THE ETA FUNCTION AND SOME NEW ANOMALIES

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The odd-dimensional parity anomaly and the induced vacuum charge in an even-dimensional space are computed in terms of the eta function of spectra geometry.

We wish to point out that two related anomalies [1,2] can be easily understood in terms of the η function of spectral geometry. For simplicity we take our manifolds to be closed, riemannian and with a spin structure and consider an elliptic self-adjoint differential operator H acting on fields (cross sections of a vector bundle) over the m-dimensional manifold M. Let the eigenvalues of H be $\{\lambda_i\}$. Then [3]

$$\eta_H(s) \equiv \sum_{\lambda_i \neq 0} (\operatorname{sign} \lambda_i)|\lambda_i|^{-s} .$$

This converges for Re $s > m/$(differential order of H) and can be analytically continued to the complex s-plane. Amazingly, $\eta_H(0)$ is always finite [3,4]. If $H(\epsilon)$ is a one-parameter family of invertible H's then

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \eta_H(s) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} \operatorname{Tr} H(H^2)^{(-s-1)/2}$$

$$= -s \operatorname{Tr} \frac{\mathrm{d}H}{\mathrm{d}\epsilon} (H^2)^{(-s-1)/2} .$$

Thus to find $\mathrm{d}\eta_H(s)/\mathrm{d}\epsilon$ as $s \to 0$ it is only necessary to find the pole term in $(H^2)^{(-s-1)/2}$. Let tr denote the trace over the matrix part of the fields (the fiber trace on End(V)). For Re s large, we can write the operator trace as the integral over M of a local matrix trace giving, in the case that $\mathrm{d}H/\mathrm{d}\epsilon$ is a zeroth order operator,

$$-s \operatorname{Tr} \frac{\mathrm{d}H}{\mathrm{d}\epsilon} (H^2)^{(-s-1)/2}$$

$$= -s \int_M \operatorname{tr} \frac{\mathrm{d}H}{\mathrm{d}\epsilon} (x) (H^2)^{(-s-1)/2}(x,x) \sqrt{g}\, \mathrm{d}^m x ,$$

where g is the determinant of the metric tensor. Now

$$(H^2)^{(-s-1)/2}(x,x) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty T^{(s-1)/2}$$

$$\times\, \mathrm{e}^{-TH^2}(x,x)\, \mathrm{d}T .$$

There is an asymptotic expansion

$$\mathrm{e}^{-TH^2}(x,x) \sim T^{-m/2} \sum_{i=0}^{\infty} a_i(x,x) T^i ,$$

which gives

$$\lim_{s\to 0} s (H^2)^{(-s-1)/2}(x,x)$$

$$= \lim_{s \to 0} \frac{1}{\sqrt{\pi}} s \int_0^1 T^{(s-m-1)/2} \sum_{i=0}^{\infty} a_i(x,x) T^i\, \mathrm{d}T$$

$$= (2/\sqrt{\pi})\, a_{(m-1)/2}(x,x) .$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \eta_H(s)$$

$$= -\frac{2}{\sqrt{\pi}} \int_M \operatorname{tr}\left(\frac{\mathrm{d}H}{\mathrm{d}\epsilon}(x)\, a_{(m-1)/2}(x,x)\right) \sqrt{g}\, \mathrm{d}^m x.$$

The $a_{(m-1)/2}$ are computable and have been tabulated for $m \leqslant 7$ [5]. That is, variations of $\eta_H(0)$ are computable for invertible H's as local expressions.

If, on the other hand, $H(\epsilon)$ passes through a noninvertible H then an eigenvalue can jump from, say, $\lambda < 0$ to $\lambda > 0$ as, say, ϵ goes from 0^- to 0^+. Because $\eta_H(0)$ essentially adds the eigenvalues by sign, the jump in $\eta_H(0)$ from $\epsilon = 0^-$ to $\epsilon = 0^+$ is 2. Similarly if an eigenvalue goes from $\lambda > 0$ to $\lambda < 0$ as ϵ goes from 0^- to 0^+ then the jump in $\eta_H(0)$ at $\epsilon = 0$ is -2. The preceding is all well known to mathematicians.

The first anomaly is that of induced vacuum charge in an even-dimensional space [1]. The renormalized charge for a quantized spinor field in a classical background field is $Q = -\frac{1}{2}\eta_H(0)$, H being the spinor hamiltonian [6]. Because $a_{(m-1)/2} = 0$ for m even, Q is given solely by the integer jumps of above. Take $H = -\mathrm{i}\gamma^j(\nabla_j + A_j) + \phi\gamma^5$. For $\phi = 0$ there is a conjugation symmetry and $\eta_H(0) = 0$. Write the kernel of H as $\operatorname{Ker} H = W^+ \oplus W^-$ such that for $V \in W^\pm$, $\gamma^5 V = \pm V$. For $\phi = 0^+$ the vectors in W^+ will acquire a positive energy while those in W^- will acquire a negative energy, giving

$$Q = -\tfrac{1}{2}(\dim W^+ - \dim W^-)$$

$$= -\tfrac{1}{2}\operatorname{Index}(-\mathrm{i}\gamma^j(\nabla_j + A_j)) ,$$

the last operator mapping positive chirality spinors to negative chirality spinors.

From the Atiyah–Singer index theorem [7],

$$Q = -\tfrac{1}{2}\int_M (\operatorname{ch} F)\hat{A}(g) ,$$

the integral of the product of the Chern character of the gauge field and the $\hat{A}$ character of the metric. On a flat space

$$Q = -\frac{1}{2}\,\frac{1}{(2\pi\mathrm{i})^{m/2}}\,\frac{1}{(m/2)!}\int_M \operatorname{tr} F^{m/2} ,$$

which is the result of ref. [1]. The $\phi\gamma^5$ term in H comes from the use of a regulator mass in the lagrangian approach.

The second anomaly is a parity-violating current in odd-dimensional space–time [2]. Let the wave equation for a massless spinor be $\not{D}\psi = 0$ and let $\not{D}$ have (real) spectrum $\{\lambda_i\}$. Formally define the effective action to be

$$\Gamma = \ln\det\not{D} \equiv \tfrac{1}{2}\ln\det\not{D}^2 + (2n+1)\mathrm{i}\pi$$

$$\times\ (\text{number of } \lambda_i < 0) .$$

Under a parity transformation,

$$\Gamma \to \Gamma_\mathrm{P} = \ln\det(-\not{D})$$

$$\equiv \tfrac{1}{2}\ln\det\not{D}^2 + (2n+1)\mathrm{i}\pi\,(\text{number of } \lambda_i > 0) .$$

We define

$$\Gamma - \Gamma_\mathrm{P} = -(2n+1)\,\mathrm{i}\pi\,\eta_{\not{D}}(0) .$$

If only gauge fields are present then the Atiyah–Patodi–Singer theorem [3] implies that

$$\tfrac{1}{2}(\eta_{\not{D}}(0) + \dim\operatorname{Ker}\not{D}) = \mathrm{CS}(\mathrm{M})\ (\mathrm{mod}\ \mathrm{Z}) ,$$

the evaluation of the Chern–Simons secondary characteristic class (CS) with values in R/Z [8]. This is the result of ref. [2]. If a mass term ϕ is added to $\not{D}$ then, with $\{\lambda_i\}$ being the new spectrum, a parity transformation takes λ_i to $2\phi - \lambda_i$. Then

$$\operatorname{Im}(\Gamma - \Gamma_\mathrm{P}) = (\text{formally})\ (2n+1)\mathrm{i}\pi$$

$$\times\ [(\text{number of } \lambda_i < 0) - (\text{number of } 2\phi - \lambda_i < 0)]$$

$$= (2n+1)\mathrm{i}\pi\,[(\text{number of } \lambda_i \in (0, 2\phi]) - \eta_{\not{D}}(0)] .$$

By the methods of the first paragraph $\frac{1}{2}(\eta_{\not{D}}(0) + \dim\operatorname{Ker}\not{D})(\mathrm{mod}\ \mathrm{Z})$ can be found. For $m = 3$,

$$\operatorname{Im}(\Gamma - \Gamma_\mathrm{P}) = (2n+1)\mathrm{i}\pi\Big((\text{number of } \lambda_i \in (0, 2\phi])$$

$$+ \dim\operatorname{Ker}\not{D} - 2\mathrm{CS}(\mathrm{M})$$

$$+ \frac{1}{3}\int \phi^3\sqrt{g}\,\mathrm{d}^m x\Big)\ (\mathrm{mod}\ 2\mathrm{i}\pi\mathrm{Z}) .$$

On an open manifold the last term may be infinite, but the currents are finite.

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