(n, \mathbb{Z})$-action on $Z = T^n$.

Now let G be either $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$. Let V be a G-Clifford module. Suppose that Z has a G-structure. If $G = \mathrm{SO}(n)$ then we have the affine holonomy homomorphism $\rho\colon \Gamma \to \mathrm{SO}(n)$. If $G = \mathrm{Spin}(n)$ then we have a given lift of it to $\rho\colon \Gamma \to \mathrm{Spin}(n)$. In either case, there is an action of Γ on V coming from $\Gamma \xrightarrow{\rho} G \to Aut(V)$. The vector bundle E^Z can now be written as $E^Z = \Gamma\backslash(N \times V)$. We see that the vector space of sections of E^Z which are parallel with respect to ∇^{aff} is isomorphic to V^Γ, the subspace of V which is fixed by the action of Γ.

If V is the spinor representation of $G = \mathrm{Spin}(n)$ then let us consider the conditions for V^Γ to be nonzero. First, as the restriction of $\rho\colon \Gamma \to \mathrm{Spin}(n)$ to $\widehat{\Gamma}$ maps $\widehat{\Gamma}$ to ± 1, we must have $\rho\big|_{\widehat{\Gamma}} = 1$. Given this, the homomorphism ρ factors through a homomorphism $F \to \mathrm{Spin}(n)$. Then we have $V^\Gamma = V^F$. This may be nonzero even if the homomorphism $F \to \mathrm{Spin}(n)$ is nontrivial.

Returning to the case of general V, as g^{TZ} is parallel with respect to ∇^{aff}, the operator D^Z preserves the space V^Γ of affine-parallel sections of E^Z. Let D^{inv} be the restriction of D^Z to V^Γ.

PROPOSITION 2. *There are positive constants A and A' depending only on* $\dim(Z)$ *and V such that if* $\| R^Z \|_\infty \operatorname{diam}(Z)^2 \leqslant A'$ *then the spectrum $\sigma(D^Z)$ of D^Z satisfies*

$$\begin{aligned}&\sigma(D^Z) \cap [-A \operatorname{diam}(Z)^{-1}, A \operatorname{diam}(Z)^{-1}]\\ &\quad = \sigma(D^{inv}) \cap [-A \operatorname{diam}(Z)^{-1}, A \operatorname{diam}(Z)^{-1}].\end{aligned} \tag{2.16}$$

Proof. As D^Z is diagonal with respect to the orthogonal decomposition

$$C^\infty(Z; E^Z) = V^\Gamma \oplus \left(V^\Gamma\right)^\perp, \tag{2.17}$$

it is enough to show that there are constants A and A' as in the statement of the proposition such that the eigenvalues of $(D^Z)^2\big|_{(V^\Gamma)^\perp}$ are greater than $A^2 \operatorname{diam}(Z)^{-2}$. As in the proof of [17, Proposition 2], we can reduce to the case when $F = \{e\}$, i.e. Z is a nilmanifold $\Gamma\backslash N$. Then

$$C^\infty(Z; E^Z) \cong (C^\infty(N) \otimes V)^\Gamma. \tag{2.18}$$

Using an orthonormal frame $\{e_i\}_{i=1}^{\dim(Z)}$ for the Lie algebra $\mathfrak{n}$ as in the proof of [17, Proposition 2], we can write

$$\nabla^{aff}_{e_i} = e_i \otimes \mathrm{Id}. \tag{2.19}$$

and

$$\nabla^Z_{e_i} = (e_i \otimes \mathrm{Id.}) + \left(\mathrm{Id.} \otimes \frac{1}{2} \sum_{a,b=1}^{\dim(Z)} \omega_{abi} \sigma^{ab}\right). \tag{2.20}$$

The rest of the proof now proceeds as in that of [17, Proposition 2], to which we refer for details. □

3. Collapsing to a Smooth Base

For background information about superconnections and their applications, we refer to [7]. Let M be a connected closed oriented Riemannian manifold which is the total space of a Riemannian submersion $\pi: M \to B$. Suppose that M has a G^M-structure and that V^M is a G^M-Clifford module, as in Section 2. If $G^M = \mathrm{SO}(n)$, put $G^Z = \mathrm{SO}(\dim(Z))$ and $G^B = \mathrm{SO}(\dim(B))$. If $G^M = \mathrm{Spin}(n)$, put $G^Z = \mathrm{Spin}(\dim(Z))$ and $G^B = \mathrm{Spin}(\dim(B))$. As a fiber Z_b has a trivial normal bundle in M, it admits a G^Z-structure. Fixing an orientation of T_bB fixes the G^Z-structure of Z_b. Note, however, that B does not necessarily have a G^B-structure. For example, if M is oriented then B is not necessarily oriented, as is shown in the example of $S^1 \times_{\mathbb{Z}_2} S^2 \to \mathbb{R}P^2$, where the generator of $\mathbb{Z}_2$ acts on S^1 by complex conjugation and on S^2 by the antipodal map. And if M is spin then B is not necessarily spin, as is shown in the example of $S^5 \to \mathbb{C}P^2$. What is true is that if the vertical tangent bundle TZ, a vector bundle on M, has a G^Z-structure then B has a G^B-structure.

Put $E^M = P \times_{G^M} V^M$. There is a Clifford bundle C on B with the property that $C^\infty(B; C) \cong C^\infty(M; E^M)$ [7, Section 9.2]. If $\dim(Z) > 0$ then $\dim(C) = \infty$. To describe C more explicitly, let $V^M = \bigoplus_{l \in L} V_l^B \otimes V_l^Z$ be the decomposition of V_M into irreducible representations of $G^B \times G^Z \subset G^M$.

EXAMPLES. (1) If $G^M = \mathrm{Spin}(n)$ and V^M is the spinor representation then V^B and V^Z are spinor representations.

(2) If $G^M = \mathrm{SO}(n)$ and $V^M = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$, then $V^B = \Lambda^*(\mathbb{R}^{\dim(B)}) \otimes_{\mathbb{R}} \mathbb{C}$ and $V^Z = \Lambda^*(\mathbb{R}^{\dim(Z)}) \otimes_{\mathbb{R}} \mathbb{C}$.

Let U be a contractible open subset of B. Choose an orientation on U. For $b \in U$, let $E_{b,l}^Z$ be the vector bundle on Z_b associated to the pair (G^Z, V_l^Z). Then $E^M\big|_{Z_b} \cong \bigoplus_{l \in L} V_l^B \otimes E_{b,l}^Z$. The vector bundles $\{E_{b,l}^Z\}_{b \in U}$ are the fiberwise restrictions of a vector bundle E_l^Z on $\pi^{-1}(U)$, a vertical 'spinor' bundle. There is a pushforward vector bundle W_l on U whose fiber $W_{l,b}$ over $b \in U$ is $C^\infty(Z_b; E_{b,l}^Z)$. If $\dim(Z) > 0$ then $\dim(W_l) = \infty$. There are Hermitian inner products $\{h^{W_l}\}_{l \in L}$ on $\{W_l\}_{l \in L}$ induced from the vertical Riemannian metric g^{TZ}. Furthermore, there are Clifford bundles $\{C_l\}_{l \in L}$ on U for which the fiber $C_{l,b}$ of C_l over $b \in U$ is isomorphic to $V_l^B \otimes W_{l,b}$. By construction, $C^\infty(Z_b; E^M\big|_{Z_b}) \cong \bigoplus_{l \in L} C_{l,b}$. The Clifford bundles $\{C_l\}_{l \in L}$ exist globally on B and $C = \bigoplus_{l \in L} C_l$. The Dirac-type operator D^M decomposes as $D^M = \bigoplus_{l \in L} D_l^M$, where D_l^M acts on $C^\infty(B; C_l)$.

In order to write D_l^M explicitly, let us recall the Bismut superconnection on W_l. We will deal with each $l \in L$ separately and so we drop the subscript l for the moment. We use the notation of [9, Section III(c)] to describe the local geometry of the fiber bundle $M \to B$, and the Einstein summation convention. Let ∇^{TZ} denote the Bismut connection on TZ [7, Proposition 10.2], which we extend to a connection on E_l^Z. The Bismut superconnection on W [7, Proposition 10.15] is of the form

$$A = D^W + \nabla^W - \tfrac{1}{4} c(T). \tag{3.1}$$

Here D^W is the fiberwise Dirac-type operator and has the form

$$D^W = -i\gamma^j \nabla_{e_j}^{TZ} = -i\gamma^j \big(e_j + \tfrac{1}{2}\omega_{pqj}\sigma^{pq}\big). \tag{3.2}$$

Next, ∇^W is a Hermitian connection on W given by

$$\nabla^W = \tau^\alpha \Big(\nabla_{e_\alpha}^{TZ} - \tfrac{1}{2}\omega_{\alpha jj}\Big) = \tau^\alpha \big(e_\alpha + \tfrac{1}{2}\omega_{jk\alpha}\sigma^{jk} - \tfrac{1}{2}\omega_{\alpha jj}\big). \tag{3.3}$$

Finally,

$$c(T) = i\omega_{\alpha\beta j}\gamma^j \tau^\alpha \tau^\beta. \tag{3.4}$$

The superconnection A can be 'quantized' into an operator D^A on $C^\infty(B; V^B \otimes W)$.

Explicitly,

$$\begin{aligned} D^A = & -i\gamma^j\left(e_j + \tfrac{1}{2}\omega_{pqj}\sigma^{pq}\right) - \\ & - i\gamma^\alpha\left(e_\alpha + \tfrac{1}{2}\omega_{\beta\gamma\alpha}\sigma^{\beta\gamma} + \tfrac{1}{2}\omega_{jk\alpha}\sigma^{jk} - \tfrac{1}{2}\omega_{\alpha jj}\right) + \\ & + i\tfrac{1}{2}\omega_{\alpha\beta j}\gamma^j\sigma^{\alpha\beta.} \end{aligned} \tag{3.5}$$

Let $\mathcal{V} \in \mathrm{End}(C_l)$ be the self-adjoint operator given by

$$\mathcal{V} = -i\left(\omega_{\alpha jk}\gamma^k\sigma^{\alpha j} + \tfrac{1}{2}\omega_{\alpha jj}\gamma^\alpha + \omega_{\alpha\beta j}(\gamma^j\sigma^{\alpha\beta} + \gamma^\alpha\sigma^{j\beta})\right). \tag{3.6}$$

Then restoring the index l everywhere,

$$D_l^M = D^{A_l} + \mathcal{V}_l. \tag{3.7}$$

EXAMPLES. (1) If $G^M = \mathrm{Spin}(n)$ and V^M is the spinor representation then $\mathcal{V} = 0$.
(2) If $G^M = \mathrm{SO}(n)$ and $V^M = \Lambda^*(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$, then

$$\mathcal{V} = -\tfrac{1}{4}i\left(\omega_{\alpha jk}\gamma^k[\widehat{\gamma}^\alpha, \widehat{\gamma}^j] + \omega_{\alpha\beta j}(\gamma^j[\widehat{\gamma}^\alpha, \widehat{\gamma}^\beta] + \gamma^\alpha[\widehat{\gamma}^j, \widehat{\gamma}^\beta])\right). \tag{3.8}$$

Now suppose that $\pi\colon M \to B$ is a Riemannian affine fiber bundle. Then $E^M\big|_{Z_b}$ inherits a flat connection from the flat affine connections on $\{E^Z_{b,l}\}_{l\in L}$. Let E^B be the Clifford bundle on B whose fiber over $b \in B$ is the space of parallel sections of $E^M\big|_{Z_b}$. Then D^M restricts to a first-order differential operator D^B on $C^\infty(B; E^B)$.

Given $b \in U$ and $l \in L$, let $W^{\mathrm{inv}}_{l,b}$ be the finite-dimensional subspace of $W_{l,b}$ consisting of affine-parallel elements of $C^\infty(Z_b; E^Z_{b,l})$. From the discussion in Section 2, $W^{\mathrm{inv}}_{l,b}$ is isomorphic to $\left(V^Z_l\right)^\Gamma$. The vector spaces $W^{\mathrm{inv}}_{l,b}$ fit together to form a finite-dimensional subbundle W^{inv}_l of W_l. There is a corresponding finite-dimensional Clifford subbundle C^{inv}_l of C_l whose fiber over $b \in U$ is isomorphic to $V^B_l \otimes W^{\mathrm{inv}}_{l,b}$. Again, C^{inv}_l exists globally on B. Then $E^B = \bigoplus_{l\in L} C^{\mathrm{inv}}_l$. Let D^B_l be the restriction of D^M_l to $C^\infty(B; C^{\mathrm{inv}}_l)$. Then

$$D^B = \bigoplus_{l\in L} D^B_l. \tag{3.9}$$

The superconnection A_l restricts to an superconnection A^{inv}_l on W^{inv}_l, the endomorphism $\mathcal{V}_l$ restricts to an endomorphism of C^{inv}_l and D^M_l restricts to the first-order differential operator

$$D^B_l = D^{A^{\mathrm{inv}}_l} + \mathcal{V}^{\mathrm{inv}}_l \tag{3.10}$$

on $C^\infty(B; C^{\mathrm{inv}}_l)$.

Proof of Theorem 1. The operator D^M_l is diagonal with respect to the orthogonal decomposition

$$C_l = C^{\mathrm{inv}}_l \oplus \left(C^{\mathrm{inv}}_l\right)^\perp. \tag{3.11}$$

Thus it suffices to show that there are constants A, A' and C such that the spectrum of $\sigma(D_l^M)$, when restricted to $\left(C_l^{\mathrm{inv}}\right)^{\perp}$, is disjoint from (1.1).

For simplicity, we drop the subscript l. Given $\eta \in C^\infty(B; (C^{\mathrm{inv}})^{\perp}) \subset C^\infty(M; E^M)$, it is enough to show that for suitable constants,

$$\langle D^M\eta, D^M\eta\rangle \geqslant \left(\text{const. diam }(Z)^{-2} - \text{const.}\left(\| R^M \|_\infty + \| \Pi \|_\infty^2 + \| T \|_\infty^2\right)\right) \langle \eta, \eta\rangle. \tag{3.12}$$

Using (2.4), it is enough to show that

$$\langle \nabla^M\eta, \nabla^M\eta\rangle \geqslant \left(\text{const. diam }(Z)^{-2} - \text{const.}\left(\| R^M \|_\infty + \| \Pi \|_\infty^2 + \| T \|_\infty^2\right)\right) \langle \eta, \eta\rangle. \tag{3.13}$$

We can write $\nabla^M = \nabla^V + \nabla^H$, where

$$\nabla^V\colon C^\infty(M; E^M) \to C^\infty(M; T^*Z \otimes E^M) \tag{3.14}$$

denotes covariant differentiation in the vertical direction and

$$\nabla^H\colon C^\infty(M; E^M) \to C^\infty(M; \pi^*T^*B \otimes E^M) \tag{3.15}$$

denotes covariant differentiation in the horizontal direction. Then

$$\begin{aligned}\langle \nabla^M\eta, \nabla^M\eta\rangle &= \langle \nabla^V\eta, \nabla^V\eta\rangle + \langle \nabla^H\eta, \nabla^H\eta\rangle \\ &\geqslant \langle \nabla^V\eta, \nabla^V\eta\rangle \\ &= \int_B \int_{Z_b} \left|\nabla^V\eta\right|^2(z)d\mathrm{vol}_{Z_b}d\mathrm{vol}_B.\end{aligned} \tag{3.16}$$

On a given fiber Z_b, we have

$$E^M\big|_{Z_b} \cong V^B \otimes E_b^Z. \tag{3.17}$$

Hence we can also use the Bismut connection ∇^{TZ} to vertically differentiate sections of E^M. That is, we can define

$$\nabla^{TZ}\colon C^\infty(M; E^M) \to C^\infty(M; T^*Z \otimes E^M). \tag{3.18}$$

Explicitly, with respect to a local framing,

$$\nabla_{e_j}^{TZ} = e_j\eta + \tfrac{1}{2}\omega_{pqj}\sigma^{pq}\eta \tag{3.19}$$

and

$$\nabla_{e_j}^{V} = e_j\eta + \tfrac{1}{2}\omega_{pqj}\sigma^{pq}\eta + \omega_{\alpha kj}\sigma^{\alpha k}\eta + \tfrac{1}{2}\omega_{\alpha\beta j}\sigma^{\alpha\beta}\eta. \tag{3.20}$$

Then from (3.16), (3.19) and (3.20),

$$\langle \nabla^M \eta, \nabla^M \eta \rangle \geqslant \int_B \left[\int_{Z_b} \left| \nabla^{TZ} \eta \right|^2 (z) - \text{const.}\left(\| T_b \|^2 + \| \Pi_b \|^2 \right) \left| \eta(z) \right|^2 \right] d\text{vol}_{Z_b} d\text{vol}_B. \tag{3.21}$$

Thus it suffices to bound $\int_{Z_b} \left| \nabla^{TZ} \eta \right|^2 (z) d\text{vol}_{Z_b}$ from below on a given fiber Z_b in terms of $\langle \eta, \eta \rangle_{Z_b}$, under the assumption that $\eta \in (W_b^{\text{inv}})^{\perp}$. Using the Gauss–Codazzi equation, we can estimate $\| R^{Z_b} \|_\infty$ in terms of $\| R^M \|_\infty$ and $\| \Pi \|_\infty^2$. Then the desired bound on $\int_{Z_b} \left| \nabla^{TZ} \eta \right|^2 (z) d\text{vol}_{Z_b}$ follows from Proposition 2. □

Proof of Theorem 2. Let g_0^{TM} denote the Riemannian metric on M. From Proposition 1, if a Riemannian metric g_1^{TM} on M is close to g_0^{TM} in $(\mathcal{M}(M, 2K), d)$ then applying the function $x \to \sinh^{-1}(x/\sqrt{2K})$ to $\sigma(D^M, g_0^{TM})$ gives a collection of numbers which is close to that obtained by applying $x \to \sinh^{-1}(x/\sqrt{2K})$ to $\sigma(D^M, g_1^{TM})$. We will use the geometric results of [11] to find a metric g_2^{TM} on M which is close to g_0^{TM} and to which we can apply Theorem 1.

First, as in [11, (2.4.1)], by the smoothing results of Abresch and others [11 , Theorem 1.12], for any $\varepsilon > 0$ we can find metrics on M and B which are ε-close in the C^1-topology to the original metrics such that the new metrics satisfy $\| \nabla^i R \|_\infty \leqslant A_i(n, \varepsilon)$ for some appropriate sequence $\{A_i(n, \varepsilon)\}_{i=0}^\infty$. Let g_1^{TM} denote the new metric on M. In the proof of the smoothing result, such as using the Ricci flow [21, Proposition 2.5], one obtains an explicit smooth 1-parameter family of metrics on M in $\mathcal{M}(M, K')$, for some $K' > K$, going from g_0^{TM} to g_1^{TM}. We can approximate this family by a piecewise-analytic family. Hence one obtains an upper bound on $d\left(g_0^{TM}, g_1^{TM}\right)$ in $\mathcal{M}(M, K')$, for some $K' > K$, which depends on K and is proportionate to ε. (Note that d is essentially the same as the C^0-metric on $\mathcal{M}(M, K')$.) By rescaling, we may assume that $\| R^M \|_\infty \leqslant 1$, $\| R^B \|_\infty \leqslant 1$ and $inj(B) \geqslant 1$. We now apply [11, Theorem 2.6], with B fixed. It implies that there are positive constants $\lambda(n)$ and $c(n, \varepsilon)$ so that if $d_{GH}(M, B) \leqslant \lambda(n)$ then there is a fibration $f : M \to B$ such that

(1) diam $\left(f^{-1}(b) \right) \leqslant c(n, \varepsilon) d_{GH}(M, B)$.
(2) f is a $c(n, \varepsilon)$-almost Riemannian submersion.
(3) $\| \Pi_{f^{-1}(b)} \|_\infty \leqslant c(n, \varepsilon)$.

As in [16], the Gauss–Codazzi equation, the curvature bound on M and the second fundamental form bound on $f^{-1}(b)$ imply a uniform bound on $\{\| R^{f^{-1}(b)} \|_\infty\}_{b \in B}$. Along with the diameter bound on $f^{-1}(b)$, this implies that if $d_{GH}(M, B)$ is sufficiently small then $f^{-1}(b)$ is almost flat.

From [11, Propositions 3.6 and 4.9], we can find another metric g_2^{TM} on M which is ε-close to g_1^{TM} in the C^1-topology so that the fibration $f : M \to B$ gives M the structure of a Riemannian affine fiber bundle. Furthermore, by [11, Proposition 4.9], there is a sequence $\{A_i'(n, \varepsilon)\}_{i=0}^\infty$ so that we may assume that g_1^{TM} and g_2^{TM} are close in the sense that

$$\| \nabla^i\big(g_1^{TM} - g_2^{TM}\big) \|_\infty \leqslant A_i'(n,\varepsilon) d_{GH}(M,B), \tag{3.22}$$

where the covariant derivative in (41) is that of the Levi-Civita connection of g_2^{TM}. Then we can interpolate linearly between g_1^{TM} and g_2^{TM} within $\mathcal{M}(M, K'')$ for some $K'' > K'$, and obtain an upper bound on $d\big(g_1^{TM}, g_2^{TM}\big)$ in $\mathcal{M}(M, K'')$ which is proportionate to ε. From [21, Theorem 2.1], we can take $K'' = 2K$ (or any number greater than K).

We now apply Theorem 1 to the Riemannian affine fiber bundle with metric g_2^{TM}. It remains to estimate the geometric terms appearing in (1.1). We have an estimate on $\| \Pi \|_\infty$ as above. Applying O'Neill's formula [8, (9.29)] to the Riemannian affine fiber bundle, we can estimate $\| T \|_\infty^2$ in terms of $\| R^M \|_\infty$ and $\| R^B \|_\infty$. Putting this together, the theorem follows. □

4. Collapsing to a Singular Base

Let $\mathfrak{p}\colon P \to M$ be the principal G-bundle of Section 2. Let $\{\mathfrak{Y}_j\}_{j=1}^n$ be the horizontal vector fields on P such that $\theta(\mathfrak{Y}_j) = e_j$. Put $D^P = -i\sum_{j=1}^n \gamma^j \mathfrak{Y}_j$, acting on $C^\infty(P) \otimes V$. There is an isomorphism $C^\infty(M; E^M) \cong (C^\infty(P) \otimes V)^G$. Under this isomorphism, $D^M \cong D^P\big|_{(C^\infty(P)\otimes V)^G}$. The Bochner-type equation (2.4) becomes

$$(D^M)^2 \cong -\sum_{j=1}^n \mathfrak{Y}_j^2 + \sum_{i,j=1}^n \omega^i_{jj}\mathfrak{Y}_i - \frac{1}{8}\sum_{a,b,i,j=1}^n (\mathfrak{p}^* R^M)_{abij}(\gamma^i\gamma^j - \gamma^j\gamma^i)\sigma^{ab} \tag{4.1}$$

when acting on $(C^\infty(P) \otimes V)^G$.

Let $\{x_a\}_{a=1}^{\dim(G)}$ be a basis for the Lie algebra $\mathfrak{g}$ which is orthonormal with respect to the negative of the Killing form. Let $\{\mathfrak{Y}_a\}_{a=1}^{\dim(G)}$ be the corresponding vector fields on P. Then $-\sum_{a=1}^{\dim(G)} \mathfrak{Y}_a^2$ acts on $(C^\infty(P) \otimes V)^G$ as $c_V \in (\mathrm{End}(V))^G$, the Casimir of the G-module V. Give P the Riemannian metric g^{TP} with the property that $\{\mathfrak{Y}_j, \mathfrak{Y}_a\}$ forms an orthonormal basis of vector fields. Let $\triangle^P$ denote the corresponding (nonnegative) scalar Laplacian on P, extended to act on $C^\infty(P) \otimes V$. Then when acting on $(C^\infty(P) \otimes V)^G$, equation (4.1) is equivalent to

$$(D^M)^2 \cong \triangle^P - \frac{1}{8}\sum_{a,b,i,j=1}^n (\mathfrak{p}^* R^M)_{abij}(\gamma^i\gamma^j - \gamma^j\gamma^i)\sigma^{ab} - c_V. \tag{4.2}$$

DEFINITION 3. A G-equivariant Riemannian affine fiber bundle structure on P consists of a Riemannian affine fiber bundle structure $\check{\pi}: P \to \check{X}$ which is G-equivariant.

In [11, Proposition 7.21] it is shown that one can make a small G-equivariant perturbation of g^{TP} in the $C^{1,\alpha}$-topology so that the new Riemannian metric is the total space of a G-equivariant Riemannian affine fiber bundle. The quotient space $M = P/G$ acquires a new quotient Riemannian metric, which is called an invariant metric [11, Section 8]. In [21, Theorem 2.1] it is shown that one can assume that the sectional curvatures of the invariant metric on M are bounded in terms of the sectional curvatures of the original metric on M. As we can take the new canonical Riemannian metric g^{TP} on P, the upshot is that we assume that the Riemannian metric on the total space of the G-equivariant affine fiber bundle $P \to \check{X}$ is the canonical metric coming from a Riemannian metric on M.

Given such a G-equivariant Riemannian affine fiber bundle, let $\check{Z}$ be the fiber of $\check{\pi}$: $P \to \check{X}$, an infranilmanifold. For collapsing purposes it suffices to take $\check{Z}$ to be a nilmanifold $\Gamma \backslash N$ [11, (7.2)]. We assume hereafter that this is the case. Put $X = \check{X}/G$, a possibly singular space. As the Lie algebra $\mathfrak{n}$ of N is represented by vector fields in a neighborhood of a point of P, and the local flow preserves the horizontal subspaces of $P \to M$, it follows that the vector fields $\{\mathfrak{Y}_j\}_{j=1}^n$ are projectable with respect to $\check{\pi}$ and push forward to vector fields $\{\mathcal{X}_j\}_{j=1}^n$ on $\check{X}$. Put $D^{\check{X}} = -i \sum_{j=1}^n \gamma^j \mathcal{X}_j$, acting on $C^\infty(\check{X}) \otimes V$. Let $v \in C^\infty(\check{X})$ be given by $v(\check{x}) = vol(\check{Z}_{\check{x}})$. We give $C^\infty(\check{X}) \otimes V$ the weighted L^2-inner product with respect to the weight function v.

We recall that there is a notion of a pseudodifferential operator being transversally elliptic with respect to the action of a Lie group G [2, Definition 1.3].

LEMMA 1. *$D^{\check{X}}$ is transversally elliptic on $\check{X}$.*

Proof. Let $s(D^{\check{X}}) \in C^\infty(T^*\check{X}) \otimes \mathrm{End}(V)$ denote the symbol of $D^{\check{X}}$. Suppose that $\xi \in T^*_{\check{x}}\check{X}$ satisfies $\xi(\check{v}) = 0$ for all $\check{v} \in T_{\check{x}}\check{X}$ which lie in the image of the representation of $\mathfrak{g}$ by vector fields on $\check{X}$. Then if $p \in \check{\pi}^{-1}(\check{x})$, we have that $(\check{\pi}^*\xi)(r) = 0$ for all $r \in T_pP$ which lie in the image of the representation of $\mathfrak{g}$ by vector fields on P. In other words, $\check{\pi}^*\xi$ is horizontal. Now $((s(D^{\check{X}}))(\xi))^2 = \sum_{j=1}^n \langle \xi, \mathcal{X}_j \rangle^2 = \sum_{j=1}^n \langle \check{\pi}^*\xi, \mathfrak{Y}_j \rangle^2$. If $(s(D^{\check{X}}))(\xi)$ fails to be an isomorphism then $\langle \check{\pi}^*\xi, \mathfrak{Y}_j \rangle = 0$ for all j. Along with the fact that $\check{\pi}^*\xi$ is horizontal, this implies that $\check{\pi}^*\xi = 0$. Thus $\xi = 0$, which proves the lemma.

DEFINITION 4. For notation, write $C^\infty(X; E^X) = (C^\infty(\check{X}) \otimes V)^G$. Let D^X be the restriction of $D^{\check{X}}$ to $C^\infty(X; E^X)$.

It will follow from the proof of the next theorem that D^X is self-adjoint on the Hilbert space completion of $C^\infty(X; E^X)$ with respect to the (weighted) inner product. As $D^{\check{X}}$ is transversally elliptic, it follows that D^X has a discrete spectrum [2, Proof of Theorem 2.2].

Let $\check{\Pi}$ denote the second fundamental forms of the fibers $\{\check{Z}_{\check{x}}\}_{\check{x} \in \check{X}}$. Let $\check{T} \in \Omega^2(P; T\check{Z})$ be the curvature of the horizontal distribution on the affine fiber bundle $P \to \check{X}$.

THEOREM 6. *There are positive constants A, A' and C which only depend on n and V such that if $\| R^{\check{Z}} \|_\infty$ diam $(\check{Z})^2 \leqslant A'$ then the intersection of $\sigma(D^M)$ with*

$$[-(A\mathrm{diam}(\check{Z})^{-2} - C(1 + \| R^M \|_\infty + \| \check{\Pi} \|_\infty^2 + \| \check{T} \|_\infty^2))^{1/2},$$
$$(A\mathrm{diam}(\check{Z})^{-2} - C(1 + \| R^M \|_\infty + \| \check{\Pi} \|_\infty^2 + \| \check{T} \|_\infty^2))^{1/2}] \quad (4.3)$$

equals the intersection of $\sigma(D^X)$ *with* (4.3).

Proof. Let us write

$$C^\infty(P) \otimes V = \left(C^\infty(\check{X}) \otimes V\right) \oplus \left(C^\infty(\check{X}) \otimes V\right)^\perp, \tag{4.4}$$

where we think of $C^\infty(\check{X}) \otimes V$ as the elements of $C^\infty(P) \otimes V$ which are constant along the fibers of the fiber bundle $\check{\pi}$: $P \to \check{X}$. Taking G-invariant subspaces, we have an orthogonal decomposition

$$C^\infty(M; E^M) = C^\infty(X; E^X) \oplus \left(C^\infty(X; E^X)\right)^\perp, \tag{4.5}$$

with respect to which D^M decomposes as

$$D^M = D^X \oplus D^M\big|_{(C^\infty(X;E^X))^\perp}. \tag{4.6}$$

As in the proof of Theorem 1, it suffices to obtain a lower bound on the spectrum of $(D^M)^2\big|_{(C^\infty(X;E^X))^\perp}$. As $(C^\infty(X; E^X))^\perp \subset (C^\infty(\check{X}) \otimes V)^\perp$, using (4.2) it suffices to obtain a lower bound on the spectrum of $\triangle^P\big|_{(C^\infty(\check{X})\otimes V)^\perp}$. This follows from the arguments of the proof of Theorem 1, using the fact that $\| R^P \|_\infty \leqslant \text{const.}(1+ \| R^M \|_\infty)$. We omit the details. In fact, it is somewhat easier than the proof of Theorem 1, since we are now only dealing with the scalar Laplacian and so can replace Proposition 2 by standard eigenvalue estimates (which just involve a lower Ricci curvature bound); see [6] and references therein.

Proof of Theorem 3. Everything in the proof will be done in a G-equivariant way, so we may omit to mention this explicitly. Let P_i be the principal G-bundle of M_i, equipped with a Riemannian metric as in the beginning of the section. From the G-equivariant version of Gromov's compactness theorem, we obtain a subsequence $\{P_i\}_{i=1}^\infty$ which converges in the equivariant Gromov–Hausdorff topology to a G-Riemannian manifold $(\check{X}, g^{T\check{X}})$ with a $C^{1,\alpha}$-regular metric. As in [14, Section 3], the measure $\chi d\mathrm{vol}_{\check{X}}$ is a weak-$*$ limit point of the pushforwards of the normalized Riemannian measures on $\{P_i\}_{i=1}^\infty$. As in [14, p. 535], after smoothing we may assume that we have G-equivariant Riemannian affine fiber bundles $\check{\pi}_i$: $P_i' \to \check{X}_i$, with G acting freely on P_i', along with G-diffeomorphisms $\check{\phi}_i$: $P_i \to P_i'$ and Φ_i: $\check{X} \to \check{X}_i$. Put $M_i' = P_i'/G$. Then $\check{\phi}_i$ descends to a diffeomorphism ϕ_i: $M_i \to M_i'$ and we may also assume, as in the proof of Theorem 2, that

(1) $\phi_i^* g^{TM_i'} \in \mathcal{M}(M_i, \text{const.}K)$,
(2) $d(\phi_i^* g^{TM_i'}, g^{TM_i}) \leqslant 2^{-i}$ in $\mathcal{M}(M_i, \text{const.}K)$ and
(3) $\lim_{i\to\infty} \Phi_i^* g^{T\check{X}_i} = g^{T\check{X}}$ in the $C^{1,\alpha}$-topology.

Using Proposition 1, we can effectively replace M_i by M_i' for the purposes of the argument. For simplicity, we relabel M_i' as M_i and P_i' as P_i. For the purposes of the limiting argument, using Theorem 6 and (4.2), we may replace the spectrum of $|D^{M_i}|$ by the spectrum of the operator $|D^{X_i}| \equiv \sqrt{\triangle^{\check{X}_i} + \mathcal{V}_i}$ acting on $C^\infty(X_i, E^{X_i}) = (C^\infty(\check{X}_i) \otimes V)^G$, where $\mathcal{V}_i$ is the restriction of

$$-\frac{1}{8} \sum_{a,b,i,j=1}^{n} (\mathfrak{p}^* R^{M_i})_{abij} (\gamma^i \gamma^j - \gamma^j \gamma^i) \sigma^{ab} - c_V \tag{4.7}$$

to the elements of $(C^\infty(P_i) \otimes V)^G$ which are constant along the fibers of $\check{\pi}_i \colon P_i \to \check{X}_i$, i.e. to $C^\infty(X_i, E^{X_i})$.

From the curvature bound, we have a uniform bound on $\{\| \mathcal{V}_i \|_\infty\}_{i=1}^\infty$. Using the weak-$*$ compactness of the unit ball, let $\mathcal{V}$ be a weak-$*$ limit point of $\{\Phi_i^* \mathcal{V}_i\}_{i=1}^\infty$ in $L^\infty(\check{X}) \otimes \mathrm{End}(V) = (L^1(\check{X}) \otimes \mathrm{End}(V))^*$. We claim that with this choice of $\check{X}$, χ and $\mathcal{V}$, equation (1.2) holds.

To see this, we use the minimax characterization of eigenvalues as in [14, Section 5]. Using the diffeomorphisms $\{\Phi_i\}_{i=1}^\infty$, we identify each $\check{X}_i$ with $\check{X}$. We denote by $\langle \cdot, \cdot \rangle_{X_i}$ an L^2-inner product constructed using $\Phi_i^* g^{T\check{X}_i}$ and the weight function $(\check{\pi}_i)_*(d\mathrm{vol}_{P_i}) / \int_{\check{X}_i} (\check{\pi}_i)_*(d\mathrm{vol}_{P_i})$. We denote by $\langle \cdot, \cdot \rangle_X$ an L^2-inner product constructed using $g^{T\check{X}}$ and the weight function $\chi d\mathrm{vol}_{\check{X}}$. As $\triangle^{\check{X}}$ has a compact resolvent, it follows that $|D^X|^2$ has a compact resolvent. Then

$$\lambda_k(|D^X|)^2 = \inf_W \sup_{\psi \in W - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X}, \tag{4.8}$$

where W ranges over the k-dimensional subspaces of the Sobolev space $H^1(X; E^X)$. Given $\varepsilon > 0$, let W_∞ be a k-dimensional subspace such that

$$\sup_{\psi \in W_\infty - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X} \leqslant \lambda_k(|D^X|)^2 + \varepsilon. \tag{4.9}$$

As $\psi \otimes \psi^*$ lies in the finite-dimensional subspace $W_\infty \otimes W_\infty^*$ of $L^1(\check{X}) \otimes \mathrm{End}(V)$, it follows that

$$\lim_{i \to \infty} \langle \psi, \mathcal{V}_i \psi \rangle_X = \langle \psi, \mathcal{V}\psi \rangle_X \tag{4.10}$$

uniformly on $\{\psi \in W_\infty \colon \langle \psi, \psi \rangle_X = 1\}$. Then

$$\lim_{i \to \infty} \sup_{\psi \in W_\infty - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_{X_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{X_i}}{\langle \psi, \psi \rangle_{X_i}} = \sup_{\psi \in W_\infty - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X}. \tag{4.11}$$

As

$$\lambda_k(|D^{X_i}|)^2 = \inf_W \sup_{\psi \in W - 0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_{X_i} + \langle \psi, \mathcal{V}_i \psi \rangle_{X_i}}{\langle \psi, \psi \rangle_{X_i}}, \tag{4.12}$$

it follows that

$$\limsup_{i \to \infty} \lambda_k(|D^{X_i}|) \leqslant \lambda_k(|D^X|). \tag{4.13}$$

We now show that

$$\liminf_{i\to\infty} \lambda_k(|D^{X_i}|) \geqslant \lambda_k(|D^X|). \tag{4.14}$$

Along with (4.13), this will prove the theorem. Suppose that (4.14) is not true. Then there is some $\varepsilon > 0$ and some infinite subsequence of $\{M_i\}_{i=1}^{\infty}$, which we relabel as $\{M_i\}_{i=1}^{\infty}$, such that for all $i \in \mathbb{Z}^+$,

$$\lambda_k(|D^{X_i}|)^2 \leqslant \lambda_k(|D^X|)^2 - 2\varepsilon. \tag{4.15}$$

For each $i \in \mathbb{Z}^+$, let W_i be a k-dimensional subspace of $H^1(X; E^X)$ such that

$$\sup_{\psi\in W_i-0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi\rangle_{X_i} + \langle \psi, \mathcal{V}_i\psi\rangle_{X_i}}{\langle \psi, \psi\rangle_{X_i}} \leqslant \lambda_k(|D^{X_i}|)^2 + \varepsilon. \tag{4.16}$$

Let $\{f_{i,j}\}_{j=1}^{k}$ be a basis for W_i which is orthonormal with respect to $\langle\cdot,\cdot\rangle_X$. Then for a given j, the sequence $\{f_{i,j}\}_{i=1}^{\infty}$ is bounded in $H^1(X; E^X)$. After taking a subsequence, which we relabel as $\{f_{i,j}\}_{i=1}^{\infty}$, we can assume that $\{f_{i,j}\}_{i=1}^{\infty}$ converges weakly in $H^1(X; E^X)$ to some $f_{\infty,j}$. Doing this successively for $j \in \{1, \ldots, k\}$, we can assume that for each j, $\lim_{i\to\infty} f_{i,j} = f_{\infty,j}$ weakly in $H^1(X; E^X)$. Then from the compactness of the embedding $H^1(X; E^X) \to L^2(X; E^X)$, we have strong convergence in $L^2(X; E^X)$. In particular, $\{f_{\infty,j}\}_{j=1}^{k}$ are orthonormal. Put $W_\infty = \mathrm{span}(f_{\infty,1}, \ldots, f_{\infty,k})$.

If $w_\infty = \sum_{j=1}^{k} c_j f_{\infty,j}$ is a nonzero element of W_∞, put $w_i = \sum_{j=1}^{k} c_j f_{i,j}$. Then $\{w_i\}_{i=1}^{\infty}$ converges weakly to w_∞ in $H^1(X; E^X)$ and hence converges strongly to w_∞ in $L^2(X; E^X)$. From a general result about weak limits, we have

$$\langle w_\infty, w_\infty\rangle_{H^1} \leqslant \limsup_{i\to\infty} \langle w_i, w_i\rangle_{H^1}. \tag{4.17}$$

Along with the L^2-convergence of $\{w_i\}_{i=1}^{\infty}$ to w_∞, this implies that

$$\langle dw_\infty, dw_\infty\rangle_X \leqslant \limsup_{i\to\infty} \langle dw_i, dw_i\rangle_{X_i}. \tag{4.18}$$

As $w_i \otimes w_i^*$ converges in $L^1(\check{X}) \otimes \mathrm{End}(E)$ to $w_\infty \otimes w_\infty^*$, we have

$$\begin{aligned}\lim_{i\to\infty} \langle w_i, \mathcal{V}_i w_i\rangle_X &= \lim_{i\to\infty} (\langle w_\infty, \mathcal{V}_i w_\infty\rangle_X + (\langle w_i, \mathcal{V}_i w_i\rangle_X - \langle w_\infty, \mathcal{V}_i w_\infty\rangle_X)) \\ &= \langle w_\infty, \mathcal{V} w_\infty\rangle_X.\end{aligned} \tag{4.19}$$

Then

$$\sup_{\psi\in W_\infty-0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi\rangle_X + \langle \psi, \mathcal{V}\psi\rangle_X}{\langle \psi, \psi\rangle_X} \leqslant \limsup_{i\to\infty} \sup_{\psi\in W_i-0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi\rangle_{X_i} + \langle \psi, \mathcal{V}_i\psi\rangle_{X_i}}{\langle \psi, \psi\rangle_{X_i}}. \tag{4.20}$$

Thus from (4.15), (4.16) and (4.20),

$$\inf_{W} \sup_{\psi \in W-0} \frac{\langle \mathrm{d}\psi, \mathrm{d}\psi \rangle_X + \langle \psi, \mathcal{V}\psi \rangle_X}{\langle \psi, \psi \rangle_X} \leqslant \lambda_k(|D^X|)^2 - \varepsilon, \tag{4.21}$$

which is a contradiction. This proves the theorem. □

Proof of Theorem 4. Let $\{g_i^{TM}\}_{i=1}^{\infty}$ be a sequence of Riemannian metrics on M as in the statement of the theorem, with respect to which $\lambda_k(|D^M|)$ goes to infinity. Let P be the principal G-bundle of M and let $\check{X}$ be the limit space of Theorem 3, a smooth manifold with a $C^{1,\alpha}$-regular metric. As the limit space $X = \check{X}/G$ has diameter 1, it has positive dimension. As in the proof of Theorem 3, after slightly smoothing the metric on $\check{X}$, there is a G-equivariant Riemannian affine fiber bundle $\check{\pi}$: $P \to \check{X}$ whose fiber is a nilmanifold $\check{Z}$. Let $\check{x}$ be a point in a principal orbit for the G-action on $\check{X}$, with isotropy group $H \subset G$. Then H acts affinely on the nilmanifold fiber $\check{Z}_{\check{x}}$. In particular, H is virtually abelian. The quotient $Z = \check{Z}_{\check{x}}/H$ is the generic fiber of the possibly-singular affine fiber bundle π: $M \to X$, the G-quotient of $\check{\pi}$: $P \to \check{X}$. Then $E^M\big|_Z = \check{Z}_{\check{x}} \times_H V$. In particular, the vector space of affine-parallel sections of $E^M\big|_Z$ is isomorphic to V^H. On the other hand, if $C^{\infty}(X; E^X) \neq 0$ then $|D^X|$ has an infinite discrete spectrum. Theorem 3 now implies that $C^{\infty}(X; E^X) \cong (C^{\infty}(\check{X}) \otimes V)^G$ must be the zero space. As the orbit $\check{x} \cdot G$ has a neighborhood consisting of principal orbits, the restriction map from $(C^{\infty}(\check{X}) \otimes V)^G$ to $(C^{\infty}(\check{x} \cdot G) \otimes V)^G$ is surjective. However, $(C^{\infty}(\check{x} \cdot G) \otimes V)^G$ is isomorphic to V^H. Thus $V^H = 0$.

Conversely, let π: $M \to B$ be an affine fiber bundle. Theorem 1 implies that if $E^M\big|_Z$ does not have any nonzero affine-parallel sections then upon collapsing M to B as in [16, Section 6], the eigenvalues of D_M go off to $\pm\infty$. This proves the theorem. □

5. Proof of Theorem 5

As the proof of Theorem 5 is similar to [19, Pf. of Theorem 2], we only indicate the structure of the proof and the necessary modifications to [19, Pf. of Theorem 2].

The closure $\overline{U_I}$ of an appropriate neighborhood of an end has the (affine) structure of an affine fiber bundle over $[0, \infty)$ with fiber Z_I. The vector bundle E_I^B is the trivial vector bundle over $[0, \infty)$ whose fiber over $s \in [0, \infty)$ consists of the affine-parallel sections of $E^M\big|_{\{s\} \times Z_I}$. As in [19, Section 4], if U_I is sufficiently far out the end then we can use Propositions 1 and 2 of the present paper to construct an embedding of $C^{\infty}([0, \infty); E_I^B)$ into $C^{\infty}(\overline{U_I}; E^M\big|_{\overline{U_I}})$ whose image consists of elements with 'bounded energy' fiberwise restrictions. Let P_0 be the Hilbert space extension of orthogonal projection from $\bigoplus_{I=1}^N C^{\infty}(\overline{U_I}; E^M\big|_{\overline{U_I}})$ to $\bigoplus_{I=1}^N C^{\infty}([0, \infty); E_I^B)$. By standard arguments as in [13, Pf. of Proposition 2.1], the essential spectrum of D^M equals that of D_{end}^M. With respect to the decomposition of the Hilbert space into

$\mathrm{Im}(P_0) \oplus \mathrm{Im}(I - P_0)$, we write

$$D^M_{\mathrm{end}} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{5.1}$$

The operators $\mathcal{B}$ and $\mathcal{C}$ are bounded, as can be seen by the method of proof of [19, Proposition 2], replacing the operator $\widehat{d} + \widehat{d}^*$ of [19, Pf. of Proposition 2] by D^{Z_I}. As in [19, Proposition 3], the operator $\mathcal{D}$ has vanishing essential spectrum. Put $\mathcal{L} = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{D} \end{pmatrix}$. To prove the theorem, it suffices to show that D^M_{end} and $\mathcal{L}$ have the same essential spectrum. For this, it suffices to show that $\left(D^M_{\mathrm{end}} + ki\right)^{-1} - (\mathcal{L} + ki)^{-1}$ is compact for some $k > 0$ [20, Vol. IV, Chapter XIII.4, Corollary 1].

We use the general identity that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^{-1} + \alpha^{-1}\beta\left(\delta - \gamma\alpha^{-1}\beta\right)^{-1}\gamma\alpha^{-1} & -\alpha^{-1}\beta\left(\delta - \gamma\alpha^{-1}\beta\right)^{-1} \\ -\left(\delta - \gamma\alpha^{-1}\beta\right)^{-1}\gamma\alpha^{-1} & \left(\delta - \gamma\alpha^{-1}\beta\right)^{-1} \end{pmatrix} \tag{5.2}$$

provided that α and $\delta - \gamma\alpha^{-1}\beta$ are invertible. Put

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = D^M_{\mathrm{end}} + ki = \begin{pmatrix} \mathcal{A} + ki & \mathcal{B} \\ \mathcal{C} & \mathcal{D} + ki \end{pmatrix}. \tag{5.3}$$

If k is positive then α and δ are invertible, with δ^{-1} being compact. If k is large enough then $\| \delta^{-1/2}\gamma\alpha^{-1}\beta\delta^{-1/2} \| < 1$. Writing

$$\delta - \gamma\alpha^{-1}\beta = \delta^{1/2}\left(I - \delta^{-1/2}\gamma\alpha^{-1}\beta\delta^{-1/2}\right)\delta^{1/2}, \tag{5.4}$$

we now see that $\delta - \gamma\alpha^{-1}\beta$ is invertible. It also follows from (5.4) that $\left(\delta - \gamma\alpha^{-1}\beta\right)^{-1}$ is compact. Using (5.2), the theorem follows.

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