3D Ricci flow since Perelman

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Outline of the talk

1. Homogeneous spaces and the geometrization conjecture
2. The geometrization conjecture and Ricci flow
3. Finiteness of the number of surgeries
4. Long-time behavior
5. Flowing through singularities

3D Ricci flow since Perelman

Homogeneous spaces and the geometrization conjecture

Geometrization conjecture and Ricci flow

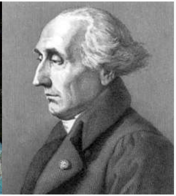
Finiteness of the number of surgeries

Long-time behavior

Flowing through singularities

A brief history of math

18^{th} century : one dimensional spaces



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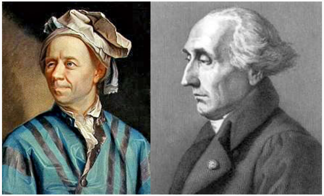

19^{th} century : two dimensional spaces

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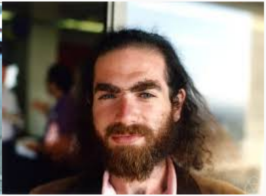
20^{th} century : three dimensional spaces

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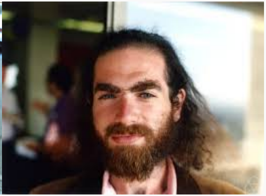

21^{st} century :

Today's talk

Remaining open questions from 20^{th} century math.

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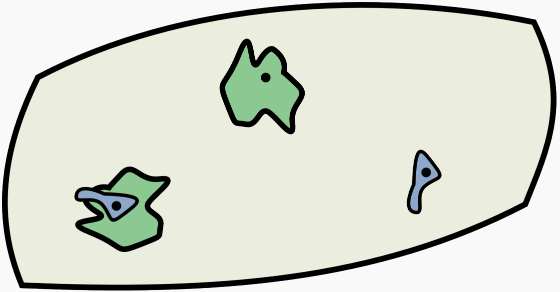
First, how do we understand three dimensional spaces?

In terms of homogeneous spaces.

Locally homogeneous metric spaces

A metric space X is *locally homogeneous* if all $x, y \in X$, there are neighbourhoods U and V of x and y and an isometric isomorphism $(U, x) \to (V, y)$.



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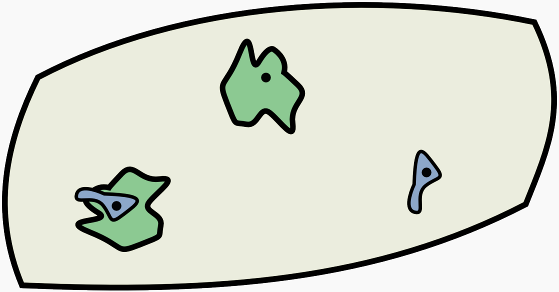
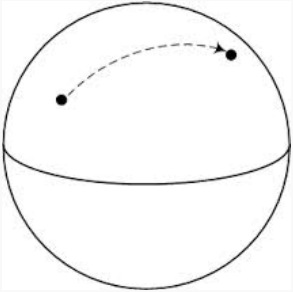


The metric space X is *globally homogeneous* if for all $x, y \in X$, there is an isometric isomorphism $\phi : X \to X$ that $\phi(x) = y$.

Locally homogeneous Riemannian manifolds

Any Riemannian manifold M gets a metric space structure.



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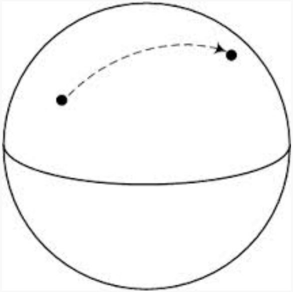

Theorem

(Singer 1960) If M is a complete, simply connected Riemannian manifold which is locally homogeneous, then M is globally homogeneous.

We will say that a smooth manifold M *admits a geometric structure* if M admits a complete, locally homogeneous Riemannian metric.

It is a theorem of Singer that such a metric on a simply connected manifold X must be homogeneous, i.e. the isometry group of X must act transitively.

From “The Geometries of 3-Manifolds” by Peter Scott

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Thus we can regard the universal cover X of M, together with its isometry group, as a geometry in the sense of Klein, and we can sensibly say that M admits a geometric structure modelled on X. Thurston has classified the 3-dimensional geometries and there are eight of them.

Two-dimensional geometries

Globally homogeneous S^2,

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Globally homogeneous S^2, locally homogeneous 

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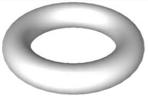
Globally homogeneous $\mathbb{R}^2$, locally homogeneous 

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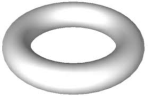
Globally homogeneous $\mathbb{R}^2$, locally homogeneous 

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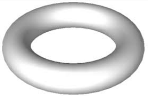
Globally homogeneous $\mathbb{R}^2$, locally homogeneous 

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Three-dimensional Thurston geometries

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Warning : unlike in two dimensions, not every compact three-dimensional manifold admits a geometric structure, i.e. admits a locally homogeneous Riemannian metric.

Geometrization conjecture

If M is a compact orientable 3-manifold then there is a canonical way to split M into pieces, using certain embedded 2-spheres and 2-tori. (The collection of 2-spheres and 2-tori could be empty.)

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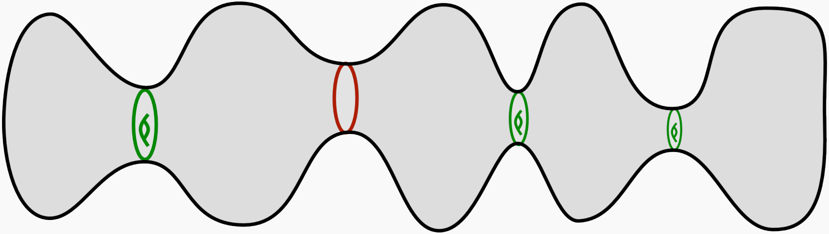

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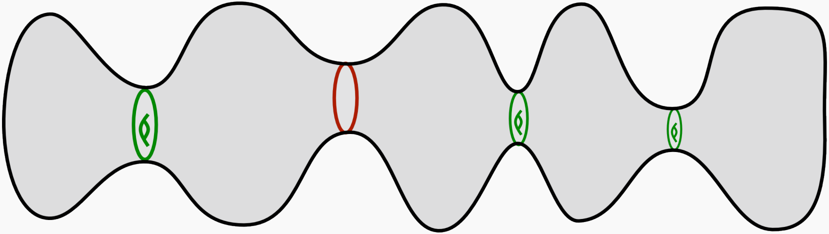

Conjecture (Thurston, 1982)

The ensuing pieces have geometric structures, i.e. admit locally homogeneous metrics

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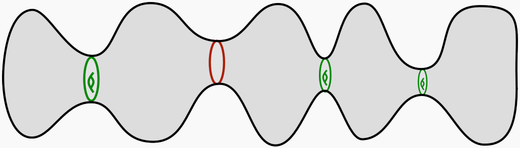

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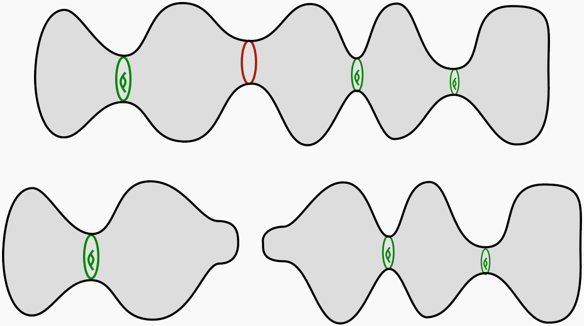

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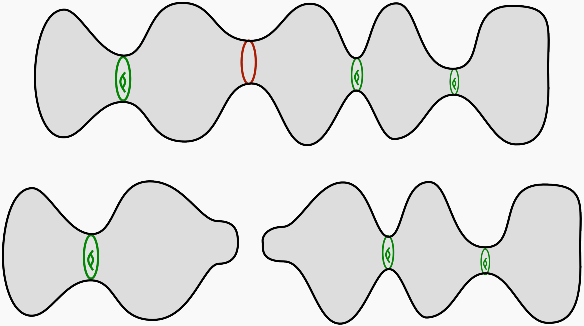

Cut along the 2-tori. The interiors of the ensuing pieces should admit complete locally homogeneous metrics.

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The geometrization conjecture now says that each M_i has a locally homogeneous metric. Since M_i is simply connected and compact, it must be diffeomorphic to S^3.

Then the original 3-manifold M is a "connected sum" of 3-spheres, and is also diffeomorphic to S^3.

Ricci flow approach to geometrization

Hamilton's Ricci flow equation

$$\frac{dg}{dt} = -2\ \mathrm{Ric}_g\,.$$

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Maybe the Ricci flow will evolve an initial Riemannian metric into something homogeneous.

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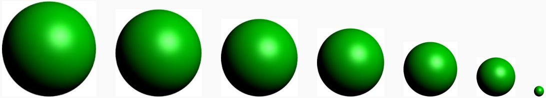

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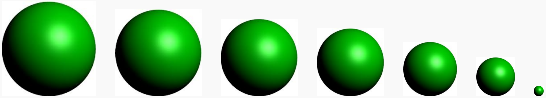

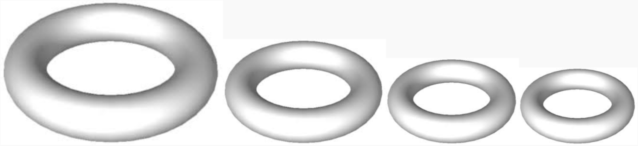

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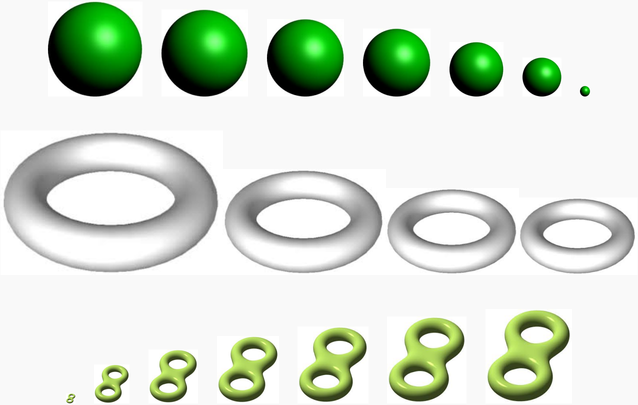
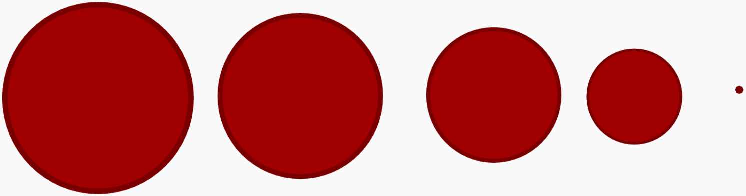


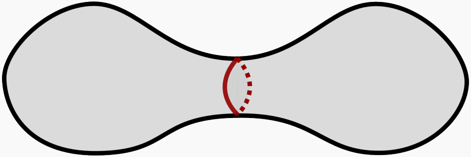
Singularities in 3D Ricci flow

Some components may disappear, e.g. a round shrinking 3-sphere.



Neckpinch

A 2-sphere pinches off. (Drawn one dimension down.)



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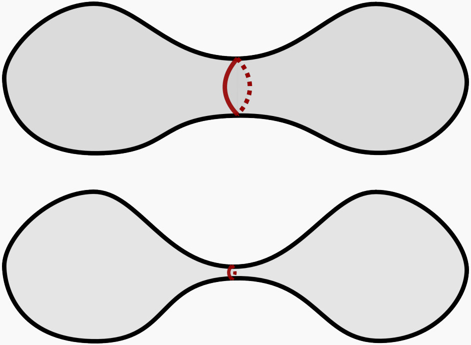

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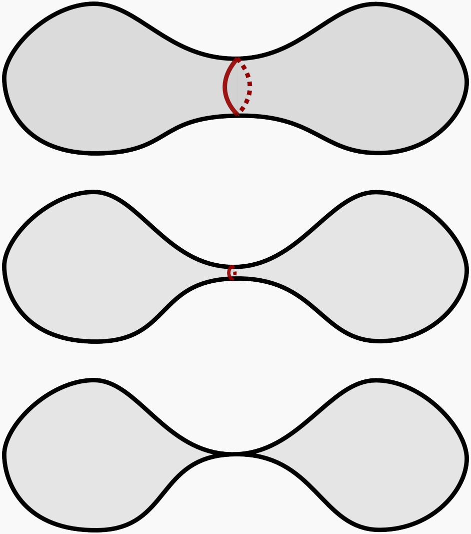
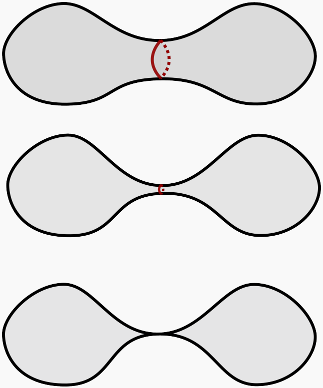


Hamilton's idea of surgery



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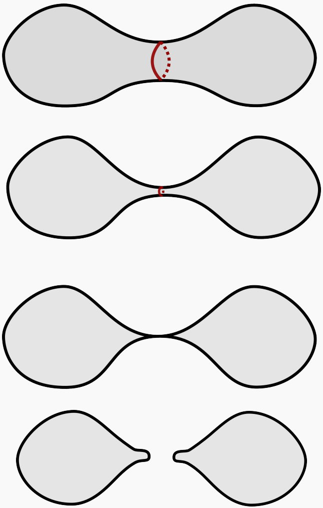

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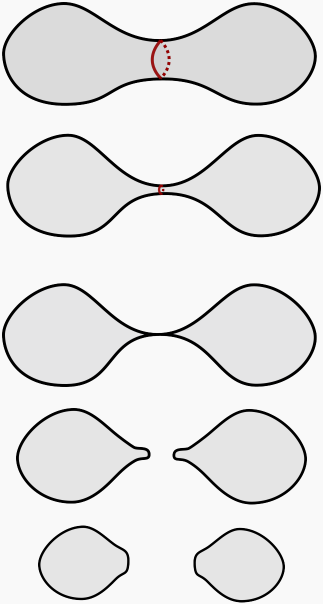

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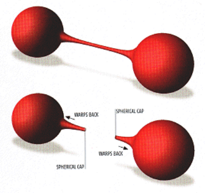

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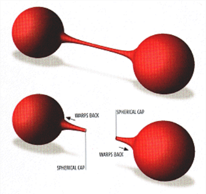

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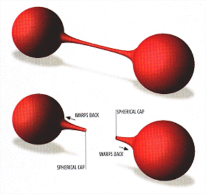

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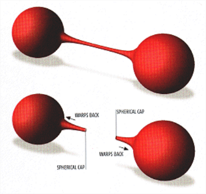

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Remark : the surgeries are done on 2-spheres, not 2-tori.

Intuitive way to prove the geometrization conjecture using Ricci flow

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(Relevant geometries : $\mathbb{R}^3$, H^3, $H^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, Sol, Nil.)

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From Perelman's first Ricci flow paper : *Moreover, it can be shown ... that the solution is smooth (if nonempty) from some finite time on.*

From Perelman's second Ricci flow paper : *This is a technical paper, which is a continuation of [I]. Here we verify most of the assertions, made in [I, §13]; the exceptions are ... the smoothness of the solution from some time on, which turned out to be unjustified, and, on the other hand, irrelevant for the other conclusions.*

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2. The thin part is a "graph manifold". (This doesn't use Ricci flow. Stated by Perelman, proofs by Shioya-Yamaguchi, Morgan-Tian, Bessières-Besson-Boileau-Maillot-Porti and Kleiner-L.)

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Graph manifolds were known to have a geometric decomposition. Along with knowledge of the topological effects of surgeries, this proved the geometrization conjecture.

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Remark : Answering these questions has no topological implication. We already know that the geometrization conjecture holds. Rather, they are *analytic* questions about the Ricci flow.

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To be more precise, there is a parameter in Perelman's Ricci-flow-with-surgery that determines the scale at which surgery is performed.

Finiteness of the number of surgeries

Theorem

(Bamler 2013) Starting from any compact Riemannian 3-manifold, Perelman's Ricci-flow-with-surgery only encounters a finite number of surgeries.

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To be more precise, there is a parameter in Perelman's Ricci-flow-with-surgery that determines the scale at which surgery is performed.

The statement is that if this parameter is small enough (which can always be achieved) then there is a finite number of surgeries.

Smooth flows

Corollary

A compact 3-manifold M admits a smooth Ricci flow that exists for all positive time if and only if $\pi_2(M) = \pi_3(M) = 0$, i.e. if the universal cover of M is diffeomorphic to $\mathbb{R}^3$.

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This is good because we know lots about metrics with bounded sectional curvature (Cheeger-Fukaya-Gromov).

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3. Use of minimal embedded 2-complexes.

Long-time behavior

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The only case that we completely understand is when M admits *some* hyperbolic metric. Then from Perelman's work, for *any* initial metric on M, as $t \to \infty$ the rescaled Riemannian metric $\widehat{g}(t)$ approaches the metric on M of constant sectional curvature $-\frac{1}{4}$.

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Question : if M doesn't admit a hyperbolic metric, what are the candidate geometries for the long-time behavior?

Quasistatic solutions

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$$\frac{dg}{dt} = -2\,\mathrm{Ric}_g$$

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Fact : On a compact 3-manifold, any such quasistatic solution has constant sectional curvature.

Apparent paradox : What happens to the Ricci flow if our 3-manifold doesn't admit a constant curvature metric?

Nil geometry

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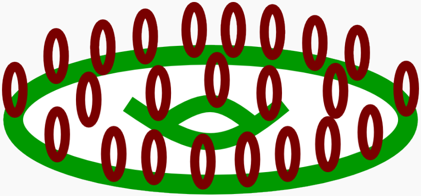

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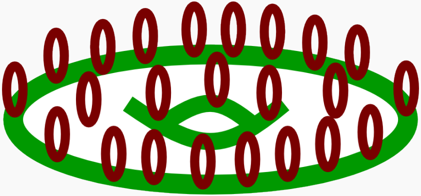

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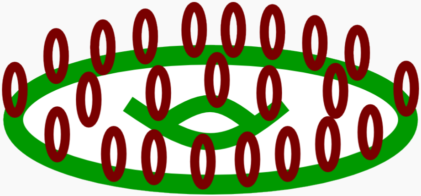

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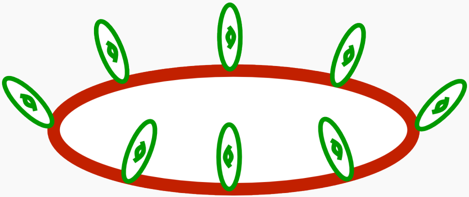
With the rescaled metric $\widehat{g}(t) = \frac{g(t)}{t}$, $(M, \widehat{g}(t))$ shrinks to a point.

Sol geometry

M fibers over a circle with 2-torus fibers. The monodromy is a hyperbolic element of $\mathrm{SL}(2, \mathbb{Z})$.



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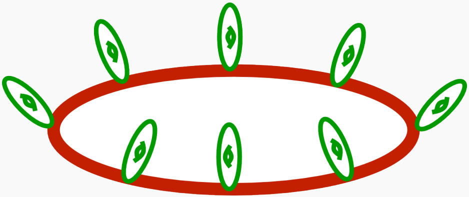

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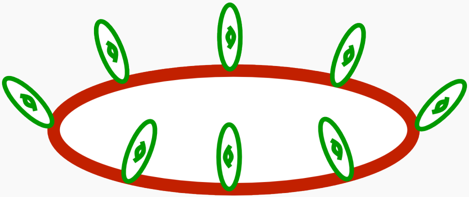

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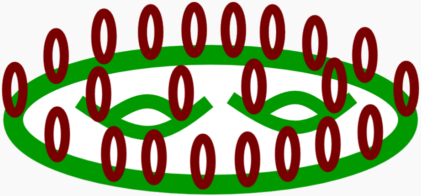
Suppose that M is the unit tangent bundle of a hyperbolic surface Σ.



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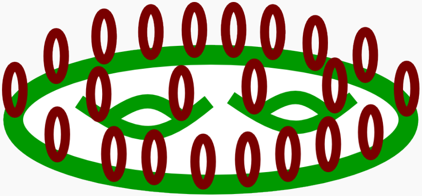

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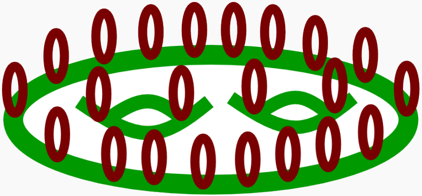

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With the rescaled metric, $(M, \widehat{g}(t))$ approaches the hyperbolic surface Σ. As the fibers shrink, the local geometry of the total space becomes more product-like.

Is there a common pattern?

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To undo the collapsing, let's pass to the universal cover. That is, we are looking at the Ricci flow on a Thurston geometry of type $\mathbb{R}^3, H^3, H^2 \times \mathbb{R}, \mathrm{Sol}, \mathrm{Nil}$ or $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

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(L. 2007) For any initial globally homogeneous metric on such a Thurston geometry, there is a limiting (blowdown) Ricci flow solution, which is an expanding soliton. There is one such soliton for each homogeneity type. It is a universal attractor.

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The limiting solitons

Thurston type	Expanding soliton
H^3	$4\,t\,g_{H^3}$
$H^2 \times \mathbb{R}$ or $\widetilde{SL(2,\mathbb{R})}$	$2\,t\,g_{H^2} + g_{\mathbb{R}}$
Sol	$e^{-2z}\,dx^2 + e^{2z}\,dy^2 + 4\,t\,dz^2$
Nil	$\frac{1}{3t^{\frac{1}{3}}}\left(dx + \frac{1}{2}ydz - \frac{1}{2}zdy\right)^2 + t^{\frac{1}{3}}\left(dy^2 + dz^2\right)$
$\mathbb{R}^3$	$g_{\mathbb{R}^3}$

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(L. 2010) Suppose that $(M, g(t))$ is a Ricci flow on a compact three-dimensional manifold, that exists for $t \in [0, \infty)$. Suppose that the sectional curvatures are $O\left(t^{-1}\right)$ in magnitude, and the diameter is $O(\sqrt{t})$. Then the pullback of the Ricci flow to $\widetilde{M}$ approaches one of the homogeneous expanding solitons.

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- The hypotheses imply that M admits a locally homogeneous metric.

Conjecture

For a long-time 3D Ricci flow, the diameter is $O(\sqrt{t})$ *if and only if* M *admits a locally homogeneous metric.*

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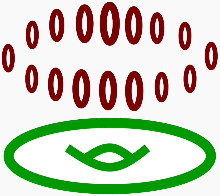

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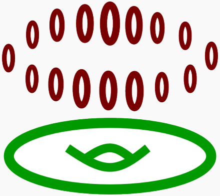

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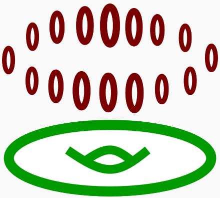

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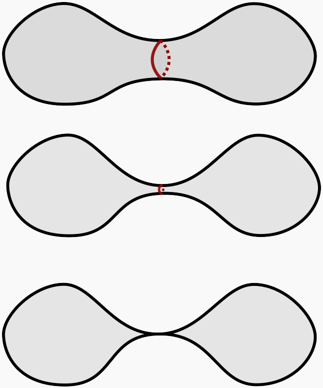
Question : is this true for *all* initial metrics on T^3?

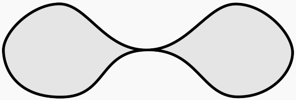
Can one flow through a singularity?



Is there a natural way to extend the flow beyond the singularity?

Deus Ex Machina

Surgery



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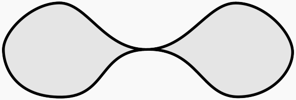

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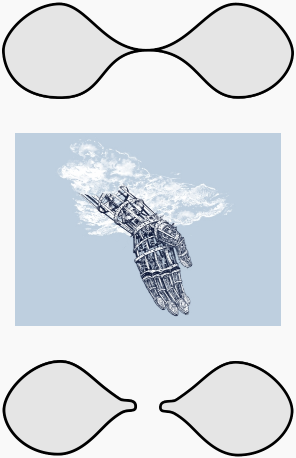

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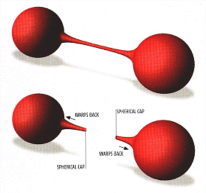

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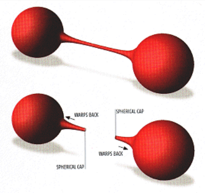

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From Perelman's first Ricci flow paper : *It is likely that by passing to the limit in this construction one would get a canonically defined Ricci flow through singularities, but at the moment I don't have a proof of that.*

A toy model

$$\frac{dx}{dt} = x^2, \quad x(0) = 1.$$

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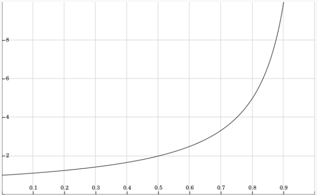

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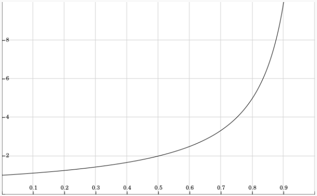

How can we extend this beyond the singularity at $t = 1$?

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Regularize the equation. For $\epsilon \neq 0$, consider

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This gives a way to extend the solution beyond (or around) $t = 1$.

Analogy

$$\frac{dx}{dt} = x^2 \longrightarrow \frac{dg}{dt} = -2\,\mathrm{Ric}_g$$

$$\frac{dx}{dt} = (x + i\epsilon)^2 \longrightarrow \text{Ricci-flow-with-surgery algorithm}$$

$$\epsilon \longrightarrow \text{Surgery parameter } h(t)$$

$$x_\epsilon(t) \longrightarrow \text{Ricci-flow-with-surgery solution}$$

$$x(t) = \frac{1}{1-t} \longrightarrow \text{A Ricci flow through singularities (?)}$$

Sublimits of Ricci flows with surgery

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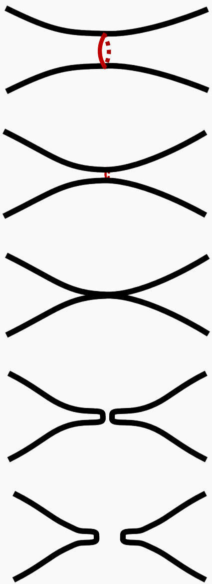
Then after passing to a subsequence, there is a limit

$$\lim_{i\to\infty} g_i(t) = g_\infty(t). \tag{3}$$

Spacetime of a flow through a neckpinch singularity



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2. In 3D Ricci flow, the scalar curvature R controls the local geometry, i.e. the curvature tensor and the injectivity radius (Hamilton-Ivey, Perelman). Fix $\overline{R} < \infty$. Consider the sublevel set

$$X^{\leq \overline{R}} = \{x \in X \ : \ R(x) \leq \overline{R}\}.$$

Take a sublimit of these regions as the surgery parameter goes to zero. Then take a sublimit as $\overline{R} \to \infty$.

Some properties of the limit

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One can study such singular Ricci flows in their own right.

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Thank you.