LOCALLY COLLAPSED 3-MANIFOLDS

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Abstract. We prove that a 3-dimensional compact Riemannian manifold which is locally collapsed, with respect to a lower curvature bound, is a graph manifold. This theorem was stated by Perelman and was used in his proof of the geometrization conjecture.

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1. Introduction

1.1. **Overview.** In this paper we prove that a 3-dimensional Riemannian manifold which is locally collapsed, with respect to a lower curvature bound, is a graph manifold. This result was stated without proof by Perelman in [Per, Theorem 7.4], where it was used to show that certain collapsed manifolds arising in his proof of the geometrization conjecture are graph manifolds. Our goal is to provide a proof of Perelman's collapsing theorem which is streamlined, self-contained and accessible. Other proofs of Perelman's theorem appear in [BBB$^+$10, CG11, MT, SY05].

In the rest of this introduction we state the main result and describe some of the issues involved in proving it. We then give an outline of the proof. We finish by discussing the history of the problem.

1.2. **Statement of results.** We begin by defining an intrinsic local scale function for a Riemannian manifold.

Definition 1.1. Let M be a complete Riemannian manifold. Given $p \in M$, the *curvature scale* R_p at p is defined as follows. If the connected component of M containing p has nonnegative sectional curvature then $R_p = \infty$. Otherwise, R_p is the (unique) number $r > 0$ such that the infimum of the sectional curvatures on $B(p,r)$ equals $-\frac{1}{r^2}$.

We need one more definition.

Definition 1.2. Let M be a compact orientable 3-manifold (possibly with boundary). Give M an arbitrary Riemannian metric. We say that M is a *graph manifold* if there is a finite disjoint collection of embedded 2-tori $\{T_j\}$ in the interior of M such that each connected component of the metric closure of $M - \bigcup_j T_j$ is the total space of a circle bundle over a surface (generally with boundary).

For simplicity, in this introduction we state the main theorem in the case of closed manifolds. For the general case of manifolds with boundary, we refer the reader to Theorem 16.1.

Theorem 1.3. *(cf.* [Per, Theorem 7.4]*) Let c_3 denote the volume of the unit ball in $\mathbb{R}^3$ and let $K \geq 10$ be a fixed integer. Fix a function $A : (0,\infty) \to (0,\infty)$. Then there is a $w_0 \in (0, c_3)$ such that the following holds.*

Suppose that (M,g) is a closed orientable Riemannian 3-manifold. Assume in addition that for every $p \in M$,

1. $\mathrm{vol}(B(p,R_p)) \leq w_0 R_p^3$ *and*
2. *For every $w' \in [w_0, c_3)$, $k \in [0,K]$, and $r \leq R_p$ such that $\mathrm{vol}(B(p,r)) \geq w' r^3$, the inequality*

$$|\nabla^k \operatorname{Rm}| \leq A(w')\, r^{-(k+2)} \tag{1.4}$$

holds in the ball $B(p,r)$.

Then M is a graph manifold.

1.3. **Motivation.** Theorem 1.3, or more precisely the version for manifolds with boundary, is essentially the same as Perelman's [Per, Theorem 7.4]. Either result can be used to complete the Ricci flow proof of Thurston's geometrization conjecture. We explain this in Section 17, following the presentation of Perelman's work in [KL08].

To give a brief explanation, let $(M, g(\cdot))$ be a Ricci flow with surgery whose initial manifold is compact, orientable and three-dimensional. Put $\widehat{g}(t) = \frac{g(t)}{t}$. Let M_t denote the time t manifold. (If t is a surgery time then we take M_t to be the postsurgery manifold.) For any $w > 0$, the Riemannian manifold $(M_t, \widehat{g}(t))$ has a decomposition into a w-thick part and a w-thin part. (Here the terms "thick" and "thin" are suggested by the Margulis thick-thin decomposition but the definition is somewhat different. In the case of hyperbolic manifolds, the two notions are essentially equivalent.) As $t \to \infty$, the w-thick part of $(M_t, \widehat{g}(t))$ approaches the w-thick part of a complete finite-volume Riemannian manifold of constant curvature $-\frac{1}{4}$, whose cusps (if any) are incompressible in M_t. Theorem 1.3 implies that for large t, the w-thin part of M_t is a graph manifold. Since graph manifolds are known to have a geometric decomposition in the sense of Thurston, this proves the geometrization conjecture.

Independent of Ricci flow considerations, Theorem 1.3 fits into the program in Riemannian geometry of understanding which manifolds can collapse. The main geometric assumption in Theorem 1.3 is the first one, which is a local collapsing statement, as we discuss in the next subsection. The second assumption of Theorem 1.3 is more technical in nature. In the application to the geometrization conjecture, the validity of the second assumption essentially arises from the smoothing effect of the Ricci flow equation.

In fact, Theorem 1.3 holds without the second assumption. In order to prove this stronger result, one must use the highly nontrivial Stability Theorem of Perelman [Per91, Kap07]. As mentioned in [Per], if one does make the second assumption then one can effectively replace the Stability Theorem by standard C^K-convergence of Riemannian manifolds. Our proof of Theorem 1.3 is set up so that it extends to a proof of the stronger theorem, without the second assumption, provided that one invokes the Stability Theorem in relevant places; see Sections 1.5.7 and 18.

1.4. **Aspects of the proof.** The strategy in proving Theorem 1.3 is to first understand the local geometry and topology of the manifold M. One then glues these local descriptions together to give an explicit decomposition of M that shows it to be a graph manifold. This strategy is common to [SY05, MT, CG11] and the present paper. In this subsection we describe the strategy in a bit more detail. Some of the new features of the present paper will be described more fully in Subsection 1.5.

1.4.1. *An example.* The following simple example gives a useful illustration of the strategy of the proof.

Let $P \subset H^2$ be a compact convex polygonal domain in the two-dimensional hyperbolic space. Embedding H^2 in the four-dimensional hyperbolic space H^4, let $N_s(P)$ be the metric s-neighborhood around P in H^4. Take M to be the boundary $\partial N_s(C)$. If s is sufficiently small then one can check that the hypotheses of Theorem 1.3 are satisfied.

Consider the structure of M when s is small. There is a region $M^{\text{2-stratum}}$, lying at distance $\geq$ const. s from the boundary ∂P, which is the total space of a circle bundle. At scale comparable to s, a suitable neighborhood of a point in $M^{\text{2-stratum}}$ is nearly isometric to a product of a planar region with S^1. There is also a region M^{edge} lying at distance $\leq$ const. s from an edge of P, but away from the vertices of P, which is the total space of a 2-disk bundle. At scale comparable to s, a suitable neighborhood of a point in M^{edge} is nearly isometric to the product of an interval with a 2-disk. Finally, there is a region $M^{\text{0-stratum}}$ lying at distance $\leq$ const. s from the vertices of P. A connected component of $M^{\text{0-stratum}}$ is diffeomorphic to a 3-disk.

We can choose $M^{\text{2-stratum}}$, M^{edge} and $M^{\text{0-stratum}}$ so that there is a decomposition $M = M^{\text{2-stratum}} \cup M^{\text{edge}} \cup M^{\text{0-stratum}}$ with the property that on interfaces, fibration structures are compatible. Now $M^{\text{edge}} \cup M^{\text{0-stratum}}$ is a finite union of 3-disks and $D^2 \times I$'s, which is homeomorphic to a solid torus. Also, $M^{\text{2-stratum}}$ is a circle bundle over a 2-disk, i.e. another solid torus, and $M^{\text{2-stratum}}$ intersects $M^{\text{edge}} \cup M^{\text{0-stratum}}$ in a 2-torus. So using this geometric decomposition, we recognize that M is a graph manifold. (In this case M is obviously diffeomorphic to S^3, being the boundary of a convex set in H^4, and so it is a graph manifold; the point is that one can recognize this using the geometric structure that comes from the local collapsing.)

1.4.2. *Local collapsing.* The statement of Theorem 1.3 is in terms of a *local* lower curvature bound, as evidenced by the appearance of the curvature scale R_p. Assumption (1) of Theorem 1.3 can be considered to be a local collapsing statement. (This is in contrast to a global collapsing condition, where one assumes that the sectional curvatures are at least -1 and $\mathrm{vol}(B(p,1)) < \epsilon$ for every $p \in M$.) To clarify the local collapsing statement, we make one more definition.

Definition 1.5. Let c_3 denote the volume of the Euclidean unit ball in $\mathbb{R}^3$. Fix $\bar{w} \in (0, c_3)$. Given $p \in M$, the *$\bar{w}$-volume scale at p* is

$$r_p(\bar{w}) = \inf\{r > 0 \ : \ \mathrm{vol}(B(p,r)) = \bar{w}\, r^3\}. \tag{1.6}$$

If there is no such r then we say that the $\bar{w}$-volume scale is infinite.

There are two ways to look at hypothesis (1) of Theorem 1.3, at the curvature scale or at the volume scale. Suppose first that we rescale the ball $B(p, R_p)$ to have radius one. Then the resulting ball will have sectional curvature bounded below by -1 and volume bounded above by w_0. As w_0 will be small, we can say that on the curvature scale, the manifold is locally volume collapsed with respect to a lower curvature bound. On the other hand, suppose that we rescale $B(p, r_p(w_0))$ to have radius one. Let $B'(p,1)$ denote the rescaled ball. Then $\mathrm{vol}(B'(p,1)) = w_0$. Hypothesis (1) of Theorem 1.3 implies that there is a big number $\mathcal{R}$ so that the sectional curvature on the radius $\mathcal{R}$-ball $B'(p, \mathcal{R})$ (in the rescaled manifold) is bounded below by $-\frac{1}{\mathcal{R}^2}$. Using this, we deduce that on the volume scale, a large neighborhood of p is well approximated by a large region in a complete nonnegatively curved 3-manifold N_p. This gives a local model for the geometry of M. Furthermore, if w_0 is small then we can say that at the volume scale, the neighborhood of p is close in a coarse sense to a space of dimension less than three.

In order to prove Theorem 1.3, one must first choose on which scale to work. We could work on the curvature scale, or the volume scale, or some intermediate scale (as is done in [MT, SY05, CG11]). In this paper we will work consistently on the volume scale. This gives a uniform and simplifying approach.

1.4.3. *Local structure.* At the volume scale, the local geometry of M is well approximated by that of a nonnegatively curved 3-manifold. (That we get a 3-manifold instead of a 3-dimensional Alexandrov space comes from the second assumption in Theorem 1.3.) The topology of nonnegatively curved 3-manifolds is known in the compact case by work of Hamilton [Ham82, Ham86] and in the noncompact case by work of Cheeger-Gromoll [CG72]. In the latter case, the geometry is also well understood. Some relevant examples of such manifolds are:

(1) $\mathbb{R}^2 \times S^1$,
(2) $\mathbb{R} \times S^2$,
(3) $\mathbb{R} \times \Sigma$, where Σ is a noncompact nonnegatively curved surface which is diffeomorphic to $\mathbb{R}^2$ and has a cylindrical end, and
(4) $\mathbb{R} \times_{\mathbb{Z}_2} S^2$.

If a neighborhood of a point $p \in M$ is modeled by $\mathbb{R}^2 \times S^1$ at the volume scale then the length of the circle fiber is comparable to w_0. Hence if w_0 is small then the neighborhood looks almost like a 2-plane. Similarly, if the neighborhood is modeled by $\mathbb{R} \times S^2$ then it looks almost like a line. On the other hand, if the neighborhood is modeled by $\mathbb{R} \times \Sigma$ then for small w_0, the surface Σ looks almost like a half-line and the neighborhood looks almost like a half-plane. Finally, if the neighborhood is modeled by $\mathbb{R} \times_{\mathbb{Z}_2} S^2$ then it looks almost like a half-line.

1.4.4. *Gluing.* The remaining issue is use the local geometry to deduce the global topology of M. This is a gluing issue, as the local models need to be glued together to obtain global information.

One must determine which local models should be glued together. We do this by means of a stratification of M. If p is a point in M then for $k \leq 2$, we say that p is a k-stratum point if on the volume scale, a large ball around p approximately splits off an $\mathbb{R}^k$-factor metrically, but not an $\mathbb{R}^{k+1}$-factor.

For $k \in \{1, 2\}$, neighborhoods of the k-stratum points will glue together in order to produce the total space of a fibration over a k-dimensional manifold. For example, neighborhoods of the 2-stratum points will glue together to form a circle bundle over a surface. Neighborhoods of the 0-stratum points play a somewhat different role. They will be inserted as "plugs"; for example, neighborhoods of the exceptional fibers in a Seifert fibration will arise in this way.

By considering how M is decomposed into these various subspaces that fiber, we will be able to show that M is a graph manifold.

1.5. **Outline of the proof.** We now indicate the overall structure of the proof of Theorem 1.3. In this subsection we suppress parameters or denote them by const. In the paper we will use some minimal facts about pointed Gromov-Hausdorff convergence and Alexandrov spaces, which are recalled in Section 3.

1.5.1. *Modified volume scale.* The first step is to replace the volume scale by a slight modification of it. The motivation for this step is the fluctuation of the volume scale. Suppose that p and q are points in overlapping local models. As these local models are at the respective volume scales, there will be a problem in gluing the local models together if $r_q(\bar{w})$ differs wildly from $r_p(\bar{w})$. We need control on how the volume scale fluctuates on a ball of the form $B(p, \text{const.}\, r_p(\bar{w}))$. We deal with this problem by replacing the volume scale $r_p(\bar{w})$ by a modified scale which has better properties. We assign a scale $\mathfrak{r}_p$ to each point $p \in M$ such that:

(1) $\mathfrak{r}_p$ is much less than the curvature scale R_p.
(2) The function $p \mapsto \mathfrak{r}_p$ is smooth and has Lipschitz constant $\Lambda \ll 1$.
(3) The ball $B(p, \mathfrak{r}_p)$ has volume lying in the interval $[w' \mathfrak{r}_p^3, \bar{w} \mathfrak{r}_p^3]$, where $w' < \bar{w}$ are suitably chosen constants lying in the interval $[w_0, c_3]$.

The proof of the existence of the scale function $p \mapsto \mathfrak{r}_p$ follows readily from the local collapsing assumption, the Bishop-Gromov volume comparison theorem, and an argument similar to McShane's extension theorem for real-valued Lipschitz functions; see Section 6.

1.5.2. *Implications of compactness.* Condition (1) above implies that the rescaled manifold $\frac{1}{\mathfrak{r}_p} M$, in the vicinity of p, has almost nonnegative curvature. Furthermore, condition (3) implies that it looks collapsed but not too collapsed, in the sense that the volume of the unit ball around p in the rescaled manifold $\frac{1}{\mathfrak{r}_p} M$ is small but not too small. Thus by working at the scale $\mathfrak{r}_p$, we are able to retain the local collapsing assumption (in a somewhat weakened form) while gaining improved behavior of the scale function.

Next, the bounds (1.4) extend to give bounds on the derivatives of the curvature tensor of the form

$$|\nabla^k \operatorname{Rm}| \leq A'(C, w') \tag{1.7}$$

for $0 \leq k \leq K$, when restricted to balls $B(p, C)$ in $\frac{1}{\mathfrak{r}_p} M$. Using (1.7) and standard compactness theorems for pointed Riemannian manifolds, we get:

(4) For every $p \in M$, the rescaled pointed manifold $(\frac{1}{\mathfrak{r}(p)} M, p)$ is close in the pointed C^K-topology to a pointed nonnegatively curved C^K-smooth Riemannian 3-manifold $(N_p, \star)$.
(5) For every $p \in M$, the pointed manifold $(\frac{1}{\mathfrak{r}(p)} M, p)$ is close in the pointed Gromov-Hausdorff topology to a pointed nonnegatively curved Alexandrov space $(X_p, \star)$ of dimension at most 2.

1.5.3. *Stratification.* The next step is to define a partition of M into k-stratum points, for $k \in \{0, 1, 2\}$. The partition is in terms of the number of $\mathbb{R}$-factors that approximately split off in $(\frac{1}{\mathfrak{r}(p)} M, p)$.

Let $0 < \beta_1 < \beta_2$ be new parameters. Working at scale $\mathfrak{r}_p$, we classify points in M as follows (see Section 7):

- A point $p \in M$ lies in the 2-**stratum** if $(\frac{1}{\mathfrak{r}(p)} M, p)$ is β_2-close to $(\mathbb{R}^2, 0)$ in the pointed Gromov-Hausdorff topology.
- A point $p \in M$ lies in the 1-**stratum**, if it does not lie in the 2-stratum, but $(\frac{1}{\mathfrak{r}(p)} M, p)$ is β_1-close to $\big(\mathbb{R} \times Y_p, \big(0, \star_{Y_p}\big)\big)$ in the pointed Gromov-Hausdorff topology, where Y_p is a point, a circle, an interval or a half-line, and $\star_{Y_p}$ is a basepoint in Y_p.
- A point lies in the 0-**stratum** if it does not lie in the k-stratum for $k \in \{1, 2\}$.

We now discuss the structure near points in the different strata in more detail, describing the model spaces X_p and N_p.

2-*stratum points.* (Section 8). If β_2 is small and $p \in M$ is a 2-stratum point then X_p is isometric to $\mathbb{R}^2$, while N_p is isometric to a product $\mathbb{R}^2 \times S^1$ where the S^1 factor is small. Since the pointed rescaled manifold $(\frac{1}{\mathfrak{r}_p} M, p)$ is close to $(N_p, \star)$, we can transfer the projection map $N_p \simeq \mathbb{R}^2 \times S^1 \to \mathbb{R}^2$ to a map η_p defined on a large ball $B(p, C) \subset \frac{1}{\mathfrak{r}_p} M$, where it defines a circle fibration.

1-*stratum points.* (Sections 9 and 10). If β_1 is small and p is a 1-stratum point then $X_p = \mathbb{R} \times Y_p$, where Y_p is a point, a circle, an interval or a half-line. The C^K-smooth model space N_p will be an isometric product $\mathbb{R} \times \bar{N}_p$, where $\bar{N}_p$ is a complete nonnegatively curved orientable surface. As in the 2-stratum case, we can transfer the projection map $N_p \simeq \mathbb{R} \times \bar{N}_p \to \mathbb{R}$ to a map η_p defined on a large ball $B(p, C) \subset \frac{1}{\mathfrak{r}_p} M$, where it defines a submersion.

We further classify the 1-stratum points according to the diameter of the cross-section Y_p. If the diameter of Y_p is not too large then we say that p lies in the *slim* 1-*stratum.* (The motivation for the terminology is that in this case $(\frac{1}{\mathfrak{r}_p} M, p)$ appears slim, being at moderate Gromov-Hausdorff distance from a line.) For slim 1-stratum points, the cross-section $\bar{N}_p$ is diffeomorphic to S^2 or T^2. Moreover, in this case the submersion η_p will be a fibration with fiber diffeomorphic to $\bar{N}_p$.

We also distinguish another type of 1-stratum point, the *edge points.* A 1-stratum point p is an edge point if $(X_p, \star) = (\mathbb{R} \times Y_p, \star)$ can be taken to be pointed isometric to a flat Euclidean half-plane whose basepoint lies on the edge. Roughly speaking, we show that near p, the set E' of edge points looks like a 1-dimensional manifold at scale $\mathfrak{r}_p$. Furthermore, there is a smooth function $\eta_{E'}$ which behaves like the "distance to the edge" and which combines with η_p to yield "half-plane coordinates" for $\frac{1}{\mathfrak{r}_p} M$ near p. When restricted to an appropriate sublevel set of $\eta_{E'}$, the map η_p defines a fibration with fibers diffeomorphic to a compact surface with boundary F_p. Using the fact that for edge points $N_p = \mathbb{R} \times \bar{N}_p$ is Gromov-Hausdorff close to a half-plane, one sees that the pointed surface $(\bar{N}_p, \star)$ is Gromov-Hausdorff close to a pointed ray $([0, \infty), \star)$. This allows one to conclude that F_p is diffeomorphic to a closed 2-disk.

0-*stratum points.* (Section 11). We know by (4) that if p is a 0-stratum point, then $(\frac{1}{\mathfrak{r}_p} M, p)$ is C^K-close to a nonnegatively curved C^K-smooth 3-manifold N_p. The idea for analyzing the structure of M near a 0-stratum point p is to use the fact that nonnegatively curved manifolds look asymptotically like cones, and are diffeomorphic to any sufficiently large ball in them (centered at a fixed basepoint). More precisely, we find a scale r^0_p with $\mathfrak{r}_p \leq r^0_p \leq$ const. $\mathfrak{r}_p$ so that:

- The pointed rescaled manifold $(\frac{1}{r^0_p} M, p)$ is close in the C^K-topology to a C^K-smooth nonnegatively curved 3-manifold $(N'_p, \star)$.
- The distance function d_p in $\frac{1}{\mathfrak{r}^0_p} M$ has no critical points in the metric annulus $A(p, \frac{1}{10}, 10) = \overline{B(p, 10)} - B(p, \frac{1}{10})$, and $B(p, 1) \subset \frac{1}{r^0_p} M$ is diffeomorphic to N'_p.
- The pointed space $(\frac{1}{r^0_p} M, p)$ is close in the pointed Gromov-Hausdorff topology to a Euclidean cone (in fact the Tits cone of N'_p).
- N'_p has at most one end.

The proof of the existence of the scale r^0_p is based on the fact that nonnegatively curved manifolds are asymptotically conical, the critical point theory of Grove-Shiohama [GS77], and a compactness argument. Using the approximately conical structure, one obtains a smooth function η_p on $\frac{1}{r^0_p} M$ which, when restricted to the metric annulus $A(p, \frac{1}{10}, 10) \subset \frac{1}{r^0_p} M$, behaves like the radial function on a cone. In particular, for $t \in [\frac{1}{10}, 10]$, the sublevel sets $\eta_p^{-1}[0, t)$ are diffeomorphic to N'_p.

The soul theorem [CG72], together with Hamilton's classification of closed nonnegatively curved 3-manifolds [Ham82, Ham86], implies that N'_p is diffeomorphic to one of the following: a manifold W/Γ where W is either S^3, $S^2 \times S^1$ or T^3 equipped with a standard Riemannian metric and Γ is a finite group of isometries; $S^1 \times \mathbb{R}^2$; $S^2 \times \mathbb{R}$, $T^2 \times \mathbb{R}$; or a twisted line bundle over $\mathbb{R}P^2$ or the Klein bottle. Thus we know the possibilities for the topology of $B(p, 1) \subset \frac{1}{r^0_p} M$.

1.5.4. *Compatibility of the local structures.* Having determined the local structure of M near each point, we examine how these local structures fit together on their overlap. For example, consider the slim 1-stratum points corresponding to an S^2-fiber. A neighborhood of the set of such points looks like a union of cylindrical regions. If the axes of overlapping cylinders are very well-aligned then the process of gluing them together will be simplified. It turns out that such compatibility is automatic from our choice of stratification.

To see this, suppose that $p, q \in M$ are 2-stratum points with $B(p, \text{const.}\, \mathfrak{r}_p) \cap B(q, \text{const.}\, \mathfrak{r}_q) \neq \emptyset$. Then provided that Λ is small, we know that $\mathfrak{r}_p \approx \mathfrak{r}_q$. Suppose now that $z \in B(p, \text{const.}\, \mathfrak{r}_p) \cap B(q, \text{const.}\, \mathfrak{r}_q)$. We have two $\mathbb{R}^2$-factors at z, coming from the approximate splittings at p and q. If the parameter β_2 is small then these $\mathbb{R}^2$-factors must align well at z. If not then we would get two misaligned $\mathbb{R}^2$-factors at p, which would generate an approximate $\mathbb{R}^3$-factor at p, contradicting the local collapsing assumption. Hence the maps η_p and η_q, which arose from approximate $\mathbb{R}^2$-splittings, are nearly "aligned" with each other on their overlap, so that η_p and η_q are affine functions of each other, up to arbitrarily small C^1-error.

Now fix β_2. Let $p, q \in M$ be 1-stratum points. At any $z \in B(p, \mathrm{const.}\, \mathfrak{r}_p) \cap B(q, \mathrm{const.}\, \mathfrak{r}_q)$, there are two $\mathbb{R}$-factors, coming from the approximate $\mathbb{R}$-splittings at p and q. If β_1 is small then these two $\mathbb{R}$-factors must align well at z, or else we would get two misaligned $\mathbb{R}$-factors at p, contradicting the fact that p is not a 2-stratum point. Hence the functions η_p and η_q are also affine functions of each other, up to arbitrarily small C^1-error.

One gets additional compatibility properties for pairs of points of different types. For example, if p lies in the 0-stratum and $q \in A(p, \frac{1}{10}, 10) \subset \frac{1}{\mathfrak{r}_p^0} M$ belongs to the 2-stratum then the radial function η_p, when appropriately rescaled, agrees with an affine function of η_q in $B(q, 10) \subset \frac{1}{\mathfrak{r}_q} M$ up to small C^1-error.

1.5.5. *Gluing the local pieces together.* (Sections 12-14). To begin the gluing process, we select a separated collection of points of each type in M: $\{p_i\}_{i \in I_{\text{2-stratum}}}$, $\{p_i\}_{i \in I_{\text{slim}}}$, $\{p_i\}_{i \in I_{\text{edge}}}$, $\{p_i\}_{i \in I_{\text{0-stratum}}}$, so that

- $\bigcup_{i \in I_{\text{2-stratum}}} B(p_i, \mathrm{const.}\, \mathfrak{r}_{p_i})$ covers the 2-stratum points,
- $\bigcup_{i \in I_{\text{slim}} \cup I_{\text{edge}}} B(p_i, \mathrm{const.}\, \mathfrak{r}_{p_i})$ covers the 1-stratum points, and
- $\bigcup_{i \in I_{zeroball}} B(p_i, \mathrm{const.}\, r_{p_i}^0)$ covers the 0-stratum points.

Our next objective is to combine the η_{p_i}'s so as to define global fibrations for each of the different types of points, and ensure that these fibrations are compatible on overlaps. To do this, we borrow an idea from the proof of the Whitney embedding theorem (as well as proofs of Gromov's compactness theorem [Gro99, Chapter 8.D], [Kat85]): we define a smooth map $\mathcal{E}^0 : M \to H$ into a high-dimensional Euclidean space H. The components of $\mathcal{E}^0$ are functions of the η_{p_i}'s, the edge function $\eta_{E'}$, and the scale function $p \mapsto \mathfrak{r}_p$, cutoff appropriately so that they define global smooth functions.

Due to the pairwise compatibility of the η_{p_i}'s discussed above, it turns out that the image under $\mathcal{E}^0$ of $\bigcup_{i \in I_{\text{2-stratum}}} B(p_i, \mathrm{const.}\, \mathfrak{r}_{p_i})$ is a subset $S \subset H$ which, when viewed at the right scale, is everywhere locally close (in the pointed Hausdorff sense) to a 2-dimensional affine subspace. We call such a set a *cloudy* 2*-manifold*. By an elementary argument, we show in Appendix B that a cloudy manifold of any dimension can be approximated by a core manifold W whose normal injectivity radius is controlled.

We adjust the map $\mathcal{E}^0$ by "pinching" it into the manifold core of S, thereby upgrading $\mathcal{E}^0$ to a new map $\mathcal{E}^1$ which is a circle fibration near the 2-stratum. The new map $\mathcal{E}^1$ is C^1-close to $\mathcal{E}^0$. We then perform similar adjustments near the edge points and slim 1-stratum points, to obtain a map $\mathcal{E} : M \to H$ which yields fibrations when restricted to certain regions in M, see Section 13. For example, we obtain

- A D^2-fibration of a region of M near the edge set E',
- S^2 or T^2-fibrations of a region containing the slim stratum, and
- A surface fibration collaring (the boundary of) the region near 0-stratum points.

Furthermore, it is a feature built into the construction that where the fibered regions overlap, they do so in surfaces with boundary along which the two fibrations are compatible. For instance, the interface between the edge fibration and the 2-stratum fibration is a surface which inherits the same circle fibration from the edge fibration and the 2-stratum fibration. Similarly, the interface between the 2-stratum fibration and the slim 1-stratum fibration is a surface with boundary which inherits a circle fibration from the 2-stratum. See Proposition 14.1 for the properties of the fibrations.

1.5.6. *Recognizing the graph manifold structure.* (Section 15) At this stage of the argument, one has a decomposition of M into domains with disjoint interiors, where each domain is a compact 3-manifold with corners carrying a fibration of a specific kind, with compatibility of fibrations on overlaps. Using the topological classification of the fibers and the 0-stratum domains, one readily reads off the graph manifold structure. This completes the proof of Theorem 1.3.

1.5.7. *Removing the bounds on derivatives of curvature.* (Section 18). The proof of Theorem 1.3 uses the derivative bounds (1.4) only for C^K-precompactness results. In turn these are essentially used only to determine the topology of the 0-stratum balls and the fibers of the edge fibration. Without the derivative bounds (1.4), one can appeal to similar compactness arguments. However, one ends up with a sequence of pointed Riemannian manifolds $\{(M_k, \star_k)\}$ which converge in the pointed Gromov-Hausdorff topology to a pointed 3-dimensional nonnegatively curved Alexandrov space $(M_\infty, \star_\infty)$, rather than having C^K-convergence to a C^K-smooth limit. By invoking Perelman's Stability Theorem [Per91, Kap07], one can relate the topology of the limit space to those of the approximators. The only remaining step is to determine the topology of the nonnegatively curved Alexandrov spaces that arise as limits in this fashion. In the case of noncompact limits, this was done by Shioya-Yamaguchi [SY05]. In the compact case, it follows from Simon [Sim09] or, alternatively, from the Ricci flow proof of the elliptization conjecture (using the finite time extinction results of Perelman and Colding-Minicozzi). For more details, we refer the reader to Section 18.

1.5.8. *What's new in this paper.* The proofs of the collapsing theorems in [SY00, SY05, BBB$^+$10, MT, CG11], as well as the proof in this paper, all begin by comparing the local geometry at a certain scale with the geometry of a nonnegatively curved manifold, and then use this structure to deduce that one has a graph manifold. The paper [BBB$^+$10] follows a rather different line from the other proofs, in that it uses the least amount of the information available from the nonnegatively curved models, and proceeds with a covering argument based on the theory of simplicial volume, as well as Thurston's proof of the geometrization theorem for Haken manifolds. The papers [SY00, SY05, MT, CG11] and this paper have a common overall strategy, which is to use more of the theory of manifolds with nonnegative sectional curvature – Cheeger-Gromoll theory [CG72] and critical point theory [GS77] – to obtain a more refined version of the local models. Then the local models are spliced together to obtain a decomposition of the manifold into fibered regions from which one can recognize a graph manifold.

Overall, our proof uses a minimum of material beyond the theory of nonnegatively curved manifolds. It is essentially elementary in flavor. We now comment on some specific new points in our approach.

The scale function $\mathfrak{r}_p$. The existence of a scale function $\mathfrak{r}_p$ with the properties indicated in Section 1.5.2 makes it apparent that the theory of local collapsing is, at least philosophically, no different than the global version of collapsing.

We work consistently at the scale $\mathfrak{r}_p$, which streamlines the argument. In particular, the structure theory of Alexandrov spaces, which enters if one works at the curvature scale, is largely eliminated. Also, in the selection argument, one considers ball covers where the radii are linked to the scale function $\mathfrak{r}$, so one easily obtains bounds on the intersection multiplicity from the fact that the radii of intersecting balls are comparable (when the scale function $p \mapsto \mathfrak{r}_p$ has small Lipschitz constant). The technique of constructing a scale function with small Lipschitz constant could help in other geometric gluing problems.

The stratification. Stratifications have a long history in geometric analysis, especially for singular spaces such as convex sets, minimal varieties, Alexandrov spaces, and Ricci limit spaces, where one typically looks at the number of $\mathbb{R}$-factors that split off in a tangent cone. The particular stratification that we use, based on the number of $\mathbb{R}$-factors that approximately split off in a manifold, was not used in collapsing theory before, to our knowledge. Its implications for achieving alignment may be useful in other settings.

The gluing procedure. Passing from local models to global fibrations involves some kind of gluing process. Complications arise from the fact one has to construct a global base space for the fibration at the same time as one glues together the fibration maps; in addition, one has to make the fibrations from the different strata compatible. The most obvious approach is to add fibration patches inductively, by using small isotopies and the fact that on overlaps, the fibration maps are nearly affinely equivalent. Then one must perform further isotopies to make fibrations from the different strata compatible with one another. We find the gluing procedure used here to be more elegant; moreover, it produces fibrations which are automatically compatible.

Embeddings into a Euclidean space were used before to construct fibrations in a collapsing setting [Fuk87]. However, there is the important difference that in the earlier work the base of the fibration was already specified, and this base was embedded into a Euclidean space. In contrast, in the present paper we must produce the base at the same time as the fibrations, so we produce it as a submanifold of the Euclidean space.

Cloudy manifolds. The notion of cloudy manifolds, and the proof that they have a good manifold core, may be of independent interest. Cloudy manifolds are similar to objects that have been encountered before, in the work of Reifenberg [Rei60] in geometric measure theory and also in [Pug02]. However the clean elementary argument for the existence of a smooth core given in Appendix B, using the universal bundle and transversality, seems to be new.

1.5.9. *A sketch of the history.* The theory of collapsing was first developed by Cheeger and Gromov [CG86, CG90], assuming both upper and lower bounds on sectional curvature. Their work characterized the degeneration that can occur when one drops the injectivity radius bound in Gromov's compactness theorem, generalizing Gromov's theorem on almost flat manifolds [Gro78]. The corresponding local collapsing structure was used by Anderson and Cheeger-Tian in work on Einstein manifolds [And92, CT06]. As far as we know, the first results on collapsing with a lower curvature bound were announced by Perelman in the early 90's, as an application of the theory of Alexandrov spaces, in particular his Stability Theorem from [Per91] (see also [Kap07]); however, these results were never published. Yamaguchi [Yam91] established a fibration theorem for manifolds close to Riemannian manifolds, under a lower curvature bound. Shioya-Yamaguchi [SY00] studied collapsed 3-manifolds with a diameter bound and showed that they are graph manifolds, apart from an exceptional case. In [Per], Perelman formulated without proof a theorem equivalent to our Theorem 1.3. A short time later, Shioya-Yamaguchi [SY05] proved that – apart from an exceptional case – sufficiently collapsed 3-manifolds are graph manifolds, this time without assuming a diameter bound. This result (or rather the localized version they discuss in their appendix) may be used in lieu of [Per, Theorem 7.4] to complete the proof of the geometrization conjecture. Subsequently, Bessières-Besson-Boileau-Maillot-Porti [BBB+10] gave a different approach to the last part of the proof of the geometrization conjecture, which involves collapsing as well as refined results from 3-dimensional topology. Morgan-Tian [MT] gave a proof of Perelman's collapsing result along the lines of Shioya-Yamaguchi [SY05]. We also mention the paper [CG] by Cao-Ge which relies on more sophisticated Alexandrov space results.

1.6. **Acknowledgements.** We thank Peter Scott for some references to the 3-manifold literature.

2. Notation and conventions

2.1. **Parameters and constraints.** The rest of the paper develops a lengthy construction, many steps of which generate new constants; we will refer to these as *parameters*. Although the parameters remain fixed after being introduced, one should view different sets of parameter values as defining different potential instances of the construction. This is necessary, because several arguments involve consideration of sequences of values for certain parameters, which one should associate with a sequence of distinct instances of the construction.

Many steps of the argument assert that certain statements hold provided that certain constraints on the parameters are satisfied. By convention, each time we refer to such a constraint, we will assume for the remainder of the paper that the inequalities in question are satisfied. Constraint functions will be denoted with a bar, e.g. $\beta_E < \overline{\beta}_E(\beta_1, \sigma)$ means that $\beta_E \in (0, \infty)$ satisfies an upper bound which is a function of β_1 and σ. By convention, all constraint functions take values in $(0, \infty)$.

At the end of the proof of Theorem 1.3, we will verify that the constraints on the various parameters can be imposed consistently. Fortunately, we do not have to carefully adjust each parameter in terms of the others; the constraints are rather of the form that one parameter is sufficiently small (or large) in terms of some others. Hence the only issue is the order in which the parameters are considered.

We follow Perelman's convention that a condition like $a > 0$ means that a should be considered to be a small parameter, while a condition like $A < \infty$ means that A should be considered to be a large parameter. This convention is only for expository purposes and may be ignored by a logically minded reader.

2.2. **Notation.** We will use the following compact notation for cutoff functions with prescribed support. Let $\phi \in C^\infty(\mathbb{R})$ be a nonincreasing function so that $\phi\Big|_{(-\infty,0]} = 1$, $\phi\Big|_{[1,\infty)} = 0$ and $\phi((0,1)) \subset (0,1)$. Given $a, b \in \mathbb{R}$ with $a < b$, we define $\Phi_{a,b} \in C^\infty(\mathbb{R})$ by

$$\Phi_{a,b}(x) = \phi(a + (b-a)x), \tag{2.1}$$

so that $\Phi_{a,b}\Big|_{(-\infty,a]} = 1$ and $\Phi_{a,b}\Big|_{[b,\infty)} = 0$. Given $a, b, c, d \in \mathbb{R}$ with $a < b < c < d$, we define $\Phi_{a,b,c,d} \in C^\infty(\mathbb{R})$ by

$$\Phi_{a,b,c,d}(x) = \phi_{-b,-a}(-x)\, \phi_{c,d}(x), \tag{2.2}$$

so that $\Phi_{a,b,c,d}\Big|_{(-\infty,a]} = 0$, $\Phi_{a,b,c,d}\Big|_{[b,c]} = 1$ and $\Phi_{a,b,c,d}\Big|_{[d,\infty)} = 0$.

If X is a metric space and $0 < r \le R$ then the annulus $A(x, r, R)$ is $\overline{B(x,R)} - B(x,r)$. The dimension of a metric space will always mean the Hausdorff dimension. For notation, if C is a metric cone with basepoint at the vertex $\star$ then we will sometimes just write C for the pointed metric space $(C, \star)$. (Recall that a metric cone is a pointed metric space $(Z, \star)$, which is a union of rays leaving the basepoint $\star$, such that the union of any two such rays is isometric to the union of two rays leaving the origin in $\mathbb{R}^2$.)

If Y is a subset of X and $t : Y \to (0, \infty)$ is a function then we write $N_t(Y)$ for the neighborhood of Y with variable thickness t: $N_t(Y) = \bigcup_{y \in Y} B(y, t(y))$.

If (X, d) is a metric space and $\lambda > 0$ then we write λX for the metric space $(X, \lambda d)$. For notational simplicity, we write $B(p, r) \subset \lambda X$ to denote the r-ball around p in the metric space λX.

Throughout the paper, a product of metric spaces $X_1 \times X_2$ will be endowed with the distance function given by the Pythagorean formula, i.e. if (x_1, x_2), $(y_1, y_2) \in X_1 \times X_2$ then $d_{X_1 \times X_2}((x_1, x_2), (y_1, y_2)) = \sqrt{d^2_{X_1}(x_1, y_1) + d^2_{X_2}(x_2, y_2)}$.

2.3. **Variables.** For the reader's convenience, we tabulate the variables in this paper, listed by the section in which they first appear.

Section 1.2: R_p
Section 1.4.2: $r_p(\cdot)$
Section 6: Λ, σ, $\mathfrak{r}_p$, $\bar{w}$, w'
Section 7.2: β_1, β_2, β_3, Δ
Section 8.1: $\varsigma_{\text{2-stratum}}$, η_p and ζ_p (for 2-stratum points)
Section 8.2: $\mathcal{M}$
Section 9.1: β_E, σ_E, $\beta_{E'}$, $\sigma_{E'}$
Section 9.2: $d_{E'}$, $\rho_{E'}$, $\varsigma_{E'}$
Section 9.3: ς_{edge}, η_p and ζ_p (for edge points)
Section 9.6: ζ_{edge}, $\zeta_{E'}$
Section 10.1: ς_{slim}, η_p and ζ_p (for slim 1-stratum points)

Section 11.1: Υ_0, Υ_0', δ_0, r_p^0
Section 11.2: $\varsigma_{\text{0-stratum}}$, η_p and ζ_p (for 0-stratum points)
Section 12.1: H, H_i, H_i', H_i'', $H_{\text{0-stratum}}$, H_{slim}, H_{edge}, $H_{\text{2-stratum}}$, Q_1, Q_2, Q_3, Q_4, π_i, $\pi_i^\perp$, π_{ij}, $\pi_{H_i'}$, $\pi_{H_i''}$, $\mathcal{E}^0$
Section 12.2: Ω_0
Section 12.3: A_1, $\widetilde{A}_1$, S_1, $\widetilde{S}_1$, Ω_1, Γ_1, Σ_1, r_1, Ω_1
Section 12.4: A_2, $\widetilde{A}_2$, S_2, $\widetilde{S}_2$, Ω_2, Γ_2, Σ_2, r_2, Ω_2
Section 12.5: A_3, $\widetilde{A}_3$, S_3, $\widetilde{S}_3$, Ω_3, Γ_3, Σ_3, r_3, Ω_3
Section 13: c_{adjust}
Section 13.2: W_1^0, Ξ_1, ψ_1, Ψ_1, Ω_1', $\mathcal{E}^1$, $c_{\text{2-stratum}}$
Section 13.3: W_2^0, Ξ_2, ψ_2, Ψ_2, Ω_2', $\mathcal{E}^2$, c_{edge}
Section 13.4: W_3^0, Ξ_3, ψ_3, Ψ_3, Ω_3', $\mathcal{E}^3$, c_{slim}
Section 13.5: W_1, W_2, W_3
Section 14.1: $M^{\text{0-stratum}}$, M_1
Section 14.2: W_3', U_3', W_3'', M^{slim}, M_2
Section 14.3: W_2', U_2', W_2'', M^{edge}, M_3
Section 14.4: W_1', U_1', W_1'', $M^{\text{2-stratum}}$
Section 15: r_∂, H_∂, M_i^∂
Appendix B: S, $\widetilde{S}$, $r(\cdot)$, W

3. Preliminaries

We refer to [BBI01] for basics about length spaces and Alexandrov spaces.

3.1. Pointed Gromov-Hausdorff approximations.

Definition 3.1. Let $(X, \star_X)$ be a pointed metric space. Given $\delta \in [0, \infty)$, two closed subspaces C_1 and C_2 are *δ-close in the pointed Hausdorff sense* if $C_1 \cap \overline{B(\star_X, \delta^{-1})}$ and $C_2 \cap \overline{B(\star_X, \delta^{-1})}$ have Hausdorff distance at most δ.

If X is complete and proper (i.e. closed bounded sets are compact) then the corresponding pointed Hausdorff topology, on the set of closed subspaces of X, is compact and metrizable.

We now recall some definitions and basic results about the pointed Gromov-Hausdorff topology [BBI01, Chapter 8.1].

Definition 3.2. Let $(X, \star_X)$ and $(Y, \star_Y)$ be pointed metric spaces. Given $\delta \in [0, 1)$, a pointed map $f : (X, \star_X) \to (Y, \star_Y)$ is a *δ-Gromov-Hausdorff approximation* if for every $x_1, x_2 \in B(\star_X, \delta^{-1})$ and $y \in B(\star_Y, \delta^{-1} - \delta)$, we have

$$|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| \le \delta \quad \text{and} \quad d_Y(y, f(B(\star_X, \delta^{-1}))) \le \delta \,. \tag{3.3}$$

Two pointed metric spaces $(X, \star_X)$ and $(Y, \star_Y)$ are *δ-close in the pointed Gromov-Hausdorff topology* (or δ-close for short) if there is a δ-Gromov-Hausdorff approximation from $(X, \star_X)$ to $(Y, \star_Y)$. We note that this does not define a metric space structure on the set of pointed metric spaces, but nevertheless defines a topology which happens to be metrizable. A sequence $\{(X_i, \star_i)\}_{i=1}^\infty$ of pointed metric spaces *Gromov-Hausdorff converges* to $(Y, \star_Y)$ if there is a sequence $\{f_i : (X_i, \star_{X_i}) \to (Y, \star_Y)\}_{i=1}^{\infty}$ of δ_i-Gromov-Hausdorff approximations, where $\delta_i \to 0$. We will denote this by $(X_i, \star_{X_i}) \stackrel{\mathrm{GH}}{\to} (Y, \star_Y)$.

Note that a δ-Gromov-Hausdorff approximation is a δ'-Gromov-Hausdorff approximation for every $\delta' \geq \delta$. A δ-Gromov-Hausdorff approximation f has a *quasi-inverse* $\widehat{f}$: $(Y, \star_Y) \to (X, \star_X)$ constructed by saying that for $y \in B(\star_Y, \delta^{-1} - \delta)$, we choose some $x \in B(\star_X, \delta^{-1})$ with $d_Y(y, f(x)) \leq \delta$ and put $\widehat{f}(y) = x$. There is a function $\delta' = \delta'(\delta) > 0$ with $\lim_{\delta \to 0} \delta' = 0$ so that if f is a δ-Gromov-Hausdorff approximation then $\widehat{f}$ is a δ'-Gromov-Hausdorff approximation and $\widehat{f} \circ f$ (resp. $f \circ \widehat{f}$) is δ'-close to the identity on $B(\star_X, (\delta')^{-1})$ (resp. $B(\star_Y, (\delta')^{-1})$). The condition $(X_i, \star_{X_i}) \stackrel{\mathrm{GH}}{\to} (Y, \star_Y)$ is equivalent to the existence of a sequence $\{f_i : (Y, \star_Y) \to (X_i, \star_{X_i})\}_{i=1}^{\infty}$ (note the reversal of domain and target) of δ_i-Gromov-Hausdorff approximations, where $\delta_i \to 0$.

The relation of being δ-close is not symmetric. However, this does not create a problem because only the associated notion of convergence (i.e. the topology) plays a role in our discussion.

The pointed Gromov-Hausdorff topology is a complete metrizable topology on the set of complete proper metric spaces (taken modulo pointed isometry). Hence we can talk about two such metric spaces being having distance at most δ from each other. There is a well-known criterion for a set of pointed metric spaces to be precompact in the pointed Gromov-Hausdorff topology [BBI01, Theorem 8.1.10]. Complete proper *length spaces*, which are the main interest of this paper, form a closed subset of the set of complete proper metric spaces.

3.2. C^K-convergence.

Definition 3.4. Given $K \in \mathbb{Z}^+$, let $(M_1, \star_{M_1})$ and $(M_2, \star_{M_2})$ be complete pointed C^K-smooth Riemannian manifolds. (That is, the manifold transition maps are C^{K+1} and the metric in local coordinates is C^K). Given $\delta \in [0, \infty)$, a pointed C^{K+1}-smooth map f : $(M_1, \star_{M_1}) \to (M_2, \star_{M_2})$ is a *δ-C^K approximation* if it is a δ-Gromov-Hausdorff approximation and the C^K-norm of $f^* g_{M_2} - g_{M_1}$, computed on $B(\star_{M_1}, \delta^{-1})$, is bounded above by δ. Two C^K-smooth Riemannian manifolds $(M_1, \star_{M_1})$ and $(M_2, \star_{M_2})$ are *δ-C^K close* if there is a δ-C^K approximation from $(M_1, \star_{M_1})$ to $(M_2, \star_{M_2})$.

In what follows, we will always take $K \geq 10$. We now state a C^K-precompactness result.

Lemma 3.5. *(cf.* [Pet06, Chapter 10]*) Given $v, r > 0$, $n \in \mathbb{Z}^+$ and a function A : $(0, \infty) \to (0, \infty)$, the set of complete pointed C^{K+2}-smooth n-dimensional Riemannian manifolds $(M, \star_M)$ such that*

(1) $\mathrm{vol}(B(\star_M, r)) \geq v$ *and*
(2) $|\nabla^k \mathrm{Rm}| \leq A(R)$ *on $B(\star_M, R)$, for all $0 \leq k \leq K$ and $R > 0$,*

is precompact in the pointed C^K-topology.

The bounds on the derivatives of curvature in Lemma 3.5 give uniform C^{K+1}-bounds on the Riemannian metric in harmonic coordinates. One then obtains limit metrics which are C^K-smooth. One can get improved regularity but we will not need it.

3.3. **Alexandrov spaces.** Recall that there is a notion of an Alexandrov space of curvature at least c, or equivalently a complete length space X having curvature bounded below by $c \in \mathbb{R}$ on an open set $U \subset X$ [BBI01, Chapter 4]. In this paper we will only be concerned with Alexandrov spaces of finite Hausdorff dimension, so this will be assumed implicitly without further mention.

We will also have occasion to work with incomplete, but locally complete spaces. This situation typically arises when one has a metric space X where $X = B(p, r)$, and every closed ball $\overline{B(p, r')}$ with $r' < r$ is complete. The version of Toponogov's theorem for Alexandrov spaces [BBI01, Chapter 10.3], in which one deduces global triangle comparison inequality from local ones, also applies in the incomplete situation, provided that the geodesics arising in the proof lie in an *a priori* complete part of the space. In particular, if all sides of a geodesic triangle have length $< D$ then triangle comparison is valid provided that the closed balls of radius $2D$ centered at the vertices are complete.

We recall the notion of a strainer (cf. [BBI01, Definition 10.8.9]).

Definition 3.6. Given a point p in an Alexandrov space X of curvature at least c, an *m-strainer at p of quality δ and scale r* is a collection $\{(a_i, b_i)\}_{i=1}^m$ of pairs of points such that $d(p, a_i) = d(p, b_i) = r$ and in terms of comparison angles,

$$\begin{aligned} \widetilde{\angle}_p(a_i, b_i) &> \pi - \delta, \\ \widetilde{\angle}_p(a_i, a_j) &> \frac{\pi}{2} - \delta, \\ \widetilde{\angle}_p(a_i, b_j) &> \frac{\pi}{2} - \delta, \\ \widetilde{\angle}_p(b_i, b_j) &> \frac{\pi}{2} - \delta \end{aligned} \tag{3.7}$$

for all $i, j \in \{1, \ldots, m\}$, $i \neq j$. Note that the comparison angles are defined using comparison triangles in the model space of constant curvature c.

For facts about strainers, we refer to [BBI01, Chapter 10.8.2]. The Hausdorff dimension of X equals its strainer number, which is defined as follows.

Definition 3.8. The *strainer number of* X is the supremum of numbers m such that there exists an m-strainer of quality $\frac{1}{100m}$ at some point and some scale.

By "dimension of X" we will mean the Hausdorff dimension; this coincides with its topological dimension, although we will not need this fact. If $(X, \star_X)$ is a pointed nonnegatively curved Alexandrov space then there is a pointed Gromov-Hausdorff limit $C_T X = \lim_{\lambda \to \infty} \left(\frac{1}{\lambda} X, \star_X \right)$ called the *Tits cone* of X. It is a nonnegatively curved Alexandrov space which is a metric cone, as defined in Subsection 2.2. We will consider Tits cones in the special case when X is a nonnegatively curved Riemannian manifold.

A *line* in a length space X is a curve $\gamma : \mathbb{R} \to X$ with the property that $d_X(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in \mathbb{R}$. The splitting theorem (see [BBI01, Chapter 10.5]) says that if a nonnegatively curved Alexandrov space X contains a line then it is isometric to $\mathbb{R} \times Y$ for some nonnegatively curved Alexandrov space Y.

If $\gamma : [0,T] \to X$ is a minimal geodesic in an Alexandrov space X, parametrized by arc-length, and $p \neq \gamma(0)$ is a point in X then the function $t \to d_p(\gamma(t))$ is right-differentiable and

$$\lim_{t\to 0^+} \frac{d_p(\gamma(t)) - d_p(\gamma(0))}{t} = -\cos\theta, \tag{3.9}$$

where θ is the minimal angle between γ and minimizing geodesics from $\gamma(0)$ to p [BBI01, Corollary 4.5.7].

The proof of the next lemma is similar to that of [BBI01, Theorem 10.7.2].

Lemma 3.10. *Given $n \in \mathbb{Z}^+$, let $\{(X_i, \star_{X_i})\}_{i=1}^\infty$ be a sequence of complete pointed length spaces. Suppose that $c_i \to 0$ and $r_i \to \infty$ are positive sequences such that for each i, the ball $B(\star_{X_i}, r_i)$ has curvature bounded below by $-c_i$ and dimension bounded above by n. Then a subsequence of the $(X_i, \star_{X_i})$'s converges in the pointed Gromov-Hausdorff topology to a pointed nonnegatively curved Alexandrov space of dimension at most n.*

3.4. **Critical point theory.** We recall a few facts about critical point theory here, and refer the reader to [GS77, Che91] for more information.

If M is a complete Riemannian manifold and $p \in M$, then a point $x \in M \setminus \{p\}$ is *noncritical* if there is a nonzero vector $v \in T_xM$ making an angle strictly larger than $\frac{\pi}{2}$ with the initial velocity of every minimizing segment from x to p. If there are no critical points in the set $d_p^{-1}(a,b)$ then the level sets $\{d_p^{-1}(t)\}_{t\in(a,b)}$, are pairwise isotopic Lipschitz hypersurfaces, and $d_p^{-1}(a,b)$ is diffeomorphic to a product, as in the usual Morse lemma for smooth functions. As with the traditional Morse lemma, the proof proceeds by constructing a smooth vector field ξ such that d_p has uniformly positive directional derivative in the direction ξ.

3.5. **Topology of nonnegatively curved 3-manifolds.** In this subsection we describe the topology of certain nonnegatively curved manifolds. We start with 3-manifolds.

Lemma 3.11. *Let M be a complete connected orientable 3-dimensional C^K-smooth Riemannian manifold with nonnegative sectional curvature. We have the following classification of the diffeomorphism type of M, based on the number of ends :*

- 0 *ends:* $S^1 \times S^2$, $S^1 \times_{\mathbb{Z}_2} S^2 = \mathbb{R}P^3 \# \mathbb{R}P^3$, T^3/Γ *(where Γ is a finite subgroup of* $\mathrm{Isom}^+(T^3)$ *which acts freely on T^3) or S^3/Γ (where Γ is a finite subgroup of* $\mathrm{SO}(4)$ *that acts freely on S^3).*
- 1 *end:* $\mathbb{R}^3$, $S^1 \times \mathbb{R}^2$, $S^2 \times_{\mathbb{Z}_2} \mathbb{R} = \mathbb{R}P^3 - B^3$ *or* $T^2 \times_{\mathbb{Z}_2} \mathbb{R} =$ *a flat* $\mathbb{R}$*-bundle over the Klein bottle.*
- 2 *ends:* $S^2 \times \mathbb{R}$ *or* $T^2 \times \mathbb{R}$.

If M has two ends then it splits off an $\mathbb{R}$-factor isometrically.

Proof. If M has no end then it is compact and the result follows for C^∞-metrics from [Ham86]. For C^K-smooth metrics, one could adapt the argument in [Ham86] or alternatively use [Sim09].

If M is noncompact then the Cheeger-Gromoll soul theorem says that M is diffeomorphic to the total space of a vector bundle over its soul, a closed lower-dimensional manifold with nonnegative sectional curvature [CG72]. (The proof in [CG72], which is for C^∞-metrics, goes through without change for C^K-smooth metrics.) The possible dimensions of the soul are 0, 1 and 2. The possible topologies of M are listed in the lemma.

If M has two ends then it contains a line and the Toponogov splitting theorem implies that M splits off an $\mathbb{R}$-factor isometrically. □

We now look at a pointed nonnegatively curved surface and describe the topology of a ball in it which is pointed Gromov-Hausdorff close to an interval.

Lemma 3.12. *Suppose that $(S, \star_S)$ is a pointed C^K-smooth nonnegatively curved complete orientable Riemannian 2-manifold. Let $\star_S \in S$ be a basepoint and suppose that the pointed ball $(B(\star_S, 10), \star_S)$ has pointed Gromov-Hausdorff distance at most δ from the pointed interval $([0, 10], 0)$.*

(1) *Given $\theta > 0$ there is some $\overline{\delta}(\theta) > 0$ so that if $\delta < \overline{\delta}(\theta)$ then for every $x \in \overline{B(\star_S, 9)} - B(\star_S, 1)$ the set V_x of initial velocities of minimizing geodesic segments from x to $\star_S$ has diameter bounded above by θ.*
(2) *There is some $\overline{\delta} > 0$ so that if $\delta < \overline{\delta}$ then for every $r \in [1, 9]$ the ball $\overline{B(\star_S, r)}$ is homeomorphic to a closed 2-disk.*

Proof. (1). Choose a point x' with $d_S(\star_S, x') = 9.5$. Fix a minimizing geodesic γ' from x to x' and a minimizing geodesic γ'' from $\star_S$ to x'. If γ is a minimizing geodesic from $\star_S$ to x, consider the geodesic triangle with edges γ, γ' and γ''. As

$$d(\star_S, x) + d(x, x') - d(\star_S, x'') \leq \text{const.}\, \delta, \tag{3.13}$$

triangle comparison implies that the angle at x between γ and γ' is bounded below by $\pi - a(\delta)$, where a is a positive monotonic function with $\lim_{\delta \to 0} a(\delta) = 0$. We take $\overline{\delta}$ so that $2a(\overline{\delta}) \leq \theta$,

(2). Suppose that $\delta < \overline{\delta}(\frac{\pi}{4})$. By critical point theory, the distance function $d_{\star_S} : A(\star_S, 1, 9) \to [1, 9]$ is a fibration with fibers diffeomorphic to a disjoint union of circles. In particular, the closed balls $\overline{B(\star_S, r)}$, for $r \in [1, 9]$, are pairwise homeomorphic. When $\delta \ll 1$, the fibers will be connected, since the diameter of $d_{\star_S}^{-1}(5)$ will be comparable to δ. Hence $\overline{B(\star_S, 1)}$ is homeomorphic to a surface with circle boundary.

Suppose that $\overline{B(\star_S, 1)}$ is not homeomorphic to a disk. A complete connected orientable nonnegatively curved surface is homeomorphic to S^2, T^2, $\mathbb{R}^2$, or $S^1 \times \mathbb{R}$. By elementary topology, the only possibility is if S is homeomorphic to a 2-torus, $\overline{B(\star_S, 1)}$ is homeomorphic to the complement of a 2-ball in S, and $S - B(\star_S, 2)$ is homeomorphic to a disk. In this case, S must be flat. However, the cylinder $A(\star_S, 1, 2)$ lifts to the universal cover of S, which is isometric to the flat $\mathbb{R}^2$. If δ is sufficiently small then the flat $\mathbb{R}^2$ would contain a metric ball of radius $\frac{1}{10}$ which is Gromov-Hausdorff close to an interval, giving a contradiction. □

3.6. **Smoothing Lipschitz functions.** The technique of smoothing Lipschitz functions was introduced in Riemannian geometry by Grove and Shiohama [GS77].

If M is a Riemannian manifold and F is a Lipschitz function on M then the generalized gradient of F at $m \in M$ can be defined as follows. Given $\epsilon \in (0, \mathrm{InjRad}_m)$, if $x \in B(m, \epsilon)$ is a point of differentiability of F then compute $\nabla_x F \in T_x M$ and parallel transport it along the minimizing geodesic to m. Take the closed convex hull of the vectors so obtained and then take the intersection as $\epsilon \to 0$. This gives a closed convex subset of $T_m M$, which is the generalized gradient of F at m [Cla90]; we will denote this set by $\nabla^{\mathrm{gen}}_m F$. The union $\bigcup_{m \in M} \nabla^{\mathrm{gen}}_m F \subset TM$ will be denoted $\nabla^{\mathrm{gen}} F$.

Lemma 3.14. *Let M be a complete Riemannian manifold and let $\pi : TM \to M$ be the projection map. Suppose that $U \subset M$ is an open set, $C \subset U$ is a compact subset and S is an open fiberwise-convex subset of $\pi^{-1}(U)$.*

Then for every $\epsilon > 0$ and any Lipschitz function $F : M \to \mathbb{R}$ whose generalized gradient over U lies in S, there is a Lipschitz function $\widehat{F} : M \to \mathbb{R}$ such that:

1. *$\widehat{F}$ is C^∞ on an open set containing C.*
2. *The generalized gradient of $\widehat{F}$, over U, lies in S. (In particular, at every point in U where $\widehat{F}$ is differentiable, the gradient lies in S.)*
3. *$|\widehat{F} - F|_\infty \le \epsilon$.*
4. *$\widehat{F}\big|_{M-U} = F\big|_{M-U}$.*

The proof of Lemma 3.14 proceeds by mollifying the Lipschitz function F, as in [GS77, Section 2]. We omit the details.

Corollary 3.15. *Suppose that M is a compact Riemannian manifold. Given $K < \infty$ and $\epsilon > 0$, for any K-Lipschitz function F on M there is a $(K + \epsilon)$-Lipschitz function $\widehat{F} \in C^\infty(M)$ with $|\widehat{F} - F|_\infty \le \epsilon$.*

Proof. Apply Lemma 3.14 with $C = U = M$, and $S = \{v \in TM : |v| < K + \epsilon\}$. □

Corollary 3.16. *For all $\epsilon > 0$ there is a $\theta > 0$ with the following property.*

Let M be a complete Riemannian manifold, let $Y \subset M$ be a closed subset and let $d_Y : M \to \mathbb{R}$ be the distance function from Y. Given $p \in M - Y$, let $V_p \subset T_p M$ be the set of initial velocities of minimizing geodesics from p to Y. Suppose that $U \subset M - Y$ is an open subset such that for all $p \in U$, one has $\mathrm{diam}(V_p) < \theta$. Let C be a compact subset of U. Then for every $\epsilon_1 > 0$ there is a Lipschitz function $\widehat{F} : M \to \mathbb{R}$ such that

1. *$\widehat{F}$ is smooth on a neighborhood of C.*
2. *$\|\widehat{F} - d_Y\|_\infty < \epsilon_1$.*
3. *$\widehat{F}\big|_{M-U} = d_Y\big|_{M-U}$.*
4. *For every $p \in C$, the angle between $-(\nabla \widehat{F})(p)$ and V_p is at most ϵ.*
5. *$\widehat{F} - d_Y$ is ϵ-Lipschitz.*

Proof. First, note that if $p \in M - Y$ is a point of differentiability of the distance function d_Y then $\nabla_p d_Y = -V_p$. Also, the assignment $x \mapsto V_x$ is semicontinuous in the sense that if $\{x_k\}_{k=1}^\infty$ is a sequence of points converging to x then by parallel transporting V_{x_k} radially to the fiber over x, we obtain a sequence $\{\bar{V}_{x_k}\}_{k=1}^\infty \subset T_x M$ which accumulates on a subset of V_x. It follows that the generalized derivative of d_Y at any point $p \in M - Y$ is precisely $-\operatorname{Hull}(V_p)$, where $\operatorname{Hull}(V_p)$ denotes the convex hull of V_p.

Put $S' = \bigcup_{p \in U} \operatorname{Hull}(-V_p)$. Then S' is a relatively closed fiberwise-convex subset of $\pi^{-1}(U)$, with fibers of diameter less than θ. We can fatten S' slightly to form an open fiberwise-convex set $S \subset \pi^{-1}(U)$ which contains S', with fibers of diameter less than 2θ.

Now take $\theta < \frac{\epsilon}{2}$ and apply Lemma 3.14 to $F = d_Y : M \to \mathbb{R}$, with S as in the preceding paragraph. The resulting function $\widehat{F} : M \to \mathbb{R}$ clearly satisfies (1)-(4). To see that (5) holds, note that if $p \in U$ is a point of differentiability of both $\widehat{F}$ and F then $\nabla \widehat{F}(p)$ and $\nabla F(p)$ both lie in the fiber $S \cap T_p M$, which has diameter less than 2θ. Hence the gradient of the difference satisfies

$$\|\nabla(\widehat{F} - F)(p)\| = \|\nabla \widehat{F}(p) - \nabla F(p)\| < 2\theta < \epsilon \,. \tag{3.17}$$

Since $\widehat{F}$ coincides with F outside U, this implies that $\widehat{F} - F$ is ϵ-Lipschitz. □

Remark 3.18. When we apply Corollary 3.16, the hypotheses will be verified using triangle comparison.

4. Splittings, strainers, and adapted coordinates

This section is about the notion of a pointed metric space approximately splitting off an $\mathbb{R}^k$-factor. We first define an approximate $\mathbb{R}^k$-splitting, along with the notion of compatibility between an approximate $\mathbb{R}^k$-splitting and an approximate $\mathbb{R}^j$-splitting. We prove basic properties about approximate splittings. In the case of a pointed Alexandrov space, we show that having an approximate $\mathbb{R}^k$-splitting is equivalent to having a good k-strainer. We show that if there is not an approximate $\mathbb{R}^{k+1}$-splitting at a point p then any approximate $\mathbb{R}^k$-splitting at p is nearly-compatible with any approximate $\mathbb{R}^j$-splitting at p, for $j \le k$.

We then introduce the notion of coordinates adapted to an approximate $\mathbb{R}^k$-splitting, in the setting of Riemannian manifolds with a lower curvature bound, proving existence and (approximate) uniqueness of such adapted coordinates.

4.1. Splittings.

We start with the notion of a splitting.

Definition 4.1. A product structure on a metric space X is an isometry $\alpha : X \to X_1 \times X_2$. A *$k$-splitting* of X is a product structure $\alpha : X \to X_1 \times X_2$ where X_1 is isometric to $\mathbb{R}^k$. A *splitting* is a k-splitting for some k. Two k-splittings $\alpha : X \to X_1 \times X_2$ and $\beta : X \to Y_1 \times Y_2$ are *equivalent* if there are isometries $\phi_i : X_i \to Y_i$ such that $\beta = (\phi_1, \phi_2) \circ \alpha$.

In addition to equivalence of splittings, we can talk about compatibility of splittings.

Definition 4.2. Suppose that $j \leq k$. A j-splitting $\alpha : X \to X_1 \times X_2$ is *compatible* with a k-splitting $\beta : X \to Y_1 \times Y_2$ if there is a j-splitting $\phi : Y_1 \to \mathbb{R}^j \times \mathbb{R}^{k-j}$ such that α is equivalent to the j-splitting given by the composition

$$X \xrightarrow{\beta} Y_1 \times Y_2 \xrightarrow{(\phi,\mathrm{Id})} (\mathbb{R}^j \times \mathbb{R}^{k-j}) \times Y_2 \cong \mathbb{R}^j \times (\mathbb{R}^{k-j} \times Y_2). \tag{4.3}$$

Lemma 4.4.

(1) *Suppose $\alpha : X \to \mathbb{R}^k \times Y$ is a k-splitting of a metric space X, and $\beta : X \to \mathbb{R} \times Z$ is a 1-splitting. Then either β is compatible with α, or there is a 1-splitting $\gamma : Y \to \mathbb{R} \times W$ such that β is compatible with the induced splitting $X \to (\mathbb{R}^k \times \mathbb{R}) \times W$.*
(2) *Any two splittings of a metric space are compatible with a third splitting.*

Before proving this, we need a sublemma.

Recall that a *line* in a metric space is a globally minimizing complete geodesic, i.e. an isometrically embedded copy of $\mathbb{R}$. We will say that two lines are *parallel* if their union is isometric to the union of two parallel lines in $\mathbb{R}^2$.

Sublemma 4.5.

(1) *A path $\gamma : \mathbb{R} \to X_1 \times X_2$ in a product is a constant speed geodesic if and only if the compositions $\pi_{X_i} \circ \gamma : \mathbb{R} \to X_i$ are constant speed geodesics.*
(2) *Two lines in a metric space X are parallel if and only if they have constant speed parametrizations $\gamma_1 : \mathbb{R} \to X$ and $\gamma_2 : \mathbb{R} \to X$ such that $d^2(\gamma_1(s), \gamma_2(t))$ is a quadratic function of $(s-t)$.*
(3) *If two lines γ_1, γ_2 in a product $\mathbb{R}^k \times X$ are parallel then either $\pi_X(\gamma_1), \pi_X(\gamma_2) \subset X$ are parallel lines, or they are both points.*
(4) *Suppose $\mathcal{L}$ is a collection of lines in a metric space X. If $\bigcup_{\gamma \in \mathcal{L}} \gamma = X$, and every pair $\gamma_1, \gamma_2 \in \mathcal{L}$ is parallel, then there is a 1-splitting $\alpha : X \to \mathbb{R} \times Y$ such that $\mathcal{L} = \{\alpha^{-1}(\mathbb{R} \times \{y\})\}_{y \in Y}$.*

Proof. (1). It follows from the Cauchy-Schwarz inequality that if $a, b, c \in X_1 \times X_2$ satisfy the triangle equation $d(a,c) = d(a,b) + d(b,c)$ then the same is true of their projections $a_1, b_1, c_1 \in X_1$ and $a_2, b_2, c_2 \in X_2$, and moreover $(d(a_1,b_1), d(a_2,b_2))$ and $(d(a_1,c_1), d(a_2,c_2))$ are linearly dependent in $\mathbb{R}^2$. This implies (1).

(2). The parallel lines $y = 0$ and $y = a$ in $\mathbb{R}^2$ can be parametrized by $\gamma_1(s) = (s, 0)$ and $\gamma_2(t) = (t, a)$, with $d^2(\gamma_1(s), \gamma_2(t)) = (s-t)^2 + a^2$. Conversely, suppose that lines γ_1 and γ_2 in a metric space are such that $d^2(\gamma_1(s), \gamma_2(t))$ is quadratic in $(s-t)$. After affine changes of s and t, we can assume that $d^2(\gamma_1(s), \gamma_2(t)) = (s-t)^2 + a^2$ for some $a \in \mathbb{R}$. Then the union of γ_1 and γ_2 is isometric to the union of the lines $y = 0$ and $y = a$ in $\mathbb{R}^2$.

(3). By (2), we may assume that for $i \in \{1, 2\}$ there are constant speed parametrizations $\gamma_i : \mathbb{R} \to \mathbb{R}^k \times X$, such that $d^2(\gamma_1(s), \gamma_2(t))$ is a quadratic function of $(s-t)$. The projections $\pi_{\mathbb{R}^k} \circ \gamma_i$ are constant speed geodesics in $\mathbb{R}^k$, and the quadratic function $d^2(\pi_{\mathbb{R}^k} \circ \gamma_1(s), \pi_{\mathbb{R}^k} \circ \gamma_2(t))$ is a function of $(s-t)$; otherwise $d^2(\pi_{\mathbb{R}^k} \circ \gamma_1(s), \pi_{\mathbb{R}^k} \circ \gamma_2(s))$ would be unbounded in s, which contradicts that $d^2(\gamma_1(s), \gamma_2(s))$ is constant in s. Therefore

$$d^2(\pi_X \circ \gamma_1(s), \pi_X \circ \gamma_2(t)) = d^2(\gamma_1(s), \gamma_2(t)) - d^2(\pi_{\mathbb{R}^k} \circ \gamma_1(s), \pi_{\mathbb{R}^k} \circ \gamma_2(t)) \tag{4.6}$$

is a quadratic function of $(s-t)$. By (2) we conclude that $\pi_X \circ \gamma_1$, $\pi_X \circ \gamma_2$ are parallel.

(4). Let $\gamma : \mathbb{R} \to X$ be a unit speed parametrization of some line in $\mathcal{L}$, and let $b : X \to \mathbb{R}$ be the Busemann function $\lim_{t\to\infty} \; d(\gamma(t), \cdot) - t$. By assumption, the elements of $\mathcal{L}$ partition X into the cosets of an equivalence relation; the quotient Y inherits a natural metric, namely the Hausdorff distance. The map $(b, \pi_Y) : X \to \mathbb{R} \times Y$ defines a 1-splitting – one verifies that it is an isometry using the fact that $\mathcal{L}$ consists of parallel lines. □

Proof of Lemma 4.4.

(1). Consider the collection of lines $\mathcal{L}_\beta = \{\beta^{-1}(\mathbb{R} \times \{z\}) \mid z \in Z\}$. By Lemma 4.5(3) it follows that $\pi_Y \circ \alpha(\mathcal{L}_\beta)$ consists of parallel lines, or consists entirely of points.

Case 1. $\pi_Y \circ \alpha(\mathcal{L}_\beta)$ consists of points. In this case, Sublemma 4.5 implies that $\pi_{\mathbb{R}^k} \circ \alpha(\mathcal{L}_\beta)$ is a family of parallel lines in $\mathbb{R}^k$. Decomposing $\mathbb{R}^k$ into a product $\mathbb{R}^k \simeq \mathbb{R}^1 \times \mathbb{R}^{k-1}$ in the direction defined by $\pi_{\mathbb{R}^k} \circ \alpha(\mathcal{L}_\beta)$, we obtain a 1-splitting of X which is easily seen to be equivalent to β.

Case 2. $\pi_Y \circ \alpha(\mathcal{L}_\beta)$ consists of parallel lines. Since $\bigcup_{\gamma \in \pi_Y \circ \alpha(\mathcal{L}_\beta)} \gamma = Y$, by Lemma 4.5(4), there is a 1-splitting $\gamma : Y \to \mathbb{R} \times W$ such that $\{\gamma^{-1}(\mathbb{R} \times \{w\}) \mid w \in W\} = \pi_Y \circ \alpha(\mathcal{L}_\beta)$. Letting $\alpha' : X \to (\mathbb{R}^k \times \mathbb{R}) \times W$ be the $(k+1)$-splitting given by $X \stackrel{\alpha}{\to} \mathbb{R}^k \times Y \to (\mathbb{R}^k \times \mathbb{R}) \times W$, we get that $\pi_W \circ \alpha'(\mathcal{L}_\beta)$ consists of points, so by Case 1 it follows that β is compatible with α'.

(2). Suppose that $\alpha : X \to \mathbb{R}^k \times Y$ and $\beta : X \to \mathbb{R}^l \times Z$ are splittings. We may apply part (1) to α and the 1-splitting obtained from the i^{th} coordinate direction of β, for successive values of $i \in \{1, \ldots, l\}$. This will enlarge α to a splitting α' which is compatible with all of these 1-splittings, and clearly α' is then compatible with β. □

4.2. **Approximate splittings.** Next, we consider approximate splittings.

Definition 4.7. Given $k \in \mathbb{Z}^{\geq 0}$ and $\delta \in [0, \infty)$, a (k, δ)-*splitting* of a pointed metric space $(X, \star_X)$ is a δ-Gromov-Hausdorff approximation $(X, \star_X) \to (X_1, \star_{X_1}) \times (X_2, \star_{X_2})$, where $(X_1, \star_{X_1})$ is isometric to $(\mathbb{R}^k, \star_{\mathbb{R}^k})$. (We allow $\mathbb{R}^k$ to have other basepoints than 0.)

There are "approximate" versions of equivalence and compatibility of splittings.

Definition 4.8. Suppose that $\alpha : (X, \star_X) \to (X_1, \star_{X_1}) \times (X_2, \star_{X_2})$ is a (j, δ_1)-splitting and $\beta : (X, \star_X) \to (Y_1, \star_{Y_1}) \times (Y_2, \star_{Y_2})$ is an (k, δ_2)-splitting. Then

(1) α *is ϵ-close to* β if $j = k$ and there are ϵ-Gromov-Hausdorff approximations $\phi_i : (X_i, \star_{X_i}) \to (Y_i, \star_{Y_i})$ such that the composition $(\phi_1, \phi_2) \circ \alpha$ is ϵ-close to β, i.e. agrees with β on $B(\star_X, \epsilon^{-1})$ up to error at most ϵ.
(2) α *is ϵ-compatible with* β if $j \leq k$ and there is a j-splitting $\gamma : (Y_1, \star_{Y_1}) \to (\mathbb{R}^j, \star_{\mathbb{R}^j}) \times (\mathbb{R}^{k-j}, \star_{\mathbb{R}^{k-j}})$ such that the (j, δ_2)-splitting defined by the composition

$$X \stackrel{\beta}{\to} Y_1 \times Y_2 \stackrel{(\gamma, \mathrm{Id})}{\to} (\mathbb{R}^j \times \mathbb{R}^{k-j}) \times Y_2 \cong \mathbb{R}^j \times (\mathbb{R}^{k-j} \times Y_2) \tag{4.9}$$

is ϵ-close to α.

Lemma 4.10. *Given $\delta > 0$ and $C < \infty$, there is a $\delta' = \delta'(\delta, C) > 0$ with the following property. Suppose that $(X, \star_X)$ is a complete pointed metric space with a (k, δ')-splitting α. Then for any $x \in B(\star_X, C)$, the pointed space (X, x) has a (k, δ)-splitting coming from a change of basepoint of α.*

Proof. In general, suppose that $f : (X, \star_X) \to (Y, \star_Y)$ is a δ'-Gromov-Hausdorff approximation. Given $x \in B(\star_X, C)$, consider x to be a new basepoint. Note that

$$d(\star_Y, f(x)) \le d(\star_X, x) + \delta' \le C + \delta'. \tag{4.11}$$

Suppose that δ satisfies

(1) $\delta^{-1} \le (\delta')^{-1} - C$,
(2) $\delta^{-1} - \delta \le (\delta')^{-1} - 2\delta' - C$ and
(3) $\delta > 2\delta'$.

We claim that f is a δ-Gromov-Hausdorff between (X, x) and $(Y, f(x))$. To see this, first if $x_1, x_2 \in B(x, \delta^{-1})$ then $x_1, x_2 \in B(\star_X, (\delta')^{-1}$ and so

$$|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| \le \delta' \le \delta. \tag{4.12}$$

Next, if $y \in B(f(x), \delta^{-1} - \delta)$ then $y \in B(\star_Y, (\delta')^{-1} - \delta'))$ and so there is some $\widehat{x} \in B(\star_X, (\delta')^{-1})$ with $d(y, f(\widehat{x})) \le \delta'$. Now

$$d(x, \widehat{x}) \le d(f(x), f(\widehat{x})) + \delta' \le d(f(x), y) + d(y, f(\widehat{x})) + \delta' \le \delta^{-1} - \delta + 2\delta' < \delta^{-1}, \tag{4.13}$$

which proves the claim.

The lemma now follows provided that we specialize to the case when Y splits off an $\mathbb{R}^k$-factor. □

Lemma 4.14. *Given $\delta > 0$ and $C < \infty$, there is a $\delta' = \delta'(\delta, C) > 0$ with the following property. Suppose that $(X, \star_X)$ is a complete pointed metric space with a (k_1, δ')-splitting α_1 and a (k_2, δ')-splitting α_2. Suppose that α_1 is δ'-compatible with α_2. Given $x \in B(\star_X, C)$, let α_1', α_2' be the approximate splittings of (X, x) coming from a change of basepoint in α_1, α_2. Then α_1' and α_2' are δ-compatible.*

Proof. The proof is similar to that of Lemma 4.10. We omit the details. □

4.3. **Approximate splittings of Alexandrov spaces.** Recall the notion of a point in an Alexandrov space having a k-strainer of a certain size and quality; see Subsection 3.3. The next lemma shows that the notions of having a good strainer and having a good approximate $\mathbb{R}^k$-splitting are essentially equivalent for Alexandrov spaces.

Lemma 4.15. (1) *Given $k \in \mathbb{Z}^+$ and $\delta > 0$, there is a $\delta' = \delta'(k, \delta) > 0$ with the following property. Suppose that $(X, \star_X)$ is a complete pointed nonnegatively curved Alexandrov space with a (k, δ')-splitting. Then $\star_X$ has a k-strainer of quality δ at a scale $\frac{1}{\delta}$.*

(2) *Given $n \in \mathbb{Z}^+$ and $\delta > 0$, there is a $\delta' = \delta'(n, \delta) > 0$ with the following property. Suppose that $(X, \star_X)$ is an complete pointed length space so that $B\left(\star_X, \frac{1}{\delta'}\right)$ has curvature bounded below by $-\delta'$ and dimension bounded above by n. Suppose that for some $k \le n$, $\star_X$ has a k-strainer $\{p_i^{\pm}\}_{i=1}^k$ of quality δ' at a scale $\frac{1}{\delta'}$. Then $(X, \star_X)$ has a (k, δ)-splitting $\phi : (X, \star_X) \to (\mathbb{R}^k \times X', (0, \star_{X'}))$ where the composition $\pi_{\mathbb{R}^k} \circ \phi$ has j^{th} component $d_X(p_j^+, \star_X) - d_X(p_j^+, \cdot)$.*

Proof. The proof of (1) is immediate from the definitions.

Suppose that (2) were false. Then for each $i \in \mathbb{Z}^+$, there is an complete pointed length space $(X_i, \star_{X_i})$ so that

(1) $B(\star_{X_i}, i)$ has dimension at most n,
(2) $B(\star_{X_i}, i)$ has curvature bounded below by $-\frac{1}{i}$ and
(3) $\star_{X_i}$ has a k-strainer of quality $\frac{1}{i}$ at a scale i but
(4) If $\Phi_i : (X_k, \star_{X_i}) \to \mathbb{R}^k$ has j^{th} component defined as above, then Φ_i is not the $\mathbb{R}^k$ part of a (k, δ)-splitting for any i.

After passing to a subsequence, we can assume that $\lim_{i\to\infty}(X_i, \star_{X_i}) = (X_\infty, \star_{X_\infty})$, for some pointed nonnegatively curved Alexandrov space $(X_\infty, \star_{X_\infty})$ of dimension at most n, the k-strainers yield k pairs $\{\gamma_j^{\pm}\}_{j=1}^k$ of opposite rays leaving $\star_{X_\infty}$, the opposite rays $\gamma_j^{\pm}$ fit together to form k orthogonal lines, and the j^{th} components of the Φ_i's converge to the negative of the Busemann function of γ_j^+. Using the Splitting Theorem [BBI01, Chapter 10.5] it follows that X_∞ splits off an $\mathbb{R}^k$-factor. This gives a contradiction. □

Lemma 4.16. *Given $k \le n \in \mathbb{Z}^+$, suppose that $\{(X_i, \star_{X_i})\}_{i=1}^\infty$ is a sequence of complete pointed length spaces and $\delta_i \to 0$ is a positive sequence such that*

(1) *Each $B\left(\star_{X_i}, \frac{1}{\delta_i}\right)$ has curvature bounded below by $-\delta_i$ and dimension bounded above by n.*
(2) *Each $(X_i, \star_{X_i})$ has a (k, δ_i)-splitting.*
(3) $\lim_{i\to\infty}(X_i, \star_{X_i}) = (X_\infty, \star_{X_\infty})$ *in the pointed Gromov-Hausdorff topology.*

Then $(X_\infty, \star_{X_\infty})$ is a nonnegatively curved Alexandrov space with a k-splitting.

Proof. This follows from Lemma 4.15. □

4.4. **Compatibility of approximate splittings.** Next, we show that the nonexistence of an approximate $(k+1)$-splitting implies that approximate j-splittings are approximately compatible with k-splittings for $j \le k$.

Lemma 4.17. *Given $j \le k \le n \in \mathbb{Z}^+$ and $\beta'_k, \beta_{k+1} > 0$, there are numbers $\delta = \delta(j, k, n, \beta'_k, \beta_{k+1}) > 0$. $\beta_j = \beta_j(j, k, n, \beta'_k, \beta_{k+1}) > 0$ and $\beta_k = \beta_k(j, k, n, \beta'_k, \beta_{k+1}) > 0$ with the following property. If $(X, \star_X)$ is a complete pointed length space such that*

(1) *The ball $B(\star_X, \delta^{-1})$ has curvature bounded below by $-\delta$ and dimension bounded above by n, and*
(2) *$(X, \star_X)$ does not admit a $(k+1, \beta_{k+1})$-splitting*

then any (j, β_j)-splitting of $(X, \star_X)$ is β'_k-compatible with any (k, β_k)-splitting.

Proof. Suppose that the lemma is false. Then for some $j \le k \le n \in \mathbb{Z}^+$ and β'_k, $\beta_{k+1} > 0$, there are

(1) A sequence $\{(X_i, \star_{X_i})\}_{i=1}^\infty$ of pointed complete length spaces,
(2) A sequence $\{\alpha_i : (X_i, \star_{X_i}) \to (X_{1,i}, \star_{X_{1,i}}) \times (X_{2,i}, \star_{X_{2,i}})\}$ of (j, i^{-1})-splittings and
(3) A sequence $\{\bar{\alpha}_i : (X_i, \star_{X_i}) \to (Y_{1,i}, \star_{Y_{1,i}}) \times (Y_{2,i}, \star_{Y_{2,i}})\}$ of (k, i^{-1})-splittings

such that

(4) $B(\star_{X_i}, i^{-1})$ has curvature bounded below by $-i^{-1}$,
(5) $B(\star_{X_i}, i^{-1})$ has dimension at most n,
(6) $(X_i, \star_{X_i})$ does not admit a $(k+1, \beta_{k+1})$-splitting for any i and
(7) α_i is not β'_k-compatible with $\bar{\alpha}_i$ for any i.

By (4), (5) and Lemma 3.10, after passing to a subsequence we can assume that there is a pointed nonnegatively curved Alexandrov space $(X_\infty, \star_{X_\infty})$, and a sequence $\{\Phi_i : (X_i, \star_{X_i}) \to (X_\infty, \star_{X_\infty})\}$ of i^{-1}-Gromov-Hausdorff approximations. In view of (2), (3) and Lemma 4.16, after passing to a further subsequence we can also assume that there is a pointed j-splitting $\alpha_\infty : (X_\infty, \star_{X_\infty}) \to (X_{\infty.1}, \star_{X_{\infty,1}}) \times (X_{\infty,2}, \star_{X_{\infty,2}})$ and a pointed k-splitting $\bar{\alpha}_\infty : (X_\infty, \star_{X_\infty}) \to (Y_{\infty,1}, \star_{Y_{\infty,1}}) \times (Y_{\infty,2}, \star_{Y_{\infty,2}})$ such that $\alpha_\infty \circ \Phi_i$ (respectively $\bar{\alpha}_\infty \circ \Phi_i$) is i^{-1}-close to α_i (respectively $\bar{\alpha}_i$). By Lemma 4.4 and the fact that $(X_\infty, \star_{X_\infty})$ does not admit a $(k+1)$-splitting, we conclude that α_∞ is compatible with $\bar{\alpha}_\infty$. It follows that α_i is β'_k-compatible with $\bar{\alpha}_i$ for large i, which is a contradiction. □

Remark 4.18. Assumption (1) in Lemma 4.17 is probably not necessary but it allows us to give a simple proof, and it will be satisfied when we apply the lemma.

4.5. **Overlapping cones.** In this subsection we prove a result about overlapping almost-conical regions that we will need later. We recall that a pointed metric space $(X, \star)$ is a *metric cone* if it is a union of rays leaving the basepoint $\star$, and the union of any two rays γ_1, γ_2 leaving $\star$ is isometric to the union of two rays $\bar{\gamma}_1$, $\bar{\gamma}_2 \subset \mathbb{R}^2$ leaving the origin $o \in \mathbb{R}^2$.

Lemma 4.19. *If $(X, \star_X)$ is a conical nonnegatively curved Alexandrov space and there is some $x \neq \star_X$ so that (X, x) is also a conical Alexandrov space then X has a 1-splitting such that the segment from $\star_X$ to x is parallel to the $\mathbb{R}$-factor.*

Proof. Let α be a segment joining $\star_X$ to x. Since X is conical with respect to both $\star_X$ and x, the segment α can be extended in both directions as a geodesic γ. The cone structure implies that γ is a line. The lemma now follows from the splitting theorem. □

Lemma 4.20. *Given $n \in \mathbb{Z}^+$ and $\delta > 0$, there is a $\delta' = \delta'(n, \delta) > 0$ with the following property. If*

- *$(X, \star_X)$ is a complete pointed length space,*
- *$x \in X$ has $d(\star_X, x) = 1$ and*
- *$(X, \star_X)$ and (X, x) have pointed Gromov-Hausdorff distance less than δ' from conical nonnegatively curved Alexandrov spaces CY and CY', respectively, of dimension at most n*

then (X, x) has a $(1, \delta)$-splitting.

Proof. If the lemma were false, then there would be a $\delta > 0$, a positive sequence $\delta'_i \to 0$, a sequence of pointed complete length spaces $\{(X_i, \star_{X_i})\}_{i=1}^{\infty}$, and points $x_i \in X_i$ such that for every i:

- $d(\star_{X_i}, x_i) = 1$.
- $(X_i, \star_{X_i})$ and (X_i, x_i) have pointed Gromov-Hausdorff distance less than δ'_i from conical nonnegatively curved Alexandrov spaces CY_i and CY'_i, respectively, of dimension at most n.
- (X_i, x_i) does not have a $(1, \delta)$-splitting.

After passing to a subsequence, we can assume that we have Gromov-Hausdorff limits $\lim_{i\to\infty}(X_i, \star_{X_i}) = (X_\infty, \star_{X_\infty})$, $\lim_{i\to\infty} x_i = x_\infty$ with $d(\star_{X_\infty}, x_\infty) = 1$, and both $(X_\infty, \star_{X_\infty})$ and (X_∞, x_∞) are conical nonnegatively curved Alexandrov spaces. By Lemma 4.19, (X_∞, x_∞) has a 1-splitting. This gives a contradiction. □

4.6. **Adapted coordinates.** In this subsection we discuss coordinate systems which arise from (k, δ)-splittings of Riemannian manifolds, in the presence of a lower curvature bound. The basic construction combines the standard construction of strainer coordinates [BGP92] with the smoothing result of Corollary 3.16.

Definition 4.21. Suppose $0 < \delta' \leq \delta$, and let α be a (k, δ')-splitting of a complete pointed Riemannian manifold $(M, \star_M)$. Let $\Phi : B\left(\star_M, \frac{1}{\delta}\right) \to \mathbb{R}^k$ be the composition $B\left(\star_M, \frac{1}{\delta}\right) \stackrel{\alpha}{\to} \mathbb{R}^k \times X_2 \to \mathbb{R}^k$. Then a map $\phi : (B(\star_M, 1), \star_M) \to (\mathbb{R}^k, \phi(\star_M))$ defines *α-adapted coordinates of quality δ* if

(1) ϕ is smooth and $(1+\delta)$-Lipschitz.
(2) The image of ϕ has Hausdorff distance at most δ from $B(\phi(\star_M), 1) \subset \mathbb{R}^k$.
(3) For all $m \in B(\star_M, 1)$ and $m' \in B\left(\star_M, \frac{1}{\delta}\right)$ with $d(m, m') > 1$, the (unit-length) initial velocity vector $v \in T_m M$ of any minimizing geodesic from m to m' satisfies

$$\left| D\phi(v) - \frac{\Phi(m') - \Phi(m)}{d(m, m')} \right| < \delta. \tag{4.22}$$

We will say that a map $\phi : (B(\star_M, 1), \star_M) \to (\mathbb{R}^k, 0)$ defines *adapted coordinates of quality* δ if there exists a (k, δ)-splitting α such that ϕ defines α-adapted coordinates of quality δ, as above. Likewise, $(M, \star_M)$ *admits k-dimensional adapted coordinates of quality δ* if there is a map ϕ as above which defines adapted coordinates of quality δ.

We now give a sufficient condition for an approximate splitting to have good adapted coordinates.

Lemma 4.23. *(Existence of adapted coordinates) For all $n \in \mathbb{Z}^+$ and $\delta > 0$, there is a $\delta' = \delta'(n, \delta) > 0$ with the following property. Suppose that $(M, \star_M)$ is an n-dimensional complete pointed Riemannian manifold with sectional curvature bounded below by $-\delta'^2$ on $B\left(\star_M, \frac{1}{\delta'}\right)$, which has a (k, δ')-splitting α. Then there exist α-adapted coordinates of quality δ.*

Proof. The idea of the proof is to use the approximate splitting to construct a strainer and then use the strainer to construct Lipschitz-regular coordinates, which can be smoothed using Corollary 3.16.

Fix $n \in \mathbb{Z}^+$ and $\delta > 0$. Suppose that $\delta' < \delta$ and $(M, \star_M)$ has a (k, δ')-splitting $\alpha : (M, \star_M) \to (\mathbb{R}^k, 0) \times (Y, \star_Y)$. Let $\{e_j\}_{j=1}^k$ be an orthonormal basis of $\mathbb{R}^k$. Given a parameter $s \in (\frac{1}{\delta}, \frac{1}{10\delta'})$, choose $p_{j\pm} \in M$ so that $\alpha(p_{j\pm})$ lies in the $10\delta'$-neighborhood of $(\pm s e_j, \star_Y)$.

Define $\phi_0 : B(\star_M, 1) \to \mathbb{R}^k$ by

$$\phi_0(m) = (\, d(\star_M, p_{1+}) - d(m, p_{1+}), \dots, d(\star_M, p_{k+}) - d(m, p_{k+}))\,. \tag{4.24}$$

We will show that if s and δ' are chosen appropriately then we can smooth the component functions of ϕ_0 using Corollary 3.16, to obtain a map $\phi : B(\star_M, 1) \to (\mathbb{R}^k, 0)$ which defines α-adapted coordinates of quality δ. (Note that α is also a (k, δ)-splitting since $\delta' < \delta$). We first estimate the left-hand side of (4.22). Recall that if m is a point of differentiability of $d_{p_{j+}}$ and $v \in T_m M$ is the initial vector of a unit-speed minimizing geodesic $\overline{mm'}$ then $Dd_{p_{j+}}(v) = -\cos(\widetilde{\angle}_m(m', p_{j+}))$.

Sublemma 4.25. *There exists $\bar{s} = \bar{s}(n, \delta) > \frac{1}{\delta}$ so that for each $s \geq \bar{s}$, there is some $\overline{\delta}' = \overline{\delta}'(n, s, \delta) < \frac{1}{10s}$ such that if $\delta' < \overline{\delta}'$ then the following holds.*

Under the hypotheses of the lemma, suppose that $m \in B(\star_M, 1)$, $m' \in B(\star_M, \frac{1}{\delta})$ and $d(m, m') \geq 1$. Let $\overline{mp_{j+}}$ and $\overline{mm'}$ be minimizing geodesics. Then

$$\left| \cos(\widetilde{\angle}_m(m', p_{j+})) - \frac{d(m, p_{j+}) - d(m', p_{j+})}{d(m, m')} \right| < \delta. \tag{4.26}$$

Proof. Suppose that the sublemma is not true. Then for each $\bar{s} > \frac{1}{\delta}$, one can find some $s \geq \bar{s}$ so that there are (see Figure 1):

- A positive sequence $\delta'_i \to 0$,
- A sequence $\{(M_i, \star_{M_i})\}_{i=1}^\infty$ of n-dimensional complete pointed Riemannian manifolds with sectional curvature bounded below by $-\delta_i'^2$ on $B(\star_{M_i}, \frac{1}{\delta'_i})$,
- A (k, δ'_i)-splitting α'_i of M_i and
- Points $m_i \in B(\star_{M_i}, 1)$ and $m'_i \in B(\star_{M_i}, \frac{1}{\delta})$ with $d(m_i, m'_i) \geq 1$ so that
- For each i, the inequality (4.26) fails.

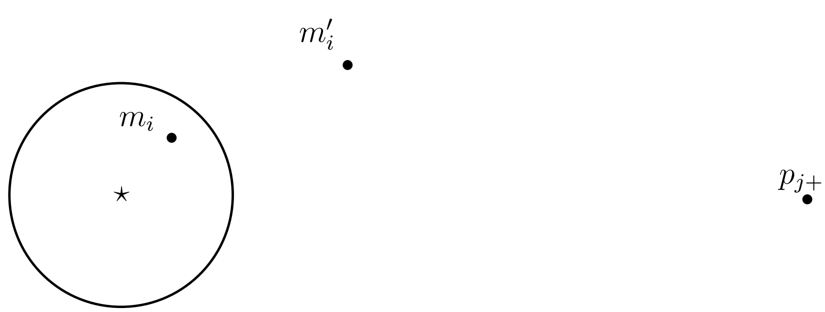

FIGURE 1

Using Lemma 4.16, after passing to a subsequence, we can assume that $\lim_{i\to\infty}(M_i, \star_{M_i}) = (X_\infty, \star_{X_\infty})$ in the pointed Gromov-Hausdorff topology for some pointed nonnegatively curved Alexandrov space $(X_\infty, \star_{X_\infty})$ which is an isometric product $\mathbb{R}^k \times Y_\infty$. After passing to a further subsequence, we can assume

- $\lim_{i\to\infty} m_i = x_\infty$ for some $x_\infty \in \overline{B(\star_{X_\infty}, 1)}$,
- $\lim_{i\to\infty} m'_i = x'_\infty$ for some $x'_\infty \in \overline{B(\star_{X_\infty}, \frac{1}{\delta})}$ with $d(x_\infty, x'_\infty) \geq 1$,
- The segments $\overline{m_i m'_i}$ converge to a segment $\overline{x_\infty x'_\infty}$ and
- $\lim_{i\to\infty} p_{i,j+} = p_{\infty,j+}$ for some points $p_{\infty,j+} \in X_\infty$ in $\mathbb{R}^k \times \{\star_{Y_\infty}\}$ of distance s from $\star_{X_\infty}$.

Then

- $\lim_{i\to\infty} \widetilde{\angle}_{m_i}(m'_i, p_{i,j+}) = \widetilde{\angle}_{x_\infty}(x'_\infty, p_{\infty,j+})$,
- $\lim_{i\to\infty} d(m_i, m'_i) = d(x_\infty, x'_\infty)$ and
- $\lim_{i\to\infty} d(m'_i, p_{i,j+}) = d(x'_\infty, p_{\infty,j+})$.

Now a straightforward verification shows that since we are in the case of an exact $\mathbb{R}^k$-splitting, there is a function $\eta = \eta(\delta, s)$ with $\lim_{s\to\infty} \eta(\delta, s) = 0$ so that

$$\left| \cos(\widetilde{\angle}_{x_\infty}({x_\infty}', p_{\infty,j+})) - \frac{d(x_\infty, p_{\infty,j+}) - d({x_\infty}', p_{\infty,j+})}{d(x_\infty, {x_\infty}')} \right| < \eta. \tag{4.27}$$

Taking s large enough gives a contradiction, thereby proving the sublemma. □

Returning to the proof of the lemma, with $s \geq \overline{s}$, define ϕ_0 as in (4.24). We have shown that if δ' is sufficiently small then ϕ_0 satisfies (4.22) with δ replaced by $\frac{1}{2}\delta$. By a similar contradiction argument, one can show that if δ' is sufficiently small, as a function of n, s and δ, then ϕ_0 is a $(1 + \frac{1}{2}\delta)$-Lipschitz map whose image is a δ-Hausdorff approximation to $B(0,1) \subset \mathbb{R}^k$.

If it were not for problems with cutpoints which could cause ϕ_0 to be nonsmooth, then we could take $\phi = \phi_0$. In general, we claim that if s is large enough then we can apply Lemma 3.16 in order to smooth $d_{p_{j+}}$ on $B(\star_M, 1)$, and thereby construct ϕ from ϕ_0. To see this, note that for any $\epsilon > 0$, by making s sufficiently large, we can arrange that for any $m \in B(\star_M, \frac{3}{\delta})$, the comparison angle $\widetilde{\angle}_m(p_{j+}, p_{j-})$ is as close to π as we wish, and hence by triangle comparison the hypotheses of Corollary 3.16 will hold with $Y = p_{j+}$, $C = \overline{B(\star_M, \frac{2}{\delta})} \subset U = B(\star_M, \frac{3}{\delta})$, and $\theta = \theta(\epsilon)$ as in the statement of Corollary 3.16. □

We now show that under certain conditions, the adapted coordinates associated to an approximate splitting are essentially unique.

Lemma 4.28. *(Uniqueness of adapted coordinates) Given $1 \leq k \leq n \in \mathbb{Z}^+$ and $\epsilon > 0$, there is an $\epsilon' = \epsilon'(n, \epsilon) > 0$ with the following property. Suppose that*

(1) *$(M, \star_M)$ is an n-dimensional complete pointed Riemannian manifold with sectional curvature bounded below by $-(\epsilon')^2$ on $B\left(\star_M, \frac{1}{\epsilon'}\right)$.*
(2) *$\alpha : (M, \star_M) \to (\mathbb{R}^k \times Z, (0, \star_Z))$ is a (k, ϵ')-splitting of $(M, \star_M)$.*
(3) *$\phi_1 : (B(\star_M, 1), \star_M) \to (\mathbb{R}^k, 0)$ is an α-adapted coordinate on $B(\star_M, 1)$ of quality ϵ'.*

(4) *Either (a)* $\phi_2 : (B(\star_M, 1), \star_M) \to (\mathbb{R}^k, 0)$ *is an* α*-adapted coordinate on* $B(\star_M, 1)$ *of quality* ϵ'*, or*
(b) ϕ_2 *has* $(1+\epsilon')$*-Lipschitz components and the following holds :*
For every $m \in B(\star_M, 1)$ *and every* $j \in \{1, \ldots, k\}$*, there is an* $m'_j \in B(\star_M, (\epsilon')^{-1})$ *with* $d(m'_j, m) > 1$ *satisfying (4.22) (with* $\phi \rightsquigarrow \phi_2$*), such that* $(\pi_{\mathbb{R}^k} \circ \alpha)(m'_j)$ *lies in the* ϵ'*-neighborhood of the line* $(\pi_{\mathbb{R}^k} \circ \alpha)(m) + \mathbb{R}\, e_j$*, and* $(\pi_Z \circ \alpha)(m'_j)$ *lies in the* ϵ'*-ball centered at* $(\pi_Z \circ \alpha)(m)$*.*

Then $\| \phi_1 - \phi_2 \|_{C^1} \leq \epsilon$ *on* $B(\star_M, 1)$*.*

Proof. We first give the proof when ϕ_2 is also an α-adapted coordinate on $B(\star_M, 1)$ of quality ϵ'.

Let $\Phi : (M, \star_M) \to \mathbb{R}^k$ be the composition $(M, \star_M) \stackrel{\alpha}{\to} \mathbb{R}^k \times Z \to \mathbb{R}^k$.

Given $\epsilon_1 > 0$, if ϵ' is sufficiently small, then by choosing points $\{p_{i,\pm}\}_{i=1}^k$ in M with $d(\alpha(p_{i,\pm}), (\pm{\epsilon_1}^{-1} e_i, \star_Z)) \leq \epsilon_1$, we obtain a strainer of quality comparable to ϵ_1 at scale ${\epsilon_1}^{-1}$. Given $m \in B(\star_M, 1)$, let γ_i be a unit speed minimizing geodesic from m to $p_{i,+}$, let $v_i \in T_m M$ be the initial velocity of γ_i, and let m_i be the point on γ_i with $d(m_i, m) = 2$. Given $\epsilon_2 > 0$, if ϵ_1 is sufficiently small then using triangle comparison, we get

$$|\langle v_i, v_j \rangle - \delta_{ij}| < \epsilon_2\,, \quad |(\Phi(m_i) - \Phi(m)) - 2e_i| < \epsilon_2. \tag{4.29}$$

for all $1 \leq i, j \leq k$. Applying (4.22) with $m' = m_i$ gives

$$\max\left(|D\phi_1(v_i) - e_i|\,,\, |D\phi_2(v_i) - e_i|\right) < \epsilon_2. \tag{4.30}$$

Finally, given $\epsilon_3 > 0$, since ϕ_1, ϕ_2 are $(1+\epsilon')$-Lipschitz, if ϵ' and ϵ_2 are small enough then we can assume that if v is a unit vector and $v \perp \mathrm{span}(v_1, \ldots, v_k)$ then $\max(|D\phi_1(v)|, |D\phi_2(v)|) < \epsilon_3$. So for any $\epsilon_4 > 0$, if ϵ_3 is sufficiently small then the operator norm of $D\phi_1 - D\phi_2$ is bounded above by ϵ_4 on $B(\star_M, 1)$.

Since $\phi_1(\star_M) - \phi_2(\star_M) = 0$, we can integrate the inequality $\| D\phi_1 - D\phi_2 \| \leq \epsilon_4$ along minimizing curves in $B(\star_M, 1)$ to conclude that $\| \phi_1 - \phi_2 \|_{C^0} \leq 2\epsilon_4$ on $B(\star_M, 1)$. Since ϵ_4 is arbitrary, the lemma follows in this case.

Now suppose ϕ_2 satisfies instead the second condition in (4). If $m \in B(\star_M, 1)$, $j \in \{1, \ldots, k\}$, m'_j is as in (4), and v_j is the initial velocity of a unit speed geodesic from m to m'_j, then (4.22) implies that $(D(\phi_2)_j)(v_j)$ is close to 1 when ϵ' is small. Since the j^{th} component $(\phi_2)_j$ is $(1+\epsilon')$-Lipschitz, this implies $\nabla(\phi_2)_j$ is close to v_j when ϵ' is small. Applying the reasoning of the above paragraphs to ϕ_1, we get that $\nabla(\phi_1)_j$ is also close to v_j when ϵ' is small. Hence $|D\phi_1 - D\phi_2|$ is small when ϵ' is small, and integrating within $B(\star_M, 1)$ as before, the lemma follows. □

Finally, we show that approximate compatibility of two approximate splittings leads to an approximate compatibility of their associated adapted coordinates.

Lemma 4.31. *Given* $1 \leq j \leq k \leq n \in \mathbb{Z}^+$ *and* $\epsilon > 0$*, there is an* $\epsilon' = \epsilon'(n, \epsilon) > 0$ *with the following property. Suppose that*

(1) $(M, \star_M)$ *is an* n*-dimensional complete pointed Riemannian manifold with sectional curvature bounded below by* $-(\epsilon')^2$ *on* $B\left(\star_M, \frac{1}{\epsilon'}\right)$*.*

(2) *α_1 is a (j,ϵ')-splitting of $(M,\star_M)$ and α_2 is a (k,ϵ')-splitting of $(M,\star_M)$.*
(3) *α_1 is ϵ'-compatible with α_2.*
(4) *$\phi_1 : (M,\star_M) \to (\mathbb{R}^j,0)$ and $\phi_2 : (M,\star_M) \to (\mathbb{R}^k,0)$ are adapted coordinates on $B(\star_M,1)$ of quality ϵ', associated to α_1 and α_2, respectively.*

Then there exists a map $T : \mathbb{R}^k \to \mathbb{R}^j$, which is a composition of an isometry with an orthogonal projection, such that $\| \phi_1 - T \circ \phi_2 \|_{C^1} \leq \epsilon$ on $B(\star_M,1)$.

Proof. Let $\alpha_1 : M \to \mathbb{R}^j \times Z_1$ and $\alpha_2 : M \to \mathbb{R}^k \times Z_2$ be the approximate splittings. Let Φ_1 be the composition $B(\star_M,(\epsilon')^{-1}) \stackrel{\alpha_1}{\to} \mathbb{R}^j \times Z_1 \to \mathbb{R}^j$ and Φ_2 be the composition $B(\star_M,(\epsilon')^{-1}) \stackrel{\alpha_2}{\to} \mathbb{R}^k \times Z_2 \to \mathbb{R}^k$.

By (3), there is a j-splitting $\gamma : \mathbb{R}^k \to \mathbb{R}^j \times \mathbb{R}^{k-j}$ and a pair of ϵ'-Gromov-Hausdorff approximations $\xi_1 : \mathbb{R}^j \to \mathbb{R}^j$, $\xi_2 : \mathbb{R}^{n-j} \times Z_2 \to Z_1$ such that the map $\widehat{\alpha}_2$ given by the composition

$$M \stackrel{\alpha_2}{\to} \mathbb{R}^k \times Z_2 \stackrel{(\gamma,\mathrm{Id}_{Z_2})}{\longrightarrow} \mathbb{R}^j \times \mathbb{R}^{k-j} \times Z_2 \stackrel{(\xi_1,\xi_2)}{\to} \mathbb{R}^j \times Z_1 \tag{4.32}$$

agrees with the map α_1 on the ball $B(\star_M,(\epsilon')^{-1})$ up to error at most ϵ'. Since Gromov-Hausdorff approximations $(\mathbb{R}^j,0) \to (\mathbb{R}^j,0)$ are close to isometries, for all $\epsilon_1 > 0$, there will be a map $T : \mathbb{R}^k \to \mathbb{R}^j$ (a composition of an isometry and a projection) which agrees with the composition

$$\mathbb{R}^k \stackrel{\gamma}{\longrightarrow} \mathbb{R}^j \times \mathbb{R}^{k-j} \stackrel{\xi_1}{\to} \mathbb{R}^j \tag{4.33}$$

up to error at most ϵ_1 on the ball $B(\star_M,\epsilon_1^{-1})$, provided ϵ' is sufficiently small. Thus for all $\epsilon_2 > 0$, the composition

$$M \stackrel{\alpha_2}{\to} \mathbb{R}^k \times Z_2 \to \mathbb{R}^k \stackrel{T}{\to} \mathbb{R}^j \tag{4.34}$$

agrees with Φ_1 up to error at most ϵ_2 on $B(\star_M,\epsilon_2^{-1})$, provided ϵ_1 and ϵ' are sufficiently small. Using Definition 4.21 for the approximate splitting α_2 and applying T, it follows that for all $\epsilon_3 > 0$, if ϵ_2 is sufficiently small then we are ensured that $T \circ \phi_2$ defines α_1-adapted coordinates on $B(\star_M,1)$ of quality ϵ_3. By Lemma 4.28 (using the first criterion in part(4) of Lemma 4.28), it follows that if ϵ_3 is sufficiently small then $\|\phi_1 - T \circ \phi_2\|_{C^1} < \epsilon$. □

Remark 4.35. In Definition 4.21 we defined adapted coordinates on the unit ball $B(\star_M,1)$. By a rescaling, we can define adapted coordinates on a ball of any specified size, and the results of this section will remain valid.

5. Standing assumptions

We now start on the proof of Theorem 1.3 in the case of closed manifolds. The proof is by contradiction.

The next lemma states that if we can get a contradiction from a certain "Standing Assumption" then we have proven Theorem 1.3. We recall from the definition of the volume scale in Definition 1.5 that if $w \leq w'$ then $r_p(w) \geq r_p(w')$.

Lemma 5.1. *If Theorem 1.3 is false then we can satisfy the following Standing Assumption, for an appropriate choice of A'.*

Standing Assumption 5.2. *Let $K \geq 10$ be a fixed integer, and let $A' : (0,\infty)\times(0,\infty) \to (0,\infty)$ be a function.*

We assume that $\{(M^\alpha, g^\alpha)\}_{\alpha=1}^\infty$ is a sequence of connected closed Riemannian 3-manifolds such that

1. *For all $p \in M^\alpha$, the ratio $\frac{R_p}{r_p(1/\alpha)}$ of the curvature scale at p to the $\frac{1}{\alpha}$-volume scale at p is bounded below by α.*
2. *For all $p \in M^\alpha$ and $w' \in [\frac{1}{\alpha}, c_3)$, let $r_p(w')$ denote the w'-volume scale at p. Then for each integer $k \in [0, K]$ and each $C \in (0,\alpha)$, we have $|\nabla^k \operatorname{Rm}| \leq A'(C, w') r_p(w')^{-(k+2)}$ on $B(p, C r_p(w'))$.*
3. *Each M^α fails to be a graph manifold.*

Proof. If Theorem 1.3 is false then for every $\alpha \in \mathbb{Z}^+$, there is a manifold (M^α, g^α) which satisfies the hypotheses of Theorem 1.3 with the parameter w_0 of the theorem set to $w_0^\alpha = \frac{1}{8\alpha^4}$, but M^α is not a graph manifold.

We claim first that for every $p^\alpha \in M^\alpha$, we have $r_{p^\alpha}(1/\alpha) < R_{p^\alpha}$. If not then for some $p^\alpha \in M^\alpha$, we have $r_{p^\alpha}(1/\alpha) \geq R_{p^\alpha}$. From the definition of $r_{p^\alpha}(1/\alpha)$, it follows that

$$\operatorname{vol}(B(p^\alpha, R_{p^\alpha})) \geq \frac{1}{\alpha} R_{p^\alpha}^3 > \frac{1}{8\alpha^4} R_{p^\alpha}^3, \tag{5.3}$$

which contradicts our choice of w_0^α.

Thus $r_{p^\alpha}(1/\alpha) < R_{p^\alpha}$. Then

$$\frac{1}{\alpha}(r_{p^\alpha}(1/\alpha))^3 = \operatorname{vol}(B(p^\alpha, r_{p^\alpha}(1/\alpha))) \leq \operatorname{vol}(B(p^\alpha, R_{p^\alpha})) \leq \frac{1}{8\alpha^4} R_{p^\alpha}^3, \tag{5.4}$$

so $\frac{R_{p^\alpha}}{r_{p^\alpha}(1/\alpha)} \geq 2\alpha$. This shows that $\{(M^\alpha, g^\alpha)\}_{\alpha=1}^\infty$ satisfies condition (1) of Standing Assumption 5.2.

To see that condition (2) of Standing Assumption 5.2 holds, for an appropriate choice of A', we first note that if suffices to just consider $C \in [1,\alpha)$, since a derivative bound on a bigger ball implies a derivative bound on a smaller ball. For $\widetilde{w}' \in \left[\frac{1}{\alpha}, c_3\right)$, we have

$$C r_{p^\alpha}(\widetilde{w}') \leq \alpha r_{p^\alpha}(1/\alpha) \leq R_p. \tag{5.5}$$

Now

$$\operatorname{vol}(B(p^\alpha, C r_{p^\alpha}(\widetilde{w}'))) \geq \operatorname{vol}(B(p^\alpha, r_{p^\alpha}(\widetilde{w}'))) = \widetilde{w}'(r_{p^\alpha}(\widetilde{w}'))^3 = C^{-3}\widetilde{w}'(C r_{p^\alpha}(\widetilde{w}'))^3. \tag{5.6}$$

Put $w' = C^{-3}\widetilde{w}'$. Then

$$w_0^\alpha = \frac{1}{8\alpha^4} \leq w' < c_3. \tag{5.7}$$

Hypothesis (2) of Theorem 1.3 implies that

$$|\nabla^k \operatorname{Rm}| \leq A(w')\left(C r_{p^\alpha}(\widetilde{w}')\right)^{-(k+2)} \tag{5.8}$$

on $B(p^\alpha, C r_{p^\alpha}(\widetilde{w}'))$. Hence condition (2) of Standing Assumption 5.2 will be satisfied, for $C \in [1,\alpha)$, if we take

$$A'(C, \widetilde{w}') = \max_{0\leq k\leq K} A\left(C^{-3}\widetilde{w}'\right) C^{-(k+2)}. \tag{5.9}$$

∎

Standing Assumption 5.2 will remain in force until Section 16, where we consider manifolds with boundary. We will eventually get a contradiction to Standing Assumption 5.2.

For the sake of notational brevity, we will usually suppress the superscript α; thus M will refer to M^α. By convention, each of the statements made in the proof is to be interpreted as being valid provided α is sufficiently large, whether or not this qualification appears explicitly.

Remark 5.10. The condition $K \geq 10$ in Standing Assumption 5.2 is clearly not optimal but it is good enough for the application to the geometrization conjecture.

Remark 5.11. We note that for fixed $\widehat{w} \in (0, c_3)$, conditions (1) and (2) of Standing Assumption 5.2 imply that for large α, the following holds for all $p \in M^\alpha$:

(1) $\frac{R_p}{r_p(\widehat{w})} \geq \alpha$ and
(2) For each integer $k \in [0, K]$ and each $C \in (0, \alpha)$, we have $|\nabla^k \operatorname{Rm}| \leq A'(C, \widehat{w}) r_p(\widehat{w})^{-(k+2)}$ on $B(p, C r_p(\widehat{w}))$.

Since in addition $\operatorname{vol}(B(p, r_p(\widehat{w}))) = \widehat{w}(r_p(\widehat{w}))^3$, we have all of the ingredients to extract convergent subsequences, at the $\widehat{w}$-volume scale, with smooth pointed limits that are non-negatively curved. This is how the hypotheses of Standing Assumption 5.2 will enter into finding a contradiction. In effect, $\widehat{w}$ will eventually become a judiciously chosen constant.

6. The scale function $\mathfrak{r}$

In this section we introduce a smooth scale function $\mathfrak{r} : M \to (0, \infty)$ which will be used throughout the rest of the paper. This function is like a volume scale in the sense that one has lower bounds on volume at scale $\mathfrak{r}$, which enables one to appeal to C^K-precompactness arguments. The advantage of $\mathfrak{r}$ over a volume scale is that $\mathfrak{r}$ can be arranged to have small Lipschitz constant, which is technically useful.

We will use the following lemma to construct slowly varying functions subject to *a priori* upper and lower bounds.

Lemma 6.1. *Suppose X is a metric space, $C \in (0, \infty)$, and $l, u : X \to (0, \infty)$ are functions. Then there is a C-Lipschitz function $r : X \to (0, \infty)$ satisfying $l \leq r \leq u$ if and only if*

$$l(p) - C d(p, q) \leq u(q) \tag{6.2}$$

for all $p, q \in X$.

Proof. Clearly if such an r exists then (6.2) must hold.

Conversely, suppose that (6.2) holds and define $r : X \to (0, \infty)$ by

$$r(q) = \sup\{l(p) - C d(p, q) \mid p \in X\}\,. \tag{6.3}$$

Then $l \leq r \leq u$. For $q, q' \in X$, since $l(p) - C d(p, q) \geq l(p) - C d(p, q') - C d(q, q')$, we obtain $r(q) \geq r(q') - C d(q, q')$, from which it follows that r is C-Lipschitz. $\square$

Recall that c_3 is the volume of the unit ball in $\mathbb{R}^3$. Let $\Lambda > 0$ and $\bar{w} \in (0, c_3)$ be new parameters, and put

$$w' = \frac{\bar{w}}{2(1+2\Lambda^{-1})^3}. \tag{6.4}$$

Recall the notion of the volume scale $r_p(\bar{w})$ from Definition 1.5.

Corollary 6.5. *There is a smooth Λ-Lipschitz function $\mathfrak{r} : M \to (0, \infty)$ such that for every $p \in M$, we have*

$$\frac{1}{2}\, r_p(\bar{w}) \;\leq\; \mathfrak{r}(p) \;\leq\; 2 r_p(w')\,. \tag{6.6}$$

Proof. We let $l : M \to (0, \infty)$ be the $\bar{w}$-volume scale, and $u : M \to (0, \infty)$ be the w'-volume scale. We first verify (6.2) with parameter $C = \frac{\Lambda}{2}$. To argue by contradiction, suppose that for some $p, q \in M$ we have $l(p) - \frac{1}{2}\Lambda d(p,q) > u(q)$. In particular, $d(p,q) < \frac{2l(p)}{\Lambda}$ and $u(q) < l(p)$. There are inclusions $B(p, l(p)) \subset B(q, (1+2\Lambda^{-1})l(p)) \subset B(p, (1+4\Lambda^{-1})l(p))$. Then

$$\mathrm{vol}(B(q,(1+2\Lambda^{-1})l(p))) \geq \mathrm{vol}(B(p, l(p))) = \bar{w} l^3(p) = 2w' \left((1+2\Lambda^{-1})l(p)\right)^3. \tag{6.7}$$

For any $c > 0$, if α is sufficiently large then the sectional curvature on $B(p, (1+4\Lambda^{-1})l(p))$, and hence on $B(q, (1+2\Lambda^{-1})l(p))$, is bounded below by $-\,c^2\, l(p)^{-2}$. As $u(q) < l(p) < (1+2\Lambda^{-1})l(p)$, the Bishop-Gromov inequality implies that

$$\begin{aligned} \frac{w'\, u(q)^3}{\int_0^{u(q)/l(p)} \sinh^2(cr)\, dr} = \frac{\mathrm{vol}(B(q,u(q)))}{\int_0^{u(q)/l(p)} \sinh^2(cr)\, dr} &\geq \frac{\mathrm{vol}(B(q,(1+2\Lambda^{-1})l(p)))}{\int_0^{!+2\Lambda^{-1}} \sinh^2(cr)\, dr} \\ &\geq \frac{2w' \left((1+2\Lambda^{-1})l(p)\right)^3}{\int_0^{1+2\Lambda^{-1}} \sinh^2(cr)\, dr}, \end{aligned} \tag{6.8}$$

or

$$\frac{c^2 \left(\frac{u(q)}{l(p)}\right)^3}{\int_0^{u(q)/l(p)} \sinh^2(cr)\, dr} \geq \frac{2c^2(1+2\Lambda^{-1})^3}{\int_0^{!+2\Lambda^{-1}} \sinh^2(cr)\, dr}. \tag{6.9}$$

As the function $x \mapsto \frac{c^2}{3} \frac{x^3}{\int_0^x \sinh^2(cr)\, dr}$ tends uniformly to 1 as $c \to 0$, for $x \in (0, 3]$, taking c small gives a contradiction. (Note the factor of 2 on the right-hand side of (6.9)).

By Lemma 6.1, there is a $\frac{\Lambda}{2}$-Lipschitz function r on M satisfying $l \leq r \leq u$. The corollary now follows from Corollary 3.15.

□

We will write $\mathfrak{r}_p$ for $\mathfrak{r}(p)$. Recall our convention that the index α in the sequence $\{M^\alpha\}_{\alpha=1}^\infty$ has been suppressed, and that all statements are to be interpreted as being valid provided α is sufficiently large. The next lemma shows C^K-precompactness at scale $\mathfrak{r}$.

Lemma 6.10.

(1) *There is a constant $\widehat{w} = \widehat{w}(w') > 0$ such that* $\mathrm{vol}(B(p, \mathfrak{r}_p)) \geq \widehat{w} (\mathfrak{r}_p)^3$ *for every $p \in M$.*

(2) *For every $p \in M$, $C < \infty$ and $k \in [0, K]$, we have*

$$|\nabla^k \operatorname{Rm}| \;\le\; 2^{k+2}\, A'(C, w')\, \mathfrak{r}_p^{-(k+2)} \quad \text{on the ball} \quad B\left(p, \frac{1}{2} C \mathfrak{r}_p\right). \tag{6.11}$$

(3) *Given $\epsilon > 0$, for sufficiently large α and for every $p \in M^\alpha$, the rescaled pointed manifold $(\frac{1}{\mathfrak{r}_p} M^\alpha, p)$ is ϵ-close in the pointed C^K-topology to a complete nonnegatively curved C^K-smooth Riemannian 3-manifold. Moreover, this manifold belongs to a family which is compact in the pointed C^K-topology.*

Proof.

(1). As $\frac{1}{2}\, \mathfrak{r}_{p^\alpha} \le r_{p^\alpha}(w')$, the Bishop-Gromov inequality gives

$$\frac{\operatorname{vol}(B(p^\alpha, \frac{1}{2}\, \mathfrak{r}_{p^\alpha}))}{\int_0^{\frac{\mathfrak{r}_{p^\alpha}}{2 r_{p^\alpha}(w')}} \sinh^2(r)\, dr} \ge \frac{\operatorname{vol}(B(p^\alpha, r_{p^\alpha}(w')))}{\int_0^1 \sinh^2(r)\, dr} = \frac{w'(r_{p^\alpha}(w'))^3}{\int_0^1 \sinh^2(r)\, dr}, \tag{6.12}$$

or

$$\frac{\operatorname{vol}(B(p^\alpha, \frac{1}{2}\, \mathfrak{r}_{p^\alpha}))}{(\frac{1}{2}\, \mathfrak{r}_{p^\alpha})^3} \;\ge\; \frac{w'}{\int_0^1 \sinh^2(r)\, dr} \frac{\int_0^{\frac{\mathfrak{r}_{p^\alpha}}{2 r_{p^\alpha}(w')}} \sinh^2(r)\, dr}{\left(\frac{\mathfrak{r}_{p^\alpha}}{2 r_{p^\alpha}(w')}\right)^3} \ge \frac{w'}{3 \int_0^1 \sinh^2(r)\, dr}. \tag{6.13}$$

Thus

$$\operatorname{vol}(B(p^\alpha, \mathfrak{r}_{p^\alpha})) \ge \operatorname{vol}\left(B\left(p^\alpha, \mathfrak{r}_{p^\alpha}/2\right)\right) \ge \frac{w'}{24 \int_0^1 \sinh^2(r)\, dr} (\mathfrak{r}_{p^\alpha})^3, \tag{6.14}$$

which gives (1).

(2). From hypothesis (2) of Standing Assumption 5.2, for each $C < \alpha$ and $k \in [0, K]$ we have

$$|\nabla^k \operatorname{Rm}| \;\le\; A'(C, w')\, r_{p^\alpha}(w')^{-(k+2)} \;\le\; 2^{k+2}\, A'(C, w')\, \mathfrak{r}_{p^\alpha}^{-(k+2)} \tag{6.15}$$

on $B(p^\alpha, C r_{p^\alpha}(w')) \supset B\left(p^\alpha, \frac{1}{2} C \mathfrak{r}_{p^\alpha}\right)$.

(3). If not, then for some $\epsilon > 0$, after passing to a subsequence, for every α we could find $p^\alpha \in M^\alpha$ such that $(\frac{1}{\mathfrak{r}_{p^\alpha}} M^\alpha, p^\alpha)$ has distance at least ϵ in the C^K-topology from a complete nonnegatively curved 3-manifold.

(1) and (2) imply that after passing to a subsequence, the sequence $\{(\frac{1}{\mathfrak{r}_{p^\alpha}} M^\alpha, p^\alpha)\}$ converges in the pointed C^K-topology to a complete Riemannian 3-manifold. But since the ratio $\frac{R_p}{\mathfrak{r}_p}$ tends uniformly to infinity as $\alpha \to \infty$, the limit manifold has nonnegative curvature, which is a contradiction. $\square$

We now extend Lemma 6.10 to provide C^K-splittings.

Lemma 6.16. *Given $\epsilon > 0$ and $0 \le j \le 3$, provided $\delta < \overline{\delta}(\epsilon, w')$ the following holds. If $p \in M$, and $\phi : \left(\frac{1}{\mathfrak{r}_p} M, p\right) \to (\mathbb{R}^j \times X, (0, \star_X))$ is a (j, δ)-splitting, then ϕ is ϵ-close to a (j, ϵ)-splitting $\widehat{\phi} : \left(\frac{1}{\mathfrak{r}_p} M, p\right) \to (\mathbb{R}^j \times \widehat{X}, (0, \star_{\widehat{X}}))$, where $\widehat{X}$ is a complete nonnegatively* *curved C^K-smooth Riemannian $(3-j)$-manifold, and $\widehat{\phi}$ is ϵ-close to an isometry on the ball $B(p,\epsilon^{-1}) \subset \frac{1}{\mathfrak{r}_p}M$, in the C^{K+1}-topology.*

Proof. Suppose not. Then for some $\epsilon > 0$, after passing to a subsequence we can assume that there are points $p^\alpha \in M^\alpha$ so that $\left(\frac{1}{\mathfrak{r}_{p^\alpha}}M, p^\alpha\right)$ admits a (j,α^{-1})-splitting ϕ_j, but the conclusion of the lemma fails.

By Lemma 6.10, a subsequence of $\left\{\left(\frac{1}{\mathfrak{r}_{p^\alpha}}M^\alpha, p^\alpha\right)\right\}_{\alpha=1}^\infty$ converges in the pointed C^K-topology to a complete pointed nonnegatively curved C^K-smooth 3-dimensional Riemannian manifold (M^∞, p^∞), such that ϕ_j converges to a $(j,0)$-splitting ϕ_∞ of (M^∞, p^∞). This is a contradiction. □

Remark 6.17. If we only assume condition (1) of Assumption 5.2 then the proof of Lemma 6.16 yields the following weaker conclusion: $\widehat{X}$ is a (nonnegatively curved) $(3-j)$-dimensional Alexandrov space, and $\widehat{\phi}$ is a homeomorphism on $B(p,\epsilon^{-1})$. See Section 18 for more discussion.

Let $\sigma > 0$ be a new parameter. In the next lemma, we show that if the parameter $\bar{w}$ is small then the pointed 3-manifold $(\frac{1}{\mathfrak{r}_p}M, p)$ is Gromov-Hausdorff close to something of lower dimension.

Lemma 6.18. *Under the constraint $\bar{w} < \overline{\overline{w}}(\sigma,\Lambda)$, the following holds. For every $p \in M$, the pointed space $(\frac{1}{\mathfrak{r}_p}M, p)$ is σ-close in the pointed Gromov-Hausdorff metric to a nonnegatively curved Alexandrov space of dimension at most 2.*

Proof. Suppose that the lemma is not true. Then for some $\sigma, \Lambda > 0$, there is a sequence $\bar{w}_i \to 0$ and for each i, a sequence $\left\{\left(M^{\alpha(i,j)}, p^{\alpha(i,j)}\right)\right\}_{j=1}^\infty$ so that for each j, $\left(\frac{1}{\mathfrak{r}_{p^{\alpha(i,j)}}}M^{\alpha(i,j)}, p^{\alpha(i,j)}\right)$ has pointed Gromov-Hausdorff distance at least σ from any nonnegatively curved Alexandrov space of dimension at most 2.

Given i, as $j \to \infty$ the curvature scale at $p^{\alpha(i,j)}$ divided by $r_{p^{\alpha(i,j)}}(w')$ goes to infinity. Hence the curvature scale at $p^{\alpha(i,j)}$ divided by $\mathfrak{r}_{p^{\alpha(i,j)}}$ also goes to infinity. Thus we can find some $j = j(i)$ so that the curvature scale at $p^{\alpha(i,j(i))}$ is at least $i\,\mathfrak{r}_{p^{\alpha(i,j(i))}}$. We relabel $M^{\alpha(i,j(i))}$ as M^i and $p^{\alpha(i,j(i))}$ as p^i. Thus we have a sequence $\{(M^i,p^i)\}_{i=1}^\infty$ so that for each i, $\left(\frac{1}{\mathfrak{r}_{p^i}}M^i, p^i\right)$ has pointed Gromov-Hausdorff distance at least σ from any nonnegatively curved Alexandrov space of dimension at most 2, and the curvature scale at p^i is at least $i\,\mathfrak{r}_{p^i}$. In particular, a subsequence of the $\left(\frac{1}{\mathfrak{r}_{p^i}}M^i, p^i\right)$'s converges in the pointed Gromov-Hausdorff topology to a nonnegatively curved Alexandrov space (X,x), necessarily of dimension 3. Hence there is a uniform positive lower bound on $\frac{\mathrm{vol}(B(p^i,2\mathfrak{r}_{p^i}))}{(2\mathfrak{r}_{p^i})^3}$.

As $r_{p^i}(\bar{w}_i) \le 2\mathfrak{r}_{p^i}$, the Bishop-Gromov inequality implies that

$$\frac{\bar{w}_i\left(r_{p^i}(\bar{w}_i)\right)^3}{\int_0^{\frac{r_{p^i}(\bar{w}_i)}{2\mathfrak{r}_{p^i}}} \sinh^2(r)\,dr} = \frac{\mathrm{vol}(B(p^i, r_{p^i}(\bar{w}_i)))}{\int_0^{\frac{r_{p^i}(\bar{w}_i)}{2\mathfrak{r}_{p^i}}} \sinh^2(r)\,dr} \ge \frac{\mathrm{vol}(B(p^i, 2\mathfrak{r}_{p^i}))}{\int_0^1 \sinh^2(r)\,dr}. \tag{6.19}$$

That is,

$$\frac{\operatorname{vol}(B(p^i, 2\mathfrak{r}_{p^i}))}{(2\mathfrak{r}_{p^i})^3} \le \bar{w}_i \left(\int_0^1 \sinh^2(r)\, dr\right) \frac{\left(\frac{r_{p^i}(\bar{w}_i)}{2\mathfrak{r}_{p^i}}\right)^3}{\int_0^{\frac{r_{p^i}(\bar{w}_i)}{2\mathfrak{r}_{p^i}}} \sinh^2(r)\, dr} \le 3\, \bar{w}_i \int_0^1 \sinh^2(r)\, dr. \tag{6.20}$$

Since $\bar{w}_i \to 0$, we obtain a contradiction. $\square$

As explained in Section 2, we will assume henceforth that the constraint

$$\bar{w} < \overline{\overline{w}}(\sigma, \Lambda) \tag{6.21}$$

is satisfied.

7. Stratification

In this section we define a rough stratification of a Riemannian 3-manifold, based on the maximal dimension of a Euclidean factor of an approximate splitting at a given point.

7.1. **Motivation.** Recall that in a metric polyhedron P, there is a natural metrically defined filtration $P_0 \subset P_1 \subset \ldots \subset P$, where $P_k \subset P$ is the set of points $p \in P$ that do not have a neighborhood that isometrically splits off a factor of $\mathbb{R}^{k+1}$. The associated strata $\{P_k - P_{k-1}\}$ are manifolds of dimension k. An approximate version of this kind of filtration/stratification will be used in the proof of Theorem 1.3.

For the proof of Theorem 1.3, we use a stratification of M so that if $p \in M$ lies in the k-stratum then there is a metrically defined fibration of an approximate ball centered at p, over an open subset of $\mathbb{R}^k$. We now give a rough description of the strata; a precise definition will be given shortly.

2-*stratum.* Here $\left(\frac{1}{\mathfrak{r}_p} M, p\right)$ is close to splitting off an $\mathbb{R}^2$-factor and, due to the collapsing assumption, it is Gromov-Hausdorff close to $\mathbb{R}^2$. One gets a circle fibration over an open subset of $\mathbb{R}^2$.

1-*stratum.* Here $\left(\frac{1}{\mathfrak{r}_p} M, p\right)$ is not close to splitting off an $\mathbb{R}^2$-factor, but is close to splitting off an $\mathbb{R}$-factor. These points fall into two types: those where $\left(\frac{1}{\mathfrak{r}_p} M, p\right)$ looks like a half-plane, and those where it look like the product of $\mathbb{R}$ with a space with controlled diameter. One gets a fibration over an open subset of $\mathbb{R}$, whose fiber is D^2, S^2, or T^2.

0-*stratum.* Here $\left(\frac{1}{\mathfrak{r}_p} M, p\right)$ is not close to splitting off an $\mathbb{R}$-factor. We will show that for some radius r comparable to $\mathfrak{r}_p$, $\left(\frac{1}{r} M, p\right)$ is Gromov-Hausdorff close to the Tits cone $C_T N$ of some complete nonnegatively curved 3-manifold N with at most one end, and the ball $B(p, r) \subset M$ is diffeomorphic to N. The possibilities for the topology of N are: $S^1 \times B^2$, B^3, $\mathbb{R}P^3 - D^3$, a twisted interval bundle over the Klein bottle, $S^1 \times S^2$, $\mathbb{R}P^3 \# \mathbb{R}P^3$, a spherical space form S^3/Γ and a isometric quotient T^3/Γ of the 3-torus.

7.2. **The k-stratum points.** To define the stratification precisely, we introduce the additional parameters $0 < \beta_1 < \beta_2 < \beta_3$, and put $\beta_0 = 0$. Recall that the parameter σ has already been introduced in Section 6.

Definition 7.1. A point $p \in M$ belongs to the k-stratum, $k \in \{0, 1, 2, 3\}$, if $(\frac{1}{\mathfrak{r}_p} M, p)$ admits a (k, β_k)-splitting, but does not admit a (j, β_j)-splitting for any $j > k$.

Note that every pointed space has a $(0, 0)$-splitting, so every $p \in M$ belongs to the k-stratum for some $k \in \{0, 1, 2, 3\}$.

Lemma 7.2. *Under the constraints $\beta_3 < \overline{\beta}_3$ and $\sigma < \overline{\sigma}$ there are no 3-stratum points.*

Proof. Let $c > 0$ be the minimal distance, in the pointed Gromov-Hausdorff metric, between $(\mathbb{R}^3, 0)$ and a nonnegatively curved Alexandrov space of dimension at most 2. Taking $\overline{\beta}_3 = \overline{\sigma} = \frac{c}{4}$, the lemma follows from Lemma 6.18. □

Let $\Delta \in (\beta_2^{-1}, \infty)$ be a new parameter.

Lemma 7.3. *Under the constraint $\Delta > \overline{\Delta}(\beta_2)$, if $p \in M$ has a 2-strainer of size $\frac{\Delta}{100}\mathfrak{r}_p$ and quality $\frac{1}{\Delta}$ at p, then $\left(\frac{1}{\mathfrak{r}_p} M, p\right)$ has a $(2, \frac{1}{2}\beta_2)$-splitting $\frac{1}{\mathfrak{r}_p} M \to \mathbb{R}^2$. In particular p is in the 2-stratum.*

Proof. This follows from Lemma 4.15. □

Definition 7.4. A 1-stratum point $p \in M$ is in the *slim 1-stratum* if there is a $(1, \beta_1)$-splitting $(\frac{1}{\mathfrak{r}_p} M, p) \to (\mathbb{R} \times X, (0, \star_X))$ where $\operatorname{diam}(X) \le 10^3\Delta$.

8. The local geometry of the 2-stratum

In the next few sections, we examine the local geometry near points of different type, introducing adapted coordinates and certain associated cutoff functions.

In this section we consider the 2-stratum points. Along with the adapted coordinates and cutoff functions, we discuss the local topology and a selection process to get a ball covering of the 2-stratum points.

8.1. **Adapted coordinates, cutoff functions and local topology near 2-stratum points.** Let p denote a point in the 2-stratum and let $\phi_p : \left(\frac{1}{r_p} M, p\right) \to \mathbb{R}^2 \times (X, \star_X)$ be a $(2, \beta_2)$-splitting.

Lemma 8.1. *Under the constraints $\beta_2 < \overline{\beta}_2$ and $\sigma < \overline{\sigma}$, the factor $(X, \star_X)$ has diameter < 1.*

Proof. If not then we could find a subsequence $\{M^{\alpha_j}\}$ of the M^α's, and $p_j \in M^{\alpha_j}$, such that with $\beta_2 = \sigma = \frac{1}{j}$, the map $\phi_{p_j} : (\frac{1}{\mathfrak{r}_{p_j}} M^{\alpha_j}, p_j) \to (\mathbb{R}^2 \times X_j, (0, \star_{X_j}))$ violates the conclusion of the lemma. We then pass to a pointed Gromov-Hausdorff sublimit (M_∞, p_∞), which will be a nonnegatively curved Alexandrov space of dimension at most 2, and extract a limiting 2-splitting $\phi_\infty : (M_\infty, p_\infty) \to \mathbb{R}^2 \times X_\infty$. The only possibility is that $\dim(M_\infty) = 2$, ϕ_∞ is an isometry and X_∞ is a point. This contradicts the diameter assumption. □

Let $\varsigma_{\text{2-stratum}} > 0$ be a new parameter.

Lemma 8.2. *Under the constraint $\beta_2 < \overline{\beta}_2(\varsigma_{\text{2-stratum}})$, there is a ϕ_p-adapted coordinate η_p of quality $\varsigma_{\text{2-stratum}}$ on $B(p, 200) \subset \left(\frac{1}{\mathfrak{r}_p} M, p\right)$*

Proof. This follows from Lemma 4.23 (see also Remark 4.35). □

Definition 8.3. Let ζ_p be the smooth function on M which is the extension by zero of $\Phi_{8,9} \circ |\eta_p|$. (See Section 2 for the definition of $\Phi_{a,b}$).

Lemma 8.4. *Under the constraints $\beta_2 < \overline{\beta}_2$, $\varsigma_{\text{2-stratum}} < \overline{\varsigma}_{\text{2-stratum}}$ and $\sigma < \overline{\sigma}$, the restriction of η_p to $\eta_p^{-1}(B(0, 100))$ is a fibration with fiber S^1. In particular, for all $R \in (0, 100)$, $|\eta_p|^{-1}[0, R]$ is diffeomorphic to $S^1 \times \overline{B(0, R)}$.*

Proof. For small β_2 and $\varsigma_{\text{2-stratum}}$, the map $\eta_p : \frac{1}{\mathfrak{r}_p} M \supset B(p, 200) \to \mathbb{R}^2$ is a submersion; this follows by applying (4.22) to an appropriate 2-strainer at $x \in B(p, 200)$ furnished by the $(2, \beta_2)$-splitting ϕ_p.

By Lemma 8.1, if $\varsigma_{\text{2-stratum}} < \overline{\varsigma}_{\text{2-stratum}}$ then $\eta_p^{-1}(B(0, 100)) \subset B(p, 102) \subset \frac{1}{\mathfrak{r}_p} M$. Therefore if $K \subset B(0, 100)$ is compact then $\eta_p^{-1}(K)$ is a closed subset of $\overline{B(p, 102)} \subset \frac{1}{\mathfrak{r}_p} M$, and hence compact. It follows that $\eta_p\Big|_{\eta_p^{-1}(B(0,100))} : \eta_p^{-1}(B(0, 100)) \to B(0, 100)$ is a proper map. Thus $\eta_p\Big|_{\eta_p^{-1}(B(0,100))}$ is a trivial fiber bundle with compact 1-dimensional fibers.

Since $\eta_p^{-1}(0)$ has diameter at most 2 by Lemma 8.1 (assuming $\varsigma_{\text{2-stratum}} < \overline{\varsigma}_{\text{2-stratum}}$), it follows that any two points in the fiber $\eta_p^{-1}(0)$ can be joined by a path in $\eta_p^{-1}(B(0, 100))$. Now the triviality of the bundle implies that the fibers are connected, i.e. diffeomorphic to S^1. □

8.2. **Selection of 2-stratum balls.** Let $\mathcal{M}$ be a new parameter, which will become a bound on intersection multiplicity of balls. The corresponding bound $\overline{\mathcal{M}}$ will describe how big $\mathcal{M}$ has to be taken in order for various assertions to be valid.

Let $\{p_i\}_{i \in I_{\text{2-stratum}}}$ be a maximal set of 2-stratum points with the property that the collection $\{B(p_i, \frac{1}{3}\mathfrak{r}_{p_i})\}_{i \in I_{\text{2-stratum}}}$ is disjoint. We write ζ_i for ζ_{p_i}.

Lemma 8.5. *Under the constraints $\mathcal{M} > \overline{\mathcal{M}}$ and $\Lambda < \overline{\Lambda}$,*

(1) *$\bigcup_{i \in I_{\text{2-stratum}}} B(p_i, \mathfrak{r}_{p_i})$ contains all 2-stratum points.*
(2) *The intersection multiplicity of the collection $\{\text{supp}(\zeta_i)\}_{i \in I_{\text{2-stratum}}}$ is bounded by $\mathcal{M}$.*

Proof. (1). Assume $1 + \frac{2}{3}\Lambda < 1.01$. If p is a 2-stratum point, there is an $i \in I_{\text{2-stratum}}$ such that $B(p, \frac{1}{3}\mathfrak{r}_p) \cap B(p_i, \frac{1}{3}\mathfrak{r}_{p_i}) \neq \emptyset$ for some $i \in I_{\text{2-stratum}}$. Then $\frac{\mathfrak{r}_p}{\mathfrak{r}_{p_i}} \in [.9, 1.1]$, and $p \in B(p_i, \mathfrak{r}_{p_i})$.

(2). From the definition of ζ_i, if $\varsigma_{\text{2-stratum}}$ is sufficiently small then we are ensured that $\text{supp}(\zeta_i) \subset B(p_i, 10\mathfrak{r}_{p_i})$.

Suppose that for some $p \in M$, we have $p \in \bigcap_{j=1}^{N} B(p_{i_j}, 10\mathfrak{r}_{p_{i_j}})$ for distinct i_j's. We relabel so that $B(p_{i_1}, \mathfrak{r}_{p_{i_1}})$ has the smallest volume among the $B(p_{i_j}, \mathfrak{r}_{p_{i_j}})$'s.

If 10Λ is sufficiently small then we can assume that for all j, $\frac{1}{2} \leq \frac{\mathfrak{r}_{p_{i_j}}}{\mathfrak{r}_{p_{i_1}}} \leq 2$. Hence the N disjoint balls $\{B(p_{i_j}, \frac{1}{3}\mathfrak{r}_{p_{i_j}})\}_{j=1}^N$ lie in $B(p_{i_1}, 100\mathfrak{r}_{p_{i_1}})$ and by Bishop-Gromov volume comparison

$$N \leq \frac{\mathrm{vol}(B(p_{i_1}, 100\mathfrak{r}_{p_{i_1}}))}{\mathrm{vol}(B(p_{i_1}, \frac{1}{3}\mathfrak{r}_{p_{i_1}}))} \leq \frac{\int_0^{100} \sinh^2(r)\, dr}{\int_0^{\frac{1}{3}} \sinh^2(r)\, dr}. \tag{8.6}$$

This proves the lemma. □

9. Edge points and associated structure

In this section we study points $p \in M$ where the pair (M, p) looks like a half-plane with a basepoint lying on the edge. Such points define an edge set E. We show that any 1-stratum point, which is not a slim 1-stratum point, is not far from E.

As a technical tool, we also introduce an approximate edge set E' consisting of points where the edge structure is of slightly lower quality than that of E. The set E' will fill in the boundary edges of the approximate half-plane regions around points in E. We construct a smoothed distance function from E', along with an associated cutoff function.

We describe the local topology around points in E and choose a ball covering of E.

9.1. Edge points.

We begin with a general lemma about 1-stratum points.

Lemma 9.1. *Given $\epsilon > 0$, if $\beta_1 < \overline{\beta}_1(\epsilon)$ and $\sigma < \overline{\sigma}(\epsilon)$ then the following holds. If $(\frac{1}{\mathfrak{r}_p}M, p)$ has a $(1, \beta_1)$-splitting then there is a $(1, \epsilon)$-splitting $(\frac{1}{\mathfrak{r}_p}M, p) \to (\mathbb{R} \times Y, (0, \star_Y))$, where Y is an Alexandrov space with $\dim(Y) \leq 1$.*

Proof. This is similar to the proof of Lemma 8.1. If the lemma were false then we could find a sequence $\alpha_j \to \infty$ so that taking $\beta_1 = j^{-1}$ and $\sigma = j^{-1}$, for every j there would be $p_j \in M^{\alpha_j}$ and a $(1, j^{-1})$-splitting of $(\frac{1}{\mathfrak{r}_{p_j}}M^{\alpha_j}, p_j)$, but no $(1, \epsilon)$-splitting as asserted. Passing to a subsequence, we obtain a pointed Gromov-Hausdorff limit (M_∞, p_∞), and the $(1, j^{-1})$-splittings converge to a 1-splitting $\phi_\infty : M_\infty \to \mathbb{R} \times Y$. It follows from Lemma 6.18 that $\dim M_\infty \leq 2$, and hence $\dim Y \leq 1$. This implies that for large j, we can find arbitrarily good splittings $(\frac{1}{\mathfrak{r}_{p_j}}M^{\alpha_j}, p_j) \to (\mathbb{R} \times Y_j, (0, \star_{Y_j}))$ where Y_j is an Alexandrov space with $\dim(Y_j) \leq 1$. This is a contradiction. □

Let $0 < \beta_E < \beta_{E'}$ and $0 < \sigma_E < \sigma_{E'}$ be new parameters.

Definition 9.2. A point $p \in M$ is an (s, t)-*edge point* if there is a $(1, s)$-splitting

$$F_p : \left(\frac{1}{\mathfrak{r}_p}M, p\right) \to (\mathbb{R} \times Y, (0, \star_Y)) \tag{9.3}$$

and a t-pointed-Gromov-Hausdorff approximation

$$G_p : (Y, \star_Y) \to ([0, C], 0), \tag{9.4}$$

with $C \geq 200\Delta$. Given F_p and G_p, we put

$$Q_p = (\mathrm{Id} \times G_p) \circ F_p : \left(\frac{1}{\mathfrak{r}_p} M, p\right) \to (\mathbb{R} \times [0, C], (0,0)). \tag{9.5}$$

We let E denote the set of (β_E, σ_E)-edge points, and E' denote the set of $(\beta_{E'}, \sigma_{E'})$-edge points. Note that $E \subset E'$. We will often refer to elements of E as *edge points.*

We emphasize that in the definition above, Q_p maps the basepoint $p \in M$ to $(0,0) \in \mathbb{R} \times [0, C]$.

Lemma 9.6. *Under the constraints $\beta_{E'} < \overline{\beta}_{E'}$, $\sigma_{E'} < \overline{\sigma}_{E'}$ and $\beta_2 < \overline{\beta}_2$, no element $p \in E'$ can be a 2-stratum point.*

Proof. By Lemma 8.1, if p is a 2-stratum point and $\phi_p : (\frac{1}{\mathfrak{r}_p} M, p) \to (\mathbb{R}^2 \times X, (0, \star_X)$ is a $(2, \beta_2)$-splitting then $\operatorname{diam} X < 1$. Thus if $\beta_{E'}$, $\sigma_{E'}$ and β_2 are all small then a large pointed ball in $(\mathbb{R}^2, 0)$ has pointed Gromov-Hausdorff distance less than two from a large pointed ball in $(\mathbb{R} \times [0, C], (0,0))$, which is a contradiction. ☐

We now show that in a neighborhood of $p \in E$, the set E' looks like the border of a half-plane.

Lemma 9.7. *Given $\epsilon > 0$, if $\beta_{E'} < \overline{\beta}_{E'}(\epsilon, \Delta)$, $\sigma_{E'} < \overline{\sigma}_{E'}(\epsilon, \Delta)$, $\beta_E < \overline{\beta}_E(\beta_{E'}, \sigma_{E'})$, $\sigma_E < \overline{\sigma}_E(\beta_{E'}, \sigma_{E'})$ and $\Lambda < \overline{\Lambda}(\epsilon, \Delta)$ then the following holds.*

For $p \in E$, if Q_p is as in Definition 9.2 and $\widehat{Q}_p : (\mathbb{R} \times [0, C], (0, 0)) \to \left(\frac{1}{\mathfrak{r}_p} M, p\right)$ is a quasi-inverse for Q_p (see Subsection 3.1), then $\widehat{Q}_p([-100\Delta, 100\Delta] \times \{0\})$ is $\frac{\epsilon}{2}$-Hausdorff close to $E' \cap Q_p^{-1}([-100\Delta, 100\Delta] \times [0, 100\Delta])$.

Proof. Suppose the lemma were false. Then for some $\epsilon > 0$, there would be sequences $\alpha_i \to \infty$, $s_i \to 0$ and $\Lambda_i \to 0$ so that for each i,

(1) The scale function $\mathfrak{r}$ of M^{α_i} has Lipschitz constant bounded above by Λ_i, and
(2) There is an (s_i^2, s_i^2)-edge point $p_i \in M^{\alpha_i}$ such that $\widehat{Q}_{p_i}([-100\Delta, 100\Delta] \times \{0\})$ is not $\frac{\epsilon}{2}$-Hausdorff close to $E_i' \cap Q_{p_i}^{-1}([-100\Delta, 100\Delta] \times [0, 100\Delta])$, where E_i' denotes the set of (s_i, s_i)-edge points in M^{α_i}.

After passing to a subsequence, we can assume that $\lim_{i\to\infty} \left(\frac{1}{\mathfrak{r}_{p_i}} M^{\alpha_i}, p_i\right) = (X^\infty, p_\infty)$ for some pointed nonnegatively curved Alexandrov space (X^∞, p_∞). We can also assume that $\lim_{i\to\infty} \widehat{Q}_{p_i}\big|_{[-200\Delta, 200\Delta] \times [0, 200\Delta]}$ exists and is an isometric embedding $\widehat{Q}_\infty : [-200\Delta, 200\Delta] \times [0, 200\Delta] \to X^\infty$, with $\widehat{Q}_\infty(0, 0) = p_\infty$. Then

$$\lim_{i\to\infty} \widehat{Q}_{p_i}([-100\Delta, 100\Delta] \times \{0\}) = \widehat{Q}_\infty([-100\Delta, 100\Delta] \times \{0\}). \tag{9.8}$$

However, since $s_i \to 0$ and $\Lambda_i \to 0$, it follows that

$$\lim_{i\to\infty} E_i' \cap Q_{p_i}^{-1}([-100\Delta, 100\Delta] \times [0, 100\Delta]) = \widehat{Q}_\infty([-100\Delta, 100\Delta] \times \{0\}). \tag{9.9}$$

Hence for large i, $\widehat{Q}_{p_i}([-100\Delta, 100\Delta]\times\{0\})$ is $\frac{\epsilon}{2}$-Hausdorff close to $E'_i \cap Q_{p_i}^{-1}([-100\Delta, 100\Delta]\times [0, 100\Delta])$. This is a contradiction. $\square$

The first part of the next lemma says that 1-stratum points are either slim 1-stratum points or lie not too far from an edge point. The second part says that E is coarsely dense in E'.

Lemma 9.10. *Under the constraints $\beta_{E'} < \overline{\beta}_{E'}(\Delta)$, $\sigma_{E'} < \overline{\sigma}_{E'}(\Delta)$, $\beta_E < \overline{\beta}_E(\beta_{E'}, \sigma_{E'})$, $\sigma_E < \overline{\sigma}_E(\beta_{E'}, \sigma_{E'})$, $\beta_1 < \overline{\beta}_1(\Delta, \beta_E)$, $\sigma < \overline{\sigma}(\Delta, \sigma_E)$ and $\Lambda < \overline{\Lambda}(\Delta)$, the following holds.*

(1) *For every 1-stratum point p which is not in the slim 1-stratum, there is some $q \in E$ with $p \in B(q, \Delta\mathfrak{r}_q)$.*
(2) *For every 1-stratum point p which is not in the slim 1-stratum and for every $p' \in E' \cap B(p, 10\Delta\mathfrak{r}_p)$, there is some $q \in E$ with $p' \in B(q, \mathfrak{r}_q)$. See Figure 2 below.*

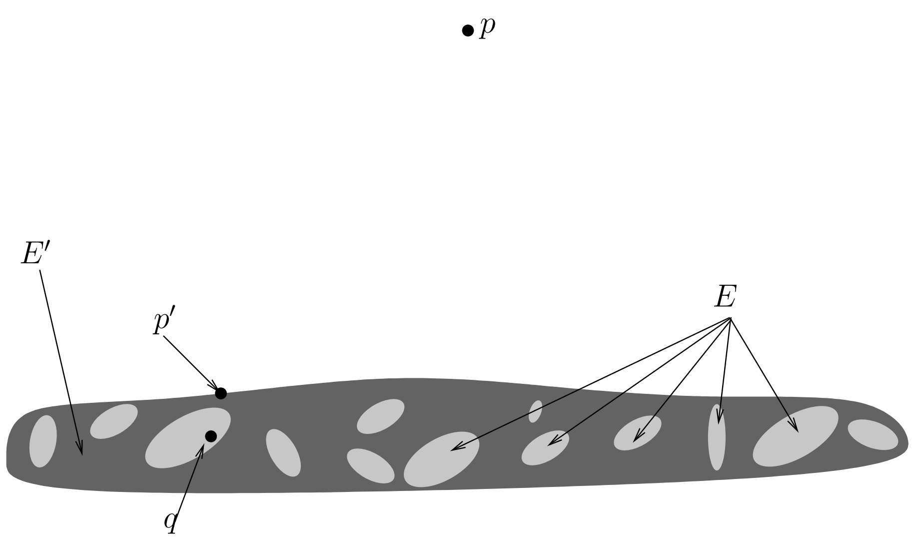

FIGURE 2

Proof. Let $\epsilon > 0$ be a constant that will be adjusted during the proof. Let p be a 1-stratum point which is not in the slim 1-stratum.

By Lemma 9.1, if $\beta_1 < \overline{\beta}_1(\Delta, \epsilon)$ and $\sigma < \overline{\sigma}(\Delta, \epsilon)$ then there is a $(1, \epsilon)$-splitting $F : (\frac{1}{\mathfrak{r}_p} M, p) \to (\mathbb{R} \times Y, (0, \star_Y))$, where Y is a nonnegatively curved Alexandrov space of dimension at most one.

Sublemma 9.11. $\operatorname{diam}(Y) \geq 500\Delta$.

Proof. Suppose that $\operatorname{diam}(Y) < 500\Delta$. Let $\phi : (\frac{1}{\mathfrak{r}_p}M, p) \to (\mathbb{R} \times X, (0, \star_X))$ be a $(1, \beta_1)$-splitting.

Since p belongs to the 1-stratum, $(\frac{1}{\mathfrak{r}_p}M, p)$ does not admit a $(2, \beta_2)$-splitting. By Lemma 4.17, it follows that if $\epsilon < \overline{\epsilon}(\Delta)$ and $\beta_1 < \overline{\beta}_1(\Delta)$ then there is a $\frac{1}{10^3\Delta}$-Gromov-Hausdorff approximation $(X, \star_X) \to (Y, \star_Y)$. Since $Y \subset B(\star_Y, 500\Delta)$, we conclude that the metric annulus $A(\star_X, 600\Delta, 900\Delta) \subset X$ is empty. But then the image of the ball $B(p, \beta_1^{-1}) \subset \frac{1}{\mathfrak{r}_p}M$ under the composition $B(p, \beta_1^{-1}) \stackrel{\phi}{\to} \mathbb{R} \times X \stackrel{\pi_X}{\to} X$ will be contained in $B(\star_X, 600\Delta) \subset X$ (because the inverse image of $B(\star_X, 600\Delta)$ under $\pi_X \circ \phi$ is open and closed in the connected set $B(p, \beta_1^{-1})$). Thus $\phi : (\frac{1}{\mathfrak{r}_p}M, p) \to (\mathbb{R} \times B(\star_X, 600\Delta), (0, \star_X))$ is a $(1, \beta_1)$-splitting, and p is in the slim 1-stratum. This is a contradiction. □

Proof of Lemma 9.10 continued. Suppose first that Y is a circle. If ϵ is sufficiently small then there is a 2-strainer of size $\frac{\Delta}{100}\mathfrak{r}_p$ and quality $\frac{1}{\Delta}$ at p. By the choice of Δ (see Section 7), p is a 2-stratum point, which is a contradiction.

Hence up to isometry, Y is an interval $[0, C]$ with $C > 500\Delta$. If $\star_Y \in \left(\frac{\Delta}{10}, C - \frac{\Delta}{10}\right)$ then the same argument as in the preceding paragraph shows that $\star_Y$ is a 2-stratum point provided ϵ is sufficiently small. Hence $\star_Y \in \left[0, \frac{\Delta}{10}\right]$ or $\star_Y \in \left[C - \frac{\Delta}{10}, C\right]$. In the second case, we can flip $[0, C]$ around its midpoint to reduce to the first case. So we can assume that $\star_Y \in \left[0, \frac{\Delta}{10}\right]$. Let $\widehat{F}$ be a quasi-inverse of F and put $q = \widehat{F}(0, 0)$. If $\Lambda\Delta$ is sufficiently small then we can ensure that $.9 \le \frac{\mathfrak{r}_p}{\mathfrak{r}_q} \le 1.1$. From Lemma 4.10, if β_1 is sufficiently small, relative to β_E, then q has a $(1, \beta_E)$-splitting. If in addition ϵ is sufficiently small, relative to σ_E, then q is guaranteed to be in E. Then $d(p, q) \le \frac{1}{2}\Delta\mathfrak{r}_p < \Delta\mathfrak{r}_q$.

To prove the second part of the lemma, Lemma 9.7 implies that if $p' \in E' \cap B(p, 10\Delta\mathfrak{r}_p)$ then we can assume that p' lies within distance $\frac{1}{2}\mathfrak{r}_p$ from $\widehat{F}([-100\Delta, 100\Delta] \times \{0\})$. (This is not a constraint on the present parameter ϵ.) Choose $q = \widehat{F}(x, 0)$ for some $x \in [-100\Delta, 100\Delta]$ so that $d(p', q) \le \frac{1}{2}\mathfrak{r}_p$. From Lemma 4.10, if β_1 is sufficiently small, relative to β_E, then q has a $(1, \beta_E)$-splitting. If in addition ϵ is sufficiently small, relative to σ_E, then q is guaranteed to be in E. If $\Lambda\Delta$ is sufficiently small then $d(p', q) \le \mathfrak{r}_q$. This proves the lemma. □

9.2. **Regularization of the distance function $d_{E'}$.** Let $d_{E'}$ be the distance function from E'. We will apply the smoothing results from Section 3.6 to $d_{E'}$. We will see that the resulting smoothing of the distance function from E' defines part of a good coordinate in a collar region near E.

Let $\varsigma_{E'} > 0$ be a new parameter.

Lemma 9.12. *Under the constraints $\beta_{E'} < \overline{\beta}_{E'}(\Delta, \varsigma_{E'})$ and $\sigma_{E'} < \overline{\sigma}_{E'}(\Delta, \varsigma_{E'})$ there is a function $\rho_{E'} : M \to [0, \infty)$ such that if $\eta_{E'} = \frac{\rho_{E'}}{\mathfrak{r}}$ then:*

(1) *We have*

$$\left| \frac{\rho_{E'}}{\mathfrak{r}} - \frac{d_{E'}}{\mathfrak{r}} \right| \le \varsigma_{E'} \, . \tag{9.13}$$

(2) *In the set* $\eta_{E'}^{-1}\left[\frac{\Delta}{10}, 10\Delta\right] \cap \left(\frac{d_E}{\mathfrak{r}}\right)^{-1}[0, 50\Delta]$, *the function* $\rho_{E'}$ *is smooth and its gradient lies in the* $\varsigma_{E'}$*-neighborhood of the generalized gradient of* $d_{E'}$.

(3) $\rho_{E'} - d_{E'}$ *is* $\varsigma_{E'}$*-Lipschitz.*

Proof. Let $\epsilon_1 \in (0, \infty)$ and $\theta \in (0, \pi)$ be constants, to be determined during the proof.

Put

$$C = \left(\frac{d_{E'}}{\mathfrak{r}}\right)^{-1}\left[\frac{\Delta}{20}, 20\Delta\right] \cap \left(\frac{d_E}{\mathfrak{r}}\right)^{-1}[0, 50\Delta]. \tag{9.14}$$

If $x \in C$ and $\Lambda < \overline{\Lambda}(\Delta)$ then there exists a $p \in E$ such that $x \in B(p, 60\Delta) \subset \frac{1}{\mathfrak{r}_p}M$. By Lemma 9.7, provided that $\beta_{E'} < \overline{\beta}_{E'}(\epsilon_1, \Delta)$, $\sigma_{E'} < \overline{\sigma}_{E'}(\epsilon_1, \Delta)$, $\beta_E < \overline{\beta}_E(\beta_{E'}, \sigma_{E'})$, $\sigma_E < \overline{\sigma}_E(\beta_{E'}, \sigma_{E'})$, and $\Lambda < \overline{\Lambda}(\epsilon_1, \Delta)$, there is a $y \in M$ such that in $\frac{1}{\mathfrak{r}_p}M$,

$$|d(y, x) - d_{E'}(x)| < \epsilon_1, \quad |d(y, E') - 2d_{E'}(x)| < \epsilon_1. \tag{9.15}$$

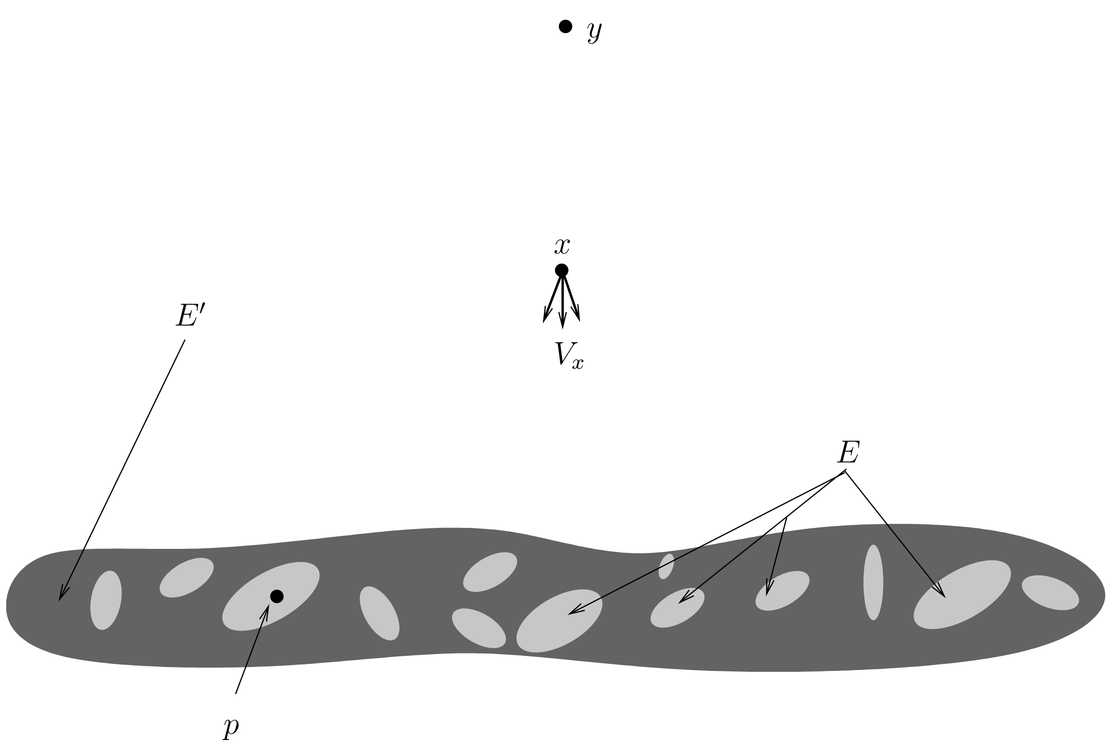

FIGURE 3

By triangle comparison, involving triangles whose vertices are at x, y and points in E' whose distance to x is almost infimal, it follows that if $\epsilon_1 < \overline{\epsilon}_1(\theta, \Delta)$ then $\operatorname{diam}(V_x) < \theta$, where V_x is the set of initial velocities of minimizing geodesic segments from x to E'. See Figure 3.

Applying Corollary 3.16, if $\theta < \overline{\theta}(\varsigma_{E'})$ then we obtain a function $\rho_{E'} : M \to [0, \infty)$ such that

(1) $\rho_{E'}$ is smooth in a neighborhood of C.
(2) $\left\| \frac{\rho_{E'}}{\mathfrak{r}} - \frac{d_{E'}}{\mathfrak{r}} \right\| < \varsigma_{E'}$.
(3) For every $x \in C$, the gradient of $\rho_{E'}$ lies in the $\varsigma_{E'}$-neighborhood of the generalized gradient of $d_{E'}$.
(4) $\rho_{E'} - d_{E'}$ is $\varsigma_{E'}$-Lipschitz.

If $\varsigma_{E'} < \frac{\Delta}{20}$ then

$$\eta_{E'}^{-1} \left[\frac{\Delta}{10}, 10\Delta \right] \cap (\frac{d_E}{\mathfrak{r}})^{-1}[0, 50\Delta] \subset C \,, \tag{9.16}$$

so the lemma follows. $\square$

9.3. **Adapted coordinates tangent to the edge.** In this subsection, $p \in E$ will denote an edge point and Q_p will denote a map as in (9.5).

Let $\varsigma_{\mathrm{edge}} > 0$ be a new parameter. Applying Lemma 4.23, we get:

Lemma 9.17. *Under the constraint $\beta_E < \overline{\beta}_E(\Delta, \varsigma_{\mathrm{edge}})$, there is a Q_p-adapted coordinate*

$$\eta_p : \left(\frac{1}{\mathfrak{r}_p} M, p \right) \supset B(p, 100\Delta) \to \mathbb{R} \tag{9.18}$$

of quality $\varsigma_{\mathrm{edge}}$.

We define a global function $\zeta_p : M \to [0,1]$ by extending

$$(\Phi_{-9\Delta, -8\Delta, 8\Delta, 9\Delta} \circ \eta_p) \cdot (\Phi_{8\Delta, 9\Delta} \circ \eta_{E'}) : B(p, 100\Delta) \to [0,1] \tag{9.19}$$

by zero.

Lemma 9.20. *The following holds:*

(1) *ζ_p is smooth.*
(2) *Under the constraints $\beta_2 < \overline{\beta}_2(\varsigma_{\text{2-stratum}})$, $\Lambda < \overline{\Lambda}(\varsigma_{\text{2-stratum}}, \Delta)$, $\beta_{E'} < \overline{\beta}_{E'}(\varsigma_{\text{2-stratum}}, \Delta)$, $\sigma_{E'} < \overline{\sigma}_{E'}(\varsigma_{\text{2-stratum}}, \Delta)$, $\beta_E < \overline{\beta}_E(\beta_2, \beta_{E'}, \sigma_{E'}, \varsigma_{\text{2-stratum}})$, $\sigma_E < \overline{\sigma}_E(\beta_2, \beta_{E'}, \sigma_{E'}, \varsigma_{\text{2-stratum}})$, $\varsigma_{E'} < \overline{\varsigma}_{E'}(\varsigma_{\text{2-stratum}})$ and $\varsigma_{\mathrm{edge}} < \overline{\varsigma}_{\mathrm{edge}}(\varsigma_{\text{2-stratum}})$, if $x \in (\eta_p, \eta_{E'})^{-1}([-10\Delta, 10\Delta] \times \left[\frac{1}{10}\Delta, 10\Delta\right])$ then x is a 2-stratum point, and there is a $(2, \beta_2)$-splitting $\phi : (\frac{1}{\mathfrak{r}_x} M, x) \to \mathbb{R}^2$ such that $(\eta_i, \eta_{E'}) : (\frac{1}{\mathfrak{r}_x} M, x) \to (\mathbb{R}^2, \phi(x))$ defines ϕ-adapted coordinates of quality $\varsigma_{\text{2-stratum}}$ on the ball $B(x, 100) \subset \frac{1}{\mathfrak{r}_x} M$.*

Proof. (1). This follows from Lemma 9.12.

(2). Let $\epsilon_1, \ldots, \epsilon_5 > 0$ be constants, to be chosen at the end of the proof. For $i \in \{1, 2\}$ choose points $x_i^\pm \in M$ such that $Q_p(x_i^\pm) \in B(Q_p(x) \pm \frac{\Delta}{20} e_i, \sigma_E) \subset \mathbb{R} \times [0, C]$. Provided that $\beta_E < \overline{\beta}_E(\beta_2, \Delta, \epsilon_1)$, $\sigma_E < \overline{\sigma}_E(\beta_2, \Delta, \epsilon_1)$ and $\Lambda < \overline{\Lambda}(\Delta)$, the tuple $\{x_i^\pm\}_{i=1}^2$ will be a 2-strainer at x of quality ϵ_1, and scale at least $\frac{\Delta}{30}$ in $\frac{1}{\mathfrak{r}_x} M$. Therefore, if $\epsilon_1 < \overline{\epsilon}_1(\beta_2)$ then x will be a 2-stratum point, with a $(2, \beta_2)$-splitting $\phi : (\frac{1}{\mathfrak{r}_x} M, x) \to (\mathbb{R}^2, 0)$ given by strainer coordinates as in Lemma 4.15.

Suppose that y is a point in $\frac{1}{\mathfrak{r}_x} M$ with $d(y, x) < \epsilon_2 \cdot \frac{\Delta}{20}$, and $z \in E'$ is a point with $d(y, z) \leq d(y, E') + \epsilon_2 \Delta$; see Figure 4. Then by Lemma 9.7, if $\beta_{E'} < \overline{\beta}_{E'}(\epsilon_3, \Delta)$, $\sigma_{E'} < \overline{\sigma}_{E'}(\epsilon_3, \Delta)$, $\beta_E <$

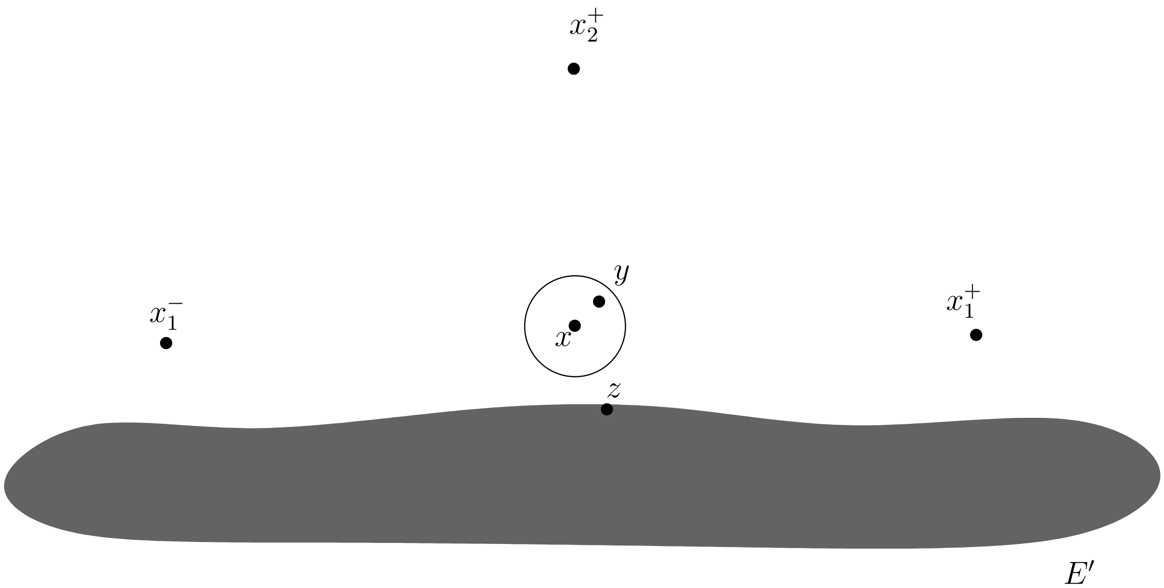

FIGURE 4

$\overline{\beta}_E(\epsilon_3, \beta_{E'}, \sigma_{E'})$, $\sigma_E < \overline{\sigma}_E(\epsilon_3, \beta_{E'}, \sigma_{E'})$, $\Lambda < \overline{\Lambda}(\epsilon_3, \Delta)$ and $\epsilon_2 < \overline{\epsilon}_2(\epsilon_3)$ then the comparison angles $\widetilde{\angle}_y(x_1^\pm, z)$, $\widetilde{\angle}_y(x_2^+, z)$ will satisfy $|\widetilde{\angle}_y(x_1^\pm, z) - \frac{\pi}{2}| < \epsilon_3$ and $|\widetilde{\angle}_y(x_2^+, z) - \pi| < \epsilon_3$. If $\gamma_i^\pm$ is a minimizing segment from y to $x_i^\pm$, and γ_z is a minimizing segment from y to z, it follows that $|\angle_y(\gamma_1^\pm, \gamma_z) - \frac{\pi}{2}| < \epsilon_4$ and $|\angle_y(\gamma_2^+, \gamma_z) - \pi| < \epsilon_4$, provided that $\epsilon_i < \overline{\epsilon}_i(\epsilon_4)$ for $i \leq 3$. Therefore $|D\eta_{E'}((\gamma_1^\pm)'(0))| < \epsilon_5$ and $|D\eta_{E'}((\gamma_2^+)'(0)) - 1| < \epsilon_5$, provided that $\epsilon_4 < \overline{\epsilon}_4(\epsilon_5)$ and $\varsigma_{E'} < \overline{\varsigma}_{E'}(\epsilon_5)$. Likewise, $|D\eta_p(\gamma_1^\pm)'(0)) - 1| < \epsilon_5$ and $|D\eta_p((\gamma_2^\pm)'(0))| < \epsilon_5$, provided that $\beta_E < \overline{\beta}_E(\epsilon_5)$ and $\varsigma_{\text{edge}} < \overline{\varsigma}_{\text{edge}}(\epsilon_5)$. It follows from Lemma 4.28 that $(\eta_p, \eta_{E'})$ defines ϕ-adapted coordinates of quality $\varsigma_{\text{2-stratum}}$ on $B(x, 100) \subset \frac{1}{\mathfrak{r}_x}M$, provided that $\epsilon_5 < \overline{\epsilon}_5(\varsigma_{\text{2-stratum}})$ and $\beta_2 < \overline{\beta}_2(\varsigma_{\text{2-stratum}})$.

We may fix the constants in the order $\epsilon_5, \ldots, \epsilon_1$. The lemma follows. □

9.4. **The topology of the edge region.** In this subsection we determine the topology of a suitable neighborhood of an edge point $p \in E$.

Lemma 9.21. *Under the constraints $\beta_{E'} < \overline{\beta}_{E'}(\Delta)$, $\sigma_{E'} < \overline{\sigma}_{E'}(\Delta)$, $\beta_E < \overline{\beta}_E(\beta_{E'}, \sigma_{E'}, w')$, $\sigma_E < \overline{\sigma}_E(\beta_{E'}, \sigma_{E'})$, $\varsigma_{\text{edge}} < \overline{\varsigma}_{\text{edge}}(\Delta)$, $\varsigma_{E'} < \overline{\varsigma}_{E'}(\Delta)$, $\Lambda < \overline{\Lambda}(\Delta)$ and $\sigma < \overline{\sigma}(\Delta)$, the map η_p restricted to $(\eta_p, \eta_{E'})^{-1}((-4\Delta, 4\Delta) \times (-\infty, 4\Delta])$ is a fibration with fiber diffeomorphic to the closed 2-disk D^2.*

Proof. Let $\epsilon > 0$ be a constant which will be internal to this proof.

By Lemma 6.16, if $\beta_E < \overline{\beta}_E(\epsilon, \Delta, w')$ then the map F_p of Definition 9.2 is ϵ-close to a $(1, \epsilon)$-splitting $\phi : \left(\frac{1}{\mathfrak{r}_p}M, p\right) \to (\mathbb{R} \times Z, (0, \star_Z))$, where Z is a complete pointed nonnegatively curved C^K-smooth surface, and ϕ is ϵ-close in the C^{K+1}-topology to an isometry between the ball $B(p, 1000\Delta) \subset \left(\frac{1}{\mathfrak{r}_p}M, p\right)$ and its image in $(\mathbb{R} \times Z, (0, \star_Z))$. If in addition $\sigma_E <$

$\overline{\sigma}_E(\Delta)$ then the pointed ball $(B(\star_Z, 10\Delta), \star_Z) \subset (Z, \star_Z)$ will have pointed-Gromov-Hausdorff distance at most δ from the pointed interval $([0,10], 0)$, where δ is the parameter of Lemma 3.12. By Lemma 3.12, we conclude that $\overline{B(\star_Z, \Delta)}$ is homeomorphic to the closed 2-disk.

Put $Y = \mathbb{R} \times \{\star_Z\} \subset \mathbb{R} \times Z$ and let $d_Y : \mathbb{R} \times Z \to \mathbb{R}$ be the distance to Y. By Lemma 3.12, if ϵ is sufficiently small then for every $x \in \mathbb{R} \times Z$ with $d_Y(x) \in [\Delta, 9\Delta]$, the set V_x of initial velocities of minimizing segments from x to Y has small diameter; moreover V_x is orthogonal to the $\mathbb{R}$-factor of $\mathbb{R} \times Z$. Thus we may apply Lemma 3.16 to find a smoothing ρ_Y of d_Y, where $\|\rho_Y - d_Y\|_\infty$ is small, and in $d_Y^{-1}(\Delta, 9\Delta)$ the gradient of ρ_Y is close to the generalized gradient of d_Y.

Note that by Lemma 9.7, we may assume that $\phi^{-1}(\mathbb{R} \times \{\star_Z\}) \cap B(p, 50\Delta)$ is Hausdorff close to $E' \cap B(p, 50\Delta)$. Since ϕ is C^{K+1}-close to an isometry, the generalized gradient of $d_Y \circ \phi$ will be close to the generalized gradient of $d_{E'}$ in the region $(\eta_p, \eta_{E'})^{-1}(-5\Delta, 5\Delta) \times (2\Delta, 5\Delta)$, where the gradients are taken with respect to the metric on $\frac{1}{\mathfrak{r}_p} M$. (One may see this by applying a compactness argument to conclude that minimizing geodesics to E' in this region are mapped by ϕ to be C^1-close to minimizing geodesics to Y.) Hence if $\varsigma_{E'}$ is small then $\eta_{E'}$ and $\rho_Y \circ \phi$ will be C^1-close in the region $(\eta_p, \eta_{E'})^{-1}(-5\Delta, 5\Delta) \times (2\Delta, 5\Delta)$. Similarly, if β_E and ς_{edge} are small then η_p will be C^1-close to $\pi_Z \circ \phi$ in the region $(\eta_p, \eta_{E'})^{-1}(-5\Delta, 5\Delta) \times (-\infty, 5\Delta)$.

For $t \in [0,1]$, define a map $f^t : (\eta_p, \eta_{E'})^{-1}\left((-5\Delta, 5\Delta) \times (-\infty, 5\Delta)\right) \to \mathbb{R}^2$ by

$$f^t = (t\,\eta_p + (1-t)\pi_Z \circ \phi\,,\; t\,\eta_{E'} + (1-t)\rho_Y \circ \phi). \tag{9.22}$$

Let $F : (\eta_p, \eta_{E'})^{-1}\left((-5\Delta, 5\Delta) \times (-\infty, 5\Delta)\right) \times [0,1] \to \mathbb{R}^2$ be the map with slices $\{f^t\}$. In view of the C^1-closeness discussed above, we may now apply Lemma C.1 to conclude that $(\eta_p, \eta_{E'})^{-1}(\{0\} \times (-\infty, 4\Delta])$ is diffeomorphic to $(\pi_Z, \rho_Y)^{-1}(\{0\} \times (-\infty, 4\Delta])$, which is a closed 2-disk.

Finally, we claim that the restriction of η_p to $(\eta_p, \eta_{E'})^{-1}(-4\Delta, 4\Delta) \times (-\infty, 4\Delta]$ yields a proper submersion to $(-4\Delta, 4\Delta)$, and is therefore a fibration. The properness follows from the fact that $(\eta_p, \eta_{E'})^{-1}((-4\Delta, 4\Delta) \times (-\infty, 4\Delta])$ is contained in a compact subset of the domain of $(\eta_p, \eta_{E'})$. The fact that it is a submersion follows from the nonvanishing of $D\eta_p$, and the linear independence of $\{D\eta_p, D\eta_{E'}\}$ at points with $(\eta_p, \eta_{E'}) \in (-4\Delta, 4\Delta) \times \{4\Delta\}$. □

Remark 9.23. Given w', we take β_E very small in the proof of Lemma 9.21 in order to get a very good 1-splitting. On the other hand, we just have to take σ_E, and hence σ, small enough to apply Lemma 3.12; the parameter δ of Lemma 3.12 is independent of w'. This will be important for the order in which we choose the parameters.

9.5. **Selection of edge balls.** Let $\{p_i\}_{i \in I_{\text{edge}}}$ be a maximal set of edge points with the property that the collection $\{B(p_i, \frac{1}{3}\Delta \mathfrak{r}_{p_i})\}_{i \in I_{\text{edge}}}$ is disjoint. We write ζ_i for ζ_{p_i}.

Lemma 9.24. *Under the constraints $\mathcal{M} > \overline{\mathcal{M}}$ and $\Lambda < \overline{\Lambda}(\Delta)$,*

- $\bigcup_{i \in I_{\text{edge}}} B(p_i, \Delta \mathfrak{r}_{p_i})$ *contains E.*
- *The intersection multiplicity of the collection $\{\text{supp}(\zeta_i)\}_{i \in I_{\text{edge}}}$ is bounded by $\mathcal{M}$.*

Proof. We omit the proof, as it is similar to the proof of Lemma 8.5. □

We now give a useful covering of the 1-stratum points.

Lemma 9.25. *Under the constraint $\Lambda < \overline{\Lambda}(\Delta)$, any 1-stratum point lies in the slim 1-stratum or lies in $\bigcup_{i \in I_{\mathrm{edge}}} B(p_i, 3\Delta \mathfrak{r}_{p_i})$.*

Proof. From Lemma 9.10, if a 1-stratum point does not lie in the slim 1-stratum then it lies in $B(p, \Delta \mathfrak{r}_p)$ for some $p \in E$. By Lemma 9.24 we have $p \in B(p_i, \Delta \mathfrak{r}_{p_i})$ for some $i \in I_{\mathrm{edge}}$. If $\Lambda\Delta$ is sufficiently small then we can assume that $.9 \le \frac{\mathfrak{r}_p}{\mathfrak{r}_{p_i}} \le 1.1$. The lemma follows. □

The next lemma will be used later for the interface between the slim stratum and the edge stratum.

Lemma 9.26. *Under the constraints $\beta_E < \overline{\beta}_E(\Delta, \beta_2)$, $\varsigma_{\mathrm{edge}} < \overline{\varsigma}_{\mathrm{edge}}(\Delta, \beta_2)$ and $\Lambda < \overline{\Lambda}(\Delta)$, the following holds. Suppose for some $i \in I_{\mathrm{edge}}$ and $q \in B(p_i, 10\Delta \mathfrak{r}_{p_i})$ we have*

$$\eta_{E'}(q) < 5\Delta\,, \quad |\eta_{p_i}(q)| < 5\Delta\,. \tag{9.27}$$

Then either p_i belongs to the slim 1-stratum, or there is a $j \in I_{\mathrm{edge}}$ such that $q \in B(p_j, 10\Delta \mathfrak{r}_{p_j})$ and $|\eta_{p_j}(q)| < 2\Delta$.

Proof. We may assume that p_i does not belong to the slim 1-stratum.

From the definition of η_i and Lemma 9.7, provided that $\beta_{E'} < \overline{\beta}_{E'}(\Delta)$, $\beta_E < \overline{\beta}_E(\beta_{E'})$, $\Lambda < \overline{\Lambda}(\Delta)$ and $\varsigma_{\mathrm{edge}} < \overline{\varsigma}_{\mathrm{edge}}(\Delta)$, there will be a $q' \in E' \cap B(p_i, 10\Delta \mathfrak{r}_{p_i})$ such that $|\eta_{p_i}(q') - \eta_{p_i}(q)| < \frac{1}{10}\Delta$. Since p_i is not in the slim 1-stratum, by Lemma 9.10(2) there is a $p \in E$ such that $q' \in B(p, \mathfrak{r}_p)$, and by Lemma 9.24 we have $p \in B(p_j, \Delta \mathfrak{r}_{p_j})$ for some $j \in I_{\mathrm{edge}}$. If $\Lambda\Delta$ is small then we will have $|\eta_{p_j}(q')| < 1.5\Delta$. Lemma 4.31 now implies that if $\beta_{\mathrm{edge}} < \overline{\beta}_{\mathrm{edge}}(\Delta, \beta_2)$ and $\varsigma_{\mathrm{edge}} < \overline{\varsigma}_{\mathrm{edge}}(\Delta, \beta_2)$ then $|\eta_{p_j}(q') - \eta_{p_j}(q)| < \frac{1}{2}\Delta$. Thus $|\eta_{p_j}(q)| < 2\Delta$. □

9.6. **Additional cutoff functions.** We define two additional cutoff functions for later use:

$$\zeta_{\mathrm{edge}} = 1 - \Phi_{\frac{1}{2},1} \circ \left(\sum_{i \in I_{\mathrm{edge}}} \zeta_i \right) \tag{9.28}$$

and

$$\zeta_{E'} = \left(\Phi_{\frac{2}{10}\Delta, \frac{3}{10}\Delta, 8\Delta, 9\Delta} \circ \eta_{E'} \right) \cdot \zeta_{\mathrm{edge}}. \tag{9.29}$$

10. The local geometry of the slim 1-stratum

In this section we consider the slim 1-stratum points. Along with the adapted coordinates and cutoff functions, we discuss the local topology and a selection process to get a ball covering of the slim 1-stratum points.

10.1. **Adapted coodinates, cutoff functions and local topology near slim 1-stratum points.** In this subsection, we let p denote a point in the slim 1-stratum, and $\phi_p : \left(\frac{1}{\mathfrak{r}_p} M, p\right) \to (\mathbb{R} \times X, (0, \star_X))$ be a $(1, \beta_1)$-splitting, with $\operatorname{diam}(X) \leq 10^3 \Delta$. Let $\varsigma_{\mathrm{slim}} > 0$ be a new parameter.

Lemma 10.1. *Under the constraint $\beta_1 < \overline{\beta}_1(\Delta, \varsigma_{\mathrm{slim}})$:*

- *There is a ϕ_p-adapted coordinate η_p of quality $\varsigma_{\mathrm{slim}}$ on $B(p, 10^6\Delta) \subset \left(\frac{1}{\mathfrak{r}_p} M, p\right)$.*
- *The cutoff function*

$$(\Phi_{-9\cdot 10^5\Delta, -8\cdot 10^5\Delta, 8\cdot 10^5\Delta, 9\cdot 10^5\Delta}) \circ \eta_p \tag{10.2}$$

extends by zero to a smooth function ζ_p on M.

Proof. This follows from Lemma 4.23 (see also Remark 4.35). □

Let η_p and ζ_p be as in Lemma 10.1.

Lemma 10.3. *Under the constraints $\beta_1 < \overline{\beta}_1(\varsigma_{\mathrm{slim}}, \Delta, w')$, $\varsigma_{\mathrm{slim}} < \overline{\varsigma}_{\mathrm{slim}}(\Delta)$, then $\eta_p^{-1}\{0\}$ is diffeomorphic to S^2 or T^2.*

Proof. From Lemma 6.16, if $\beta_1 < \overline{\beta}_1(\Delta, w')$ then close to ϕ_p, there is a $(1, \beta_1)$-splitting $\phi : \left(\frac{1}{\mathfrak{r}_p} M, p\right) \to (\mathbb{R} \times Z, (0, \star_Z))$ for some complete pointed nonnegatively curved C^K-smooth surface $(Z, \star_Z)$, with the map being C^{K+1}-close to an isometry on $B(p, 10^6\Delta)$. From Definition 7.4, the diameters of the Z-fibers are at most $10^4\Delta$. In particular, since Z is compact and M is orientable, Z must be diffeomorphic to S^2 or T^2. Furthermore, we may assume that for any pair of points $m, m' \in M$ with $m \in B(p, 10^6\Delta) \subset \frac{1}{\mathfrak{r}_p} M$, $d(m, m') \in [2, 10]$, and $\pi_Z(\phi(m)) = \pi_Z(\phi(m'))$, the initial velocity v of a minimizing segment γ from m to m' maps under ϕ_* to a vector almost tangent to the $\mathbb{R}$-factor of $\mathbb{R} \times Z$.

As ϕ is close to ϕ_p, we may assume that η_p is a ϕ-adapted coordinate of quality $2\varsigma_{\mathrm{slim}}$. If $\varsigma_{\mathrm{slim}} < \overline{\varsigma}_{\mathrm{slim}}(\Delta)$, then we may apply the estimate from the preceding paragraph, and the definition of adapted coordinates (specifically (4.22)), to conclude that η_p is C^1-close to the composition $\frac{1}{\mathfrak{r}_p} M \xrightarrow{\phi} \mathbb{R} \times Z \to \mathbb{R}$ on the ball $B(p, 10^6\Delta)$. The lemma now follows from Lemma C.3. □

10.2. **Selection of slim 1-stratum balls.** Let $\{p_i\}_{i \in I_{\mathrm{slim}}}$ be a maximal set of slim stratum points with the property that the collection $\{B(p_i, \frac{1}{3}\Delta\mathfrak{r}_{p_i})\}_{i \in I_{\mathrm{slim}}}$ is disjoint. We write ζ_i for ζ_{p_i}.

Lemma 10.4. *Under the constraints $\mathcal{M} > \overline{\mathcal{M}}$ and $\Lambda < \overline{\Lambda}(\Delta)$,*

- *$\bigcup_{i \in I_{\mathrm{slim}}} B(p_i, \Delta\mathfrak{r}_{p_i})$ contains all slim stratum points.*
- *The intersection multiplicity of the collection $\{\operatorname{supp}(\zeta_i)\}_{i \in I_{\mathrm{slim}}}$ is bounded by $\mathcal{M}$.*

We omit the proof, as it is similar to the proof of Lemma 8.5.

11. The local geometry of the 0-stratum

Thus far, points in the 0-stratum have been defined by a process of elimination (they are points that are neither 2-stratum points nor 1-stratum points) rather than by the presence of some particular geometric structure. We now discuss their geometry. We show in Lemma 11.1 that M has conical structure near every point – not just the 0-stratum points – provided one looks at an appropriate scale larger than $\mathfrak{r}$. We then use this to define radial and cutoff functions near 0-stratum points.

Let $\delta_0 > 0$ and $\Upsilon_0, \Upsilon'_0, \tau_0 > 1$ be new parameters.

11.1. **The Good Annulus Lemma.** We now show that for every point p in M, there is a scale at which a neighborhood of p is well approximated by a model geometry in two different ways: by a nonnegatively curved 3-manifold in the pointed C^K-topology, and by the Tits cone of this manifold in the pointed Gromov-Hausdorff topology.

Lemma 11.1. *Under the constraint $\Upsilon'_0 > \overline{\Upsilon}'_0(\delta_0, \Upsilon_0, w')$, if $p \in M$ then there exists $r^0_p \in [\Upsilon_0 \mathfrak{r}_p, \Upsilon'_0 \mathfrak{r}_p]$ and a complete 3-dimensional nonnegatively curved C^K-smooth Riemannian manifold N_p such that:*

1. *$(\frac{1}{r^0_p} M, p)$ is δ_0-close in the pointed Gromov-Hausdorff topology to the Tits cone $C_T N_p$ of N_p.*
2. *The ball $B(p, r^0_p) \subset M$ is diffeomorphic to N_p.*
3. *The distance function from p has no critical points in the annulus $A(p, \frac{r^0_p}{100}, r^0_p)$.*

Proof. Suppose that conclusion (1) does not hold. Then for each j, if we take $\Upsilon'_0 = j\Upsilon_0$, it is not true that conclusion (1) holds for sufficiently large α. Hence we can find a sequence $\alpha_j \to \infty$ so that for each j, $(M^{\alpha_j}, p_{\alpha_j})$ provides a counterexample with $\Upsilon'_0 = j\Upsilon_0$.

For convenience of notation, we relabel $(M^{\alpha_j}, p_{\alpha_j})$ as (M_j, p_j) and write $\mathfrak{r}_j$ for $\mathfrak{r}_{p_{\alpha_j}}$. Then by assumption, for each $r^0_j \in [\Upsilon_0 \mathfrak{r}_j, j\Upsilon_0 \mathfrak{r}_j]$ there is no 3-dimensional nonnegatively curved Riemannian manifold N_j such that conclusion (1) holds.

Assumption 5.2 implies that a subsequence of $\{(\frac{1}{\mathfrak{r}_j} M_j, p_j)\}_{j=1}^\infty$, which we relabel as $\{(\frac{1}{\mathfrak{r}_j} M_j, p_j)\}_{j=1}^\infty$, converges in the pointed C^K-topology to a pointed 3-dimensional nonnegatively curved C^K-smooth Riemannian manifold (N, p_∞). Now N is asymptotically conical. That is, there is some $R > 0$ so that if $R' > R$ then $(\frac{1}{R'} N, p_\infty)$ is $\frac{\delta_0}{2}$-close in the pointed Gromov-Hausdorff topology to the Tits cone $C_T N$.

By critical point theory, large open balls in N are diffeomorphic to N itself. Hence we can find $R' > 10^3 \max(\Upsilon_0, R)$ so that for any $R'' \in \left(\frac{1}{2}R', 2R'\right)$, there are no critical points of the distance function from p_∞ in $A(p_\infty, \frac{R''}{10^3}, 10R'') \subset N$, and the ball $B(p_\infty, R'')$ is diffeomorphic to N. In view of the convergence $(\frac{1}{\mathfrak{r}_j} M_j, p_j) \to (N, p_\infty)$ in the pointed C^K-topology, it follows that for large j there are no critical points of the distance function in $A(p_j, \frac{R''\mathfrak{r}_j}{100)}, R''\mathfrak{r}_j) \subset N_j$, and $B(p_j, R''\mathfrak{r}_j) \subset M_j$ is diffeomorphic to $B(p_\infty, R'') \subset N$. Taking $r^0_j = R'\mathfrak{r}_j$ gives a contradiction. $\square$

Remark 11.2. If we take the parameter σ of Lemma 6.18 to be small then we can additionally conclude that $C_T N_p$ is pointed Gromov-Hausdorff close to a conical nonnegatively curved Alexandrov space of dimension at most two.

11.2. **The radial function near a 0-stratum point.** For every $p \in M$, we apply Lemma 11.1 to get a scale $r_p^0 \in [\Upsilon_0 \mathfrak{r}_p, \Upsilon_0' \mathfrak{r}_p]$ for which the conclusion of Lemma 11.1 holds. In particular, $\left(\frac{1}{r_p^0} M, p\right)$ is δ_0-close in the pointed Gromov-Hausdorff topology to the Tits cone $C_T N_p$ of a nonnegatively curved 3-manifold N_p.

Let d_p be the distance function from p in $\left(\frac{1}{r_p^0} M, p\right)$. Let $\varsigma_{\text{0-stratum}} > 0$ be a new parameter.

Lemma 11.3. *Under the constraint $\delta_0 < \overline{\delta}_0(\varsigma_{\text{0-stratum}})$, there is a function $\eta_p : \frac{1}{\mathfrak{r}_p^0} M \to [0, \infty)$ such that:*

1. η_p *is smooth on* $A\left(p, \frac{1}{10}, 10\right) \subset \frac{1}{r_p^0} M$.
2. $\| \eta_p - d_p \|_\infty < \varsigma_{\text{0-stratum}}$.
3. $\eta_p - d_p : \frac{1}{\mathfrak{r}_p^0} M \to [0, \infty)$ *is* $\varsigma_{\text{0-stratum}}$*-Lipschitz.*
4. η_p *is smooth and has no critical points in* $\eta_p^{-1}([\frac{2}{10}, 2])$*, and for every* $\rho \in [\frac{2}{10}, 2]$*, the sublevel set* $\eta_p^{-1}([0, \rho])$ *is diffeomorphic to either the closed disk bundle in the normal bundle* νS *of the soul* $S \subset N_p$*, if* N_p *is noncompact, or to* N_p *itself when* N_p *is compact.*
5. *The composition* $\Phi_{\frac{2}{10}, \frac{3}{10}, \frac{8}{10}, \frac{9}{10}} \circ \eta_p$ *extends by zero to a smooth cutoff function* $\zeta_p : M \to [0, 1]$.

Proof. We apply Lemma 3.16 with $Y = \{p\}$, $U = A\left(p, \frac{1}{20}, 20\right)$ and $C = \overline{A\left(p, \frac{1}{10}, 10\right)}$. To verify the hypotheses of Lemma 3.16, suppose that $q \in U$. From Lemma 11.1, for any $\mu > 0$, there is an $\overline{\delta}_0 = \overline{\delta}_0(\mu)$ so that if $\delta_0 < \overline{\delta}_0$ then we can find some $q' \in M$ with $d(p, q') = 2d(p, q)$ and $d(q, q') \geq (1 - \mu) d(p, q)$. Fix a minimizing geodesic γ_1 from q to q'. By triangle comparison, for any $\theta > 0$, if μ is sufficiently small then we can ensure that for any minimizing geodesic γ from q to p, the angle between $\gamma'(0)$ and $\gamma_1'(0)$ is at least $\pi - \frac{\theta}{2}$. Parts (1), (2) and (3) of the lemma now follow from Lemma 3.16.

(4). Using the same proof as the "Morse lemma" for distance functions, one gets a smooth vector field ξ in $A(p, \frac{1}{10}, 10)$, such that ξd_p and $\xi \eta_p$ are both close to 1. Using the flow of ξ, if $\varsigma_{\text{0-stratum}}$ is sufficiently small then for every $\rho \in [\frac{2}{10}, 2]$, the sublevel sets $d_p^{-1}([0, \rho])$ and $\eta_p^{-1}([0, \rho])$ are homeomorphic. Let $\bar{N}$ be the closed disk bundle in νS. Then $\operatorname{int}(\bar{N}) \stackrel{homeo}{\simeq} N_p \stackrel{homeo}{\simeq} \operatorname{int}(d_p^{-1}([0, \rho])) \stackrel{homeo}{\simeq} \operatorname{int}(\eta_p^{-1}([0, \rho]))$. Since two compact orientable 3-manifolds with boundary are homeomorphic provided that their interiors are homeomorphic, we have $\bar{N} \stackrel{homeo}{\simeq} \eta_p^{-1}([0, \rho])$. (This may be readily deduced from the fact that if S is a closed orientable surface then any smooth embedding $S \to S \times \mathbb{R}$, which is also a homotopy equivalence, is isotopic to the fiber $S \times \{0\}$, as follows from the Schoenflies theorem when $S = S^2$ and from [Sta62] when $\operatorname{genus}(S) > 0$.)

(5) follows from the fact that the composition $\Phi_{\frac{2}{10}, \frac{3}{10}, \frac{8}{10}, \frac{9}{10}} \circ \eta_p$ is compactly supported in the annulus $A(p, \frac{1}{10}, 10)$. $\square$

Remark 11.4. One may avoid the Schoenflies and Stallings theorems in the proof of Lemma 11.3(4). If M is a complete noncompact nonnegatively curved manifold, and $p \in M$, then the distance function d_p has no critical points outside $B(p, r_0)$ for some $r_0 \in (0, \infty)$. In fact, for every $r > r_0$, the closed ball $\overline{B(p,r)}$ is isotopic, by an isotopy with arbitrarily small tracks, to a compact domain with smooth boundary D; moreover, the smooth isotopy class $[D]$ is canonical and independent of $r \in (r_0, \infty)$. (These assertions are true in general for noncritical sublevel sets of proper distance functions. They are proved by showing that one may smooth d_p near $S(p,r)$ without introducing critical points.) The proof of the soul theorem actually shows that the isotopy class $[D]$ is the same as that of a closed smooth tubular neighborhood of the soul, which is diffeomorphic to the unit normal bundle of the soul.

11.3. **Selecting the 0-stratum balls.** The next lemma has a statement about an adapted coordinate for the radial splitting in an annular region of a 0-stratum ball. We use the parameter ς_{slim} for the quality of this splitting, even though there is no *a priori* relationship to slim 1-stratum points. Our use of this parameter will simplify the later parameter ordering.

Lemma 11.5. *Under the constraints $\delta_0 < \bar{\delta}_0(\beta_1, \varsigma_{\text{slim}})$, $\Upsilon_0 > \overline{\Upsilon}_0(\beta_1)$, $\beta_1 < \overline{\beta}_1(\varsigma_{\text{slim}})$ and $\varsigma_{\text{0-stratum}} < \bar{\varsigma}_{\text{0-stratum}}(\varsigma_{\text{slim}})$, there is a finite collection $\{p_i\}_{i \in I_{\text{0-stratum}}}$ of points in M so that*

1. *The balls $\{B(p_i, r^0_{p_i})\}_{i \in I_{\text{0-stratum}}}$ are disjoint.*
2. *If $q \in B(p_i, 10r^0_{p_i})$, for some $i \in I_{\text{0-stratum}}$, then $r^0_q \le 20 r^0_{p_i}$ and $\frac{r^0_{p_i}}{\mathfrak{r}_q} \ge \frac{1}{20}\Upsilon_0$.*
3. *For each i, every $q \in A(p_i, \frac{1}{10}r^0_{p_i}, 10r^0_{p_i})$ belongs to the 1-stratum or 2-stratum, and there is a $(1, \beta_1)$-splitting of $\left(\frac{1}{\mathfrak{r}_q}M, q\right)$, for which $\frac{r^0_{p_i}}{\mathfrak{r}_q}\,\eta_{p_i}$ is an adapted coordinate of quality ς_{slim}.*
4. *$\bigcup_{i \in I_{\text{0-stratum}}} B(p_i, \frac{1}{10}r^0_{p_i})$ contains all the 0-stratum points.*
5. *For each $i \in I_{\text{0-stratum}}$, the manifold N_{p_i} has at most one end.*

Proof. (1). Let $V_0 \subset M$ be the set of points $p \in M$ such that the ball $B(p, r^0_p)$ contains a 0-stratum point. We partially order V_0 by declaring that $p_1 \prec p_2$ if and only if ($2r^0_{p_1} < r^0_{p_2}$ and $B(p_1, r^0_{p_1}) \subset B(p_2, r^0_{p_2})$). Note that every chain in the poset $(V_0, \prec)$ has an upper bound, since $r^0_p < \Upsilon'_0 \mathfrak{r}_p$ is bounded above. Let $V \subset V_0$ be the subset of elements which are maximal with respect to $\prec$, and apply Lemma A.1 with $\mathcal{R}_p = r^0_p$ to get the finite disjoint collection of balls $\{B(p_i, r^0_{p_i})\}_{i \in I_{\text{0-stratum}}}$. Thus (1) holds.

(2). If $q \in B(p_i, 10r^0_{p_i})$ then $r^0_q \le 20r^0_{p_i}$, for otherwise we would have $q \in V_0$ and $p_i \prec q$, contradicting the maximality of p_i. Thus $\frac{r^0_{p_i}}{\mathfrak{r}_q} \ge \frac{r^0_q}{20\mathfrak{r}_q} \ge \frac{1}{20}\Upsilon_0$.

(3). Suppose that $i \in I_{\text{0-stratum}}$ and $q \in A(p_i, \frac{1}{10}r^0_{p_i}, 10r^0_{p_i})$. Recall that $\left(\frac{1}{r^0_{p_i}}M, p_i\right)$ is δ_0-close in the pointed Gromov-Hausdorff topology to the Tits cone $C_T N_{p_i}$. If the Tits cone $C_T N_{p_i}$ were a single point then $\operatorname{diam}(M)$ would be bounded above by $\delta_0 r^0_{p_i}$; taking $\delta_0 < \frac{1}{10}$ we get $q \in B(p_i, \frac{1}{10}r^0_{p_i})$, which is a contradiction. Therefore $C_T N_{p_i}$ is not a point. It follows that there is a 1-strainer at q of scale comparable to $r^0_{p_i}$ and quality comparable to δ_0, where one of the strainer points is p_i. By (2), if $\Upsilon_0 > \overline{\Upsilon}_0(\beta_1)$ and $\delta_0 < \bar{\delta}(\beta_1)$ then Lemma 4.15

implies there is a $(1, \beta_1)$-splitting $\alpha : \left(\frac{1}{\mathfrak{r}_q} M, q\right) \to (\mathbb{R} \times X, (0, \star_X))$, where the first component is given by $d_{p_i} - d_{p_i}(q)$. In particular q is a 1-stratum point or a 2-stratum point. By Lemma 11.3, the smooth radial function η_{p_i} is $\varsigma_{\text{0-stratum}}$-Lipschitz close to d_{p_i}. Lemma 4.28 implies that if $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\varsigma_{\text{slim}})$ then we are ensured that η_{p_i} is an α-adapted coordinate of quality ς_{slim}. Hence (3) holds.

(4). If q is in the 0-stratum then $q \in V_0$, so $q \prec \bar{q}$ for some $\bar{q} \in V$. By Lemma A.1, for some $i \in I_{\text{0-stratum}}$ we have $B(\bar{q}, r^0_{\bar{q}}) \cap B(p_i, r^0_{p_i}) \neq \emptyset$ and $r^0_{\bar{q}} \leq 2r^0_{p_i}$. Therefore $q \in B(p_i, 5r^0_{p_i})$ and by (3), we have $q \in B(p_i, \frac{1}{10} r^0_{p_i})$.

(5). Let $\epsilon > 0$ be a new constant. Suppose that N_{p_i} has more than one end. Then $C_T N_{p_i} \simeq \mathbb{R}$. If $\delta_0 < \overline{\delta}_0(\epsilon)$ then every point $q \in B(p_i, 1) \subset \frac{1}{r^0_{p_i}} M$ will have a strainer of quality ϵ and scale ϵ^{-1}. By (2) and Lemma 4.15, if $\epsilon < \overline{\epsilon}(\beta_1)$ and $\Upsilon_0 > \overline{\Upsilon}_0(\beta_1)$ then there is a $(1, \beta_1)$ splitting of $(\frac{1}{\mathfrak{r}_q} M, q)$. Thus every point in $B(p_i, r^0_{p_i})$ is in the 1-stratum or 2-stratum. This contradicts the definition of V, and hence N_{p_i} has at most one end. □

12. Mapping into Euclidean space

12.1. **The definition of the map $\mathcal{E}^0 : M \to H$.** We will now use the ball collections defined in Sections 8-11, and the geometrically defined functions discussed in earlier sections, to construct a smooth map $\mathcal{E}^0 : M \to H = \bigoplus_{i \in I} H_i$, where

- $I = I_{\mathfrak{r}} \cup I_{E'} \cup I_{\text{0-stratum}} \cup I_{\text{slim}} \cup I_{\text{edge}} \cup I_{\text{2-stratum}}$, where the two index sets $I_{\mathfrak{r}}$ and $I_{E'}$ are singletons $I_{\mathfrak{r}} = \{\mathfrak{r}\}$ and $I_{E'} = \{E'\}$ respectively,
- H_i is a copy of $\mathbb{R}$ when $i = \mathfrak{r}$,
- H_i is a copy of $\mathbb{R} \oplus \mathbb{R}$ when $i \in I_{E'} \cup I_{\text{0-stratum}} \cup I_{\text{slim}} \cup I_{\text{edge}}$, and
- H_i is a copy of $\mathbb{R}^2 \oplus \mathbb{R}$ when $i \in I_{\text{2-stratum}}$.

We also put

- $H_{\text{0-stratum}} = \bigoplus_{i \in I_{\text{0-stratum}}} H_i$,
- $H_{\text{slim}} = \bigoplus_{i \in I_{\text{slim}}} H_i$,
- $H_{\text{edge}} = \bigoplus_{i \in I_{\text{edge}}} H_i$,
- $H_{\text{2-stratum}} = \bigoplus_{i \in I_{\text{2-stratum}}} H_i$,
- $Q_1 = H$,
- $Q_2 = H_{\text{0-stratum}} \bigoplus H_{\text{slim}} \bigoplus H_{\text{edge}}$,
- $Q_3 = H_{\text{0-stratum}} \bigoplus H_{\text{slim}}$,
- $Q_4 = H_{\text{0-stratum}}$, and
- $\pi_{i,j} : Q_i \to Q_j$, $\pi_i = \pi_{1,i} : H \to Q_i$, $\pi_i^\perp : H \to Q_i^\perp$ are the orthogonal projections, for $1 \leq i \leq j \leq 4$.

If $x \in Q_j$, we denote the projection to a summand H_i by $\pi_{H_i}(x) = x_i$. When $i \neq \mathfrak{r}$, we write $H_i = H_i' \oplus H_i'' \cong \mathbb{R}^{k_i} \oplus \mathbb{R}$, where $k_i \in \{1, 2\}$, and we denote the decomposition of $x_i \in H_i$ into its components by $x_i = (x_i', x_i'') \in H_i' \oplus H_i''$. We denote orthogonal projection onto H_i' and H_i'' by $\pi_{H_i'}$ and $\pi_{H_i''}$, respectively.

In Sections 8-11, we defined adapted coordinates η_p, and cutoff functions ζ_p corresponding to points $p \in M$ of different types. If $\{p_i\}$ is a collection of points used to define a ball cover, as in Sections 8-11, then we write η_i for η_{p_i} and ζ_i for ζ_{p_i}. Recall that we also defined $\eta_{E'}$ and $\zeta_{E'}$ in Sections 9.2 and 9.6, respectively. For $i \in I \setminus \{\mathfrak{r}\}$, we will also define a new scale parameter R_i, as follows:

- If $i \in I_{0\text{-stratum}}$ we put $R_i = r^0_{p_i}$, where $r^0_{p_i}$ is as in Lemma 11.5;
- If $i \in I_{\text{slim}} \cup I_{\text{edge}} \cup I_{2\text{-stratum}}$, then $R_i = \mathfrak{r}_{p_i}$;
- If $i = E'$, then $R_i = \mathfrak{r}$; note that unlike in the other cases, R_i is not a constant.

The component $\mathcal{E}^0_i : M \to H_i$ of the map $\mathcal{E}^0 : M \to H$ is defined to be $\mathfrak{r}$ when $i = \mathfrak{r}$, and

$$(R_i \eta_i \zeta_i, R_i \zeta_i) \tag{12.1}$$

otherwise.

In the remainder of this section we prepare for the adjustment procedure in Section 13 by examining the behavior of $\mathcal{E}^0$ near the different strata.

12.2. **The image of $\mathcal{E}^0$.** Before proceeding, we make some observations about the image of $\mathcal{E}^0$, to facilitate the choice of cutoff functions. Let $x = \mathcal{E}^0(p) \in H$. Then the components of x satisfy the following inequalities:

$$x_{\mathfrak{r}} > 0 \tag{12.2}$$

and for every $i \in I_{0\text{-stratum}} \cup I_{\text{slim}} \cup I_{\text{edge}} \cup I_{2\text{-stratum}}$,

$$x''_i \in [0, R_i] \quad \text{and} \quad |x'_i| \leq c_i \, x''_i \,, \tag{12.3}$$

where

$$c_i = \begin{cases} 9\Delta & \text{when } i \in I_{E'}, \\ \frac{9}{10} & \text{when } i \in I_{0\text{-stratum}}, \\ 10^5\Delta & \text{when } i \in I_{\text{slim}}, \\ 9\Delta & \text{when } i \in I_{\text{edge}}, \\ 9 & \text{when } i \in I_{2\text{-stratum}}. \end{cases} \tag{12.4}$$

Lemma 12.5. *Under the constraint $\Lambda \leq \overline{\Lambda}(\mathcal{M})$, there is a number $\Omega_0 = \Omega_0(\mathcal{M})$ so that for all $p \in M$, $|D\mathcal{E}^0_p| \leq \Omega_0$.*

Proof. This follows from the definition of $\mathcal{E}^0$. □

12.3. **Structure of $\mathcal{E}^0$ near the 2-stratum.** Put

$$\tilde{A}_1 = \bigcup_{i \in I_{2\text{-stratum}}} \{|\eta_i| \leq 8\} \,, \quad A_1 = \bigcup_{i \in I_{2\text{-stratum}}} \{|\eta_i| \leq 7\} \,. \tag{12.6}$$

We refer to Definition B.1 for the definition of a cloudy manifold. We will see that on a scale which is sufficiently small compared with $\mathfrak{r}$, the pair $(\tilde{S}_1, S_1) = (\mathcal{E}^0(\tilde{A}_1), \mathcal{E}^0(A_1)) \subset H$ is a cloudy 2-manifold. In brief, this is because, on a scale small compared with $\mathfrak{r}$, near any point in A_1 the map $\mathcal{E}^0$ is well approximated in the C^1 topology by an affine function of η_i, for some $i \in I_{2\text{-stratum}}$.

Let $\Sigma_1, \Gamma_1 > 0$ be new parameters. Define $r_1 : \tilde{S}_1 \to (0, \infty)$ by putting $r_1(x) = \Sigma_1 \, \mathfrak{r}_p$ for some $p \in (\mathcal{E}^0)^{-1}(x) \cap \tilde{A}_1$.

Lemma 12.7. *There is a constant $\Omega_1 = \Omega_1(\mathcal{M})$ so that under the constraints $\Sigma_1 < \overline{\Sigma}_1(\Gamma_1, \mathcal{M})$, $\beta_2 < \overline{\beta}_2(\Gamma_1, \Sigma_1, \mathcal{M})$, $\varsigma_{\text{2-stratum}} < \overline{\varsigma}_{\text{2-stratum}}(\Gamma_1, \Sigma_1, \mathcal{M})$, $\beta_E < \overline{\beta}_E(\Gamma_1, \Sigma_1, \Delta, \mathcal{M})$, $\sigma_E < \overline{\sigma}_E(\Gamma_1, \Sigma_1, \Delta, \mathcal{M})$, $\varsigma_{\text{edge}} < \overline{\varsigma}_{\text{edge}}(\Gamma_1, \Sigma_1, \Delta, \mathcal{M})$, $\varsigma_{E'} < \overline{\varsigma}_E(\Gamma_1, \Sigma_1, \Delta, \mathcal{M})$, $\beta_1 < \overline{\beta}_1(\Gamma_1, \Sigma_1, \Delta, \mathcal{M})$, $\varsigma_{\text{slim}} < \overline{\varsigma}_{\text{slim}}(\Gamma_1, \Sigma_1, \Delta, \mathcal{M})$, $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\Gamma_1, \Sigma_1, \Delta, \mathcal{M})$, $\Upsilon_0 \geq \overline{\Upsilon}_0(\Gamma_1, \Sigma_1, \Delta, \mathcal{M})$ and $\Lambda < \overline{\Lambda}(\Gamma_1, \Sigma_1, \Delta, \mathcal{M})$, the following holds.*

1. *The triple $(\tilde{S}_1, S_1, r_1)$ is a $(2, \Gamma_1)$ cloudy 2-manifold.*
2. *The affine subspaces $\{A_x\}_{x \in S_1}$ inherent in the definition of the cloudy 2-manifold can be chosen to have the following property. Pick $p \in A_1$ and put $x = \mathcal{E}^0(p) \in S_1$. Let $A^0_x \subset H$ be the linear subspace parallel to A_x (i.e. $A_x = A^0_x + x$) and let $\pi_{A^0_x} : H \to A^0_x$ denote orthogonal projection onto A^0_x. Then*

$$\| D\mathcal{E}^0_p - \pi_{A^0_x} \circ D\mathcal{E}^0_p \| < \Gamma_1 \, , \tag{12.8}$$

and

$$\Omega_1^{-1} \|v\| \leq \| \pi_{A^0_x} \circ D\mathcal{E}^0(v) \| \leq \Omega_1 \|v\| \tag{12.9}$$

for every $v \in T_pM$ which is orthogonal to $\ker(\pi_{A^0_x} \circ D\mathcal{E}^0_p)$.

3. *Given $i \in I_{\text{2-stratum}}$, there is a smooth map $\widehat{\mathcal{E}}^0_i : (\overline{B(0,8)} \subset \mathbb{R}^2) \to (H'_i)^\perp$ such that*

$$\| D\widehat{\mathcal{E}}^0_i \| \leq \Omega_1 R_i \tag{12.10}$$

and on the subset $\{|\eta_i| \leq 8\} \subset \frac{1}{R_i} M$, we have

$$\left\| \frac{1}{R_i} \mathcal{E}^0 - \left(\eta_i, \frac{1}{R_i} \widehat{\mathcal{E}}^0_i \circ \eta_i \right) \right\|_{C^1} < \Gamma_1. \tag{12.11}$$

Furthermore, if $x \in S_1$ then there are some $i \in I_{\text{2-stratum}}$ and $p \in \{|\eta_i| \leq 7\}$ such that $x = \mathcal{E}^0(p)$ and $A^0_x = \operatorname{Im}\left(I, \frac{1}{R_i} (D\widehat{\mathcal{E}}^0_i)_{\eta_i(p)} \right)$.

The parameters ϵ_1, $\epsilon_2 > 0$ will be internal to this subsection, which is devoted to the proof of Lemma 12.7. Until further notice, the index i will denote a fixed element of $I_{\text{2-stratum}}$.

Put $J = \{ j \in I_{E'} \cup I_{\text{0-stratum}} \cup I_{\text{slim}} \cup I_{\text{edge}} \cup I_{\text{2-stratum}} \mid \operatorname{supp} \zeta_j \cap B(p_i, 10R_i) \neq \emptyset \}$.

Sublemma 12.12. *Under the constraints $\beta_2 < \overline{\beta}_2(\epsilon_1)$, $\varsigma_{\text{2-stratum}} < \overline{\varsigma}_{\text{2-stratum}}(\epsilon_1)$, $\beta_E < \overline{\beta}_E(\epsilon_1, \Delta)$, $\varsigma_{\text{edge}} < \overline{\varsigma}_{\text{edge}}(\epsilon_1, \Delta)$, $\varsigma_{E'} < \overline{\varsigma}_E(\epsilon_1, \Delta)$, $\beta_1 < \overline{\beta}_1(\epsilon_1, \Delta)$, $\varsigma_{\text{slim}} < \overline{\varsigma}_{\text{slim}}(\epsilon_1, \Delta)$, $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}(\epsilon_1, \Delta)$ and $\Lambda < \overline{\Lambda}(\epsilon_1, \Delta)$, the following holds.*

For each $j \in J$, there is a map $T_{ij} : \mathbb{R}^2 \to \mathbb{R}^{k_j}$ which is a composition of an isometry and an orthogonal projection, such that on the ball $B(p_i, 10) \subset \frac{1}{R_i} M$, the map η_j is defined and satisfies

$$\left\| \frac{R_j}{R_i} \eta_j - (T_{ij} \circ \eta_i) \right\|_{C^1} < \epsilon_1 \, . \tag{12.13}$$

Proof. As we are assuming the hypotheses of Lemma 7.2, there are no 3-stratum points.

Suppose first that $j \in I_{\text{2-stratum}}$. Then $d(p_j, p_i) \leq 10(R_i + R_j)$. If Λ is sufficiently small then we can assume that $\frac{R_j}{R_i}$ is arbitrarily close to 1, so in particular $d(p_j, p_i) \leq 40 R_j$. By Lemma 4.10, if β_2 is sufficiently small then the $(2, \beta_2)$-splitting of $\left(\frac{1}{R_j} M, p_j \right)$ gives an arbitrarily good 2-splitting of $\left(\frac{1}{R_j}M, p_i\right)$. By Lemma 4.17, if β_2 is sufficiently small then this splitting of $\left(\frac{1}{R_j}M, p_i\right)$ is compatible, to an arbitrarily degree of closeness, with the $(2, \beta_2)$-splitting of $\left(\frac{1}{R_i}M, p_i\right)$ coming from the fact that p_i is a 2-stratum point. Hence in this case, if β_2 and $\varsigma_{\text{2-stratum}}$ are sufficiently small (as functions of ϵ_1) then the sublemma follows from Lemma 4.31, along with Remark 4.35.

If $j \in I_{\text{edge}} \cup I_{\text{slim}}$ then $d(p_i, p_j) \le 10R_i + 10^5 \Delta R_j$. We now have an approximate 1-splitting at p_j, which gives an approximate 1-splitting at p_i. As before, if β_2, $\varsigma_{\text{2-stratum}}$, β_1, ς_{edge}, ς_{slim}, and Λ are sufficiently small (as functions of ϵ_1 and Δ) then we can apply Lemmas 4.17 and 4.31 to deduce the conclusion of the sublemma. Note that in this case, we have to allow Λ to depend on Δ.

If $j \in I_{E'}$ then since $\operatorname{supp} \zeta_{E'} \cap B(p_i, 10R_i) \neq \emptyset$, we know that $\eta_{E'}(q) \in [\frac{2}{10}\Delta, 9\Delta]$ for some $q \in B(p_i, 10R_i)$. As $\Delta >> 10$, it follows from Lemma 9.12 that if Λ is sufficiently small then $B(p_i, 10R_i) \subset \eta_{E'}^{-1}\left(\frac{1}{10}\Delta, 10\Delta\right)$. From the definition of E', it follows that if $\beta_{E'}$ and Λ are sufficiently small then there is a 1-splitting at p_i of arbitrarily good quality, coming from the $[0, C]$-factor in Definition 9.2. As before, if $\beta_{E'}$, Λ, β_2, $\varsigma_{\text{2-stratum}}$, $\beta_{E'}$ and ς_E are sufficiently small (as functions of ϵ_1 and Δ) then we can apply Lemmas 4.17 and 4.31 to deduce the conclusion of the sublemma.

If $j \in I_{\text{0-stratum}}$ then since $\operatorname{supp} \zeta_j \cap B(p_i, 10R_i) \neq \emptyset$, we know that $\eta_j(q) \in [\frac{2}{10}, \frac{9}{10}]$ for some $q \in B(p_i, 10R_i)$. From Lemma 11.5, $\frac{r^0_{p_j}}{R_i} \ge \frac{1}{20}\Upsilon_0$. Hence we may assume that $B(p_i, 10R_i) \subset A(p_j, \frac{1}{10}r^0_{p_j}, r^0_{p_j})$. Lemma 11.5 also gives a $(1, \beta_1)$-splitting of $\left(\frac{R_i}{R_j}M, p_i\right)$. If β_1 and β_2 are sufficiently small then by Lemma 4.17, this 1-splitting is compatible with the $(2, \beta_2)$-splitting of $\left(\frac{1}{R_i}M, p_i\right)$ to an arbitrary degree of closeness. As before, if β_1, β_2, $\varsigma_{\text{2-stratum}}$ and $\overline{\varsigma}_{\text{0-stratum}}$ are sufficiently small (as functions of ϵ_1 and Δ) then the sublemma follows from Lemma 4.31. ∎

We retain the hypotheses of Sublemma 12.12.

For $j \in J$, the cutoff function ζ_j is a function of the $\eta_{j'}$'s for $j' \in J$, i.e. there is a smooth function $\Phi_j \in C^\infty_c(\mathbb{R}^J)$ such that $\zeta_j(\cdot) = \Phi_j\left(\{\eta_{j'}(\cdot)\}_{j' \in J}\right)$. (Note from (9.29) that $\zeta_{E'}$ depends on $\eta_{E'}$ and $\{\zeta_k\}_{k \in I_{\text{edge}}}$.) The H_j-component of $\mathcal{E}^0$, after dividing by R_i, can be written as

$$\frac{1}{R_i}\mathcal{E}^0_j = \left(\frac{R_j}{R_i}\eta_j\zeta_j, \frac{R_j}{R_i}\zeta_j\right) = \left(\frac{R_j}{R_i}\eta_j \cdot (\Phi_j \circ \{\eta_{j'}\}_{j' \in J}), \frac{R_j}{R_i}\Phi_j \circ \{\eta_{j'}\}_{j' \in J}\right). \tag{12.14}$$

Let $\mathcal{F}^0 : \mathbb{R}^2 \to H$ be the map so that the H_j-component of $\mathcal{F}^0 \circ \eta_i$, for $j \in J$, is obtained from the preceding formula by replacing each occurrence of η_j with the approximation $\frac{R_i}{R_j}(T_{ij} \circ \eta_i)$, i.e.

$$\frac{1}{R_i}\mathcal{F}^0_j(u) = \left(T_{ij}(u) \cdot \left(\Phi_j\left(\left\{\frac{R_i}{R_{j'}}T_{ij'}(u)\right\}_{j' \in J}\right)\right), \frac{R_j}{R_i}\Phi_j\left(\left\{\frac{R_i}{R_{j'}}T_{ij'}(u)\right\}_{j' \in J}\right)\right), \tag{12.15}$$

whose $H_{\mathfrak{r}}$-component is the constant function R_i, and whose other components vanish. That is,

(12.16)
$$\frac{1}{R_i}\mathcal{F}^0_j \circ \eta_i = \left((T_{ij} \circ \eta_i) \cdot \left(\Phi_j \left(\left\{ \frac{R_i}{R_{j'}} T_{ij'} \circ \eta_i \right\}_{j' \in J} \right) \right), \frac{R_j}{R_i} \Phi_j \left(\left\{ \frac{R_i}{R_{j'}} T_{ij'} \circ \eta_i \right\}_{j' \in J} \right) \right).$$

Sublemma 12.17. *Under the constraints $\epsilon_1 \le \overline{\epsilon}_1(\epsilon_2, \mathcal{M})$, $\Upsilon_0 \ge \overline{\Upsilon}_0(\epsilon_2, \mathcal{M})$ and $\Lambda \le \overline{\Lambda}(\epsilon_2, \mathcal{M})$,*

$$\left\| \frac{1}{R_i} \mathcal{E}^0 - \frac{1}{R_i} \mathcal{F}^0 \circ \eta_i \right\|_{C^1} < \epsilon_2 \tag{12.18}$$

on $B(p_i, 10) \subset \frac{1}{R_i} M$.

Proof. First note that $\mathcal{E}_{\mathfrak{r}}(p_i) = \mathcal{F}^0_{\mathfrak{r}}(p_i) = R_i$ and the $\mathcal{E}_{\mathfrak{r}}$-component of $\mathcal{E}$ has Lipschitz constant Λ, so it suffices to control the remaining components. For $j \in J$ and $j \in I_{E'} \cup I_{\text{slim}} \cup I_{\text{edge}} \cup I_{2\text{-stratum}}$, if Λ is sufficiently small then we can assume that $\frac{R_i}{R_j}$ is arbitrarily close to one. Then the H_j-component of $\frac{1}{R_i}\mathcal{E}^0 - \frac{1}{R_i}\mathcal{F}^0 \circ \eta_i$ can be estimated in C^1-norm by using (12.13) to estimate $\eta_{j'}$, plugging this into (12.14) and applying the chain rule. In applying the chain rule, we use the fact that the functions Φ_j have explicit bounds on their derivatives of order up to 2.

If $j \in J \cap I_{0\text{-stratum}}$ then the only relevant argument of Φ_j is when $j' = j$. Hence in this case we can write

$$\frac{1}{R_i} \mathcal{E}^0_j = \left(\frac{R_j}{R_i} \eta_j \cdot \Phi_j(\eta_j), \frac{R_j}{R_i} \Phi_j(\eta_j) \right) \tag{12.19}$$

and

$$\frac{1}{R_i}\mathcal{F}^0_j \circ \eta_i = \left((T_{ij} \circ \eta_i) \cdot \left(\Phi_j \left(\frac{R_i}{R_j} T_{ij} \circ \eta_i \right) \right), \frac{R_j}{R_i} \Phi_j \left(\frac{R_i}{R_j} T_{ij} \circ \eta_i \right) \right). \tag{12.20}$$

From part (2) of Lemma 11.5, $\frac{R_i}{R_j} \le \frac{20}{\Upsilon_0}$. Then the H_j-component of $\frac{1}{R_i}\mathcal{E}^0 - \frac{1}{R_i}\mathcal{F}^0 \circ \eta_i$ can be estimated in C^1-norm by using (12.13). Note when we use the chain rule to estimate the second component of $\frac{1}{R_i}\mathcal{E}^0 - \frac{1}{R_i}\mathcal{F}^0 \circ \eta_i$, namely $\frac{R_j}{R_i}\left(\Phi_j(\eta_j) - \Phi_j\left(\frac{R_i}{R_j} T_{ij} \circ \eta_i \right) \right)$, we differentiate Φ_j and this brings down a factor of $\frac{R_i}{R_j}$ when estimating norms. □

Sublemma 12.21. *Given $\Sigma \in (0, \frac{1}{10})$, suppose that $|\eta_i(p)| < 8$ for some $p \in M$. Put $x = \mathcal{E}^0(p)$. For any $q \in M$, if $\mathcal{E}^0(q) \in B(x, \Sigma R_i)$ then $|\eta_i(p) - \eta_i(q)| < 20\Sigma$.*

Proof. We know that $\zeta_i(p) = 1$. By hypothesis, $|\mathcal{E}^0(p) - \mathcal{E}^0(q)| < \Sigma R_i$. In particular, $|\zeta_i(p) - \zeta_i(q)| < \Sigma$ and $|\zeta_i(p)\eta_i(p) - \zeta_i(q)\eta_i(q)| < \Sigma$. Then

$$
\begin{aligned}
(12.22) \qquad |\eta_i(p) - \eta_i(q)| &= \frac{1}{\zeta_i(q)} |\zeta_i(q)\eta_i(p) - \zeta_i(q)\eta_i(q)| \\
&\le \frac{1}{\zeta_i(q)} \left[|\zeta_i(p)\eta_i(p) - \zeta_i(q)\eta_i(q)| + |\zeta_i(p) - \zeta_i(q)||\eta_i(p)| \right] \\
&\le \frac{10\Sigma}{1-\Sigma} \le 20\Sigma.
\end{aligned}
$$

This proves the sublemma. □

We now prove Lemma 12.7. We no longer fix $i \in I_{\text{2-stratum}}$. Given $x \in S_1$, choose $p \in A_1$ and $i \in I_{\text{2-stratum}}$ so that $\mathcal{E}^0(p) = x$ and $|\eta_i(p)| \le 8$. Put $A^0_x = \operatorname{Im}(d\mathcal{F}^0_{\eta_i(p)})$, a 2-plane in H, and let $A_x = x + A^0_x$ be the corresponding affine subspace through x. We first show that under the constraints $\Sigma_1 \le \overline{\Sigma}_1(\Gamma_1, \mathcal{M})$, $\epsilon_2 \le \overline{\epsilon}_2(\Gamma_1, \mathcal{M})$ and $\Lambda \le \overline{\Lambda}(\Gamma_1, \mathcal{M})$, the triple $(\tilde{S}_1, S_1, r_1)$ is a $(2, \Gamma_1)$ cloudy 2-manifold.

We verify condition (1) of Definition B.2. Pick $x, y \in \widetilde{S}_1$, and choose $p \in (\mathcal{E}^0)^{-1}(x) \cap \bigcup_{i \in I_{\text{2-stratum}}} |\eta_i|^{-1}[0, 8)$ (respectively $q \in (\mathcal{E}^0)^{-1}(y) \cap \bigcup_{i \in I_{\text{2-stratum}}} |\eta_i|^{-1}[0, 8)$) satisfying $r_1(x) = \Sigma_1 \mathfrak{r}_p$ (respectively $r_1(y) = \Sigma_1 \mathfrak{r}_q$).

We can assume that $\Lambda < \frac{1}{100}$. Suppose first that $d(p, q) \le \frac{\mathfrak{r}_p}{\Lambda}$. Then since $\mathfrak{r}$ is Λ-Lipschitz, we get $|\mathfrak{r}_p - \mathfrak{r}_q| \le \mathfrak{r}_p$, so in this case

$$
(12.23) \qquad |r_1(x) - r_1(y)| = \Sigma_1 |\mathfrak{r}_p - \mathfrak{r}_q| \le \Sigma_1 \mathfrak{r}_p = r_1(x)\,.
$$

Now suppose that $d(p, q) \ge 20\mathfrak{r}_p$. We claim that if Λ is sufficiently small then this implies that $d(p, q) \ge 19\mathfrak{r}_q$ as well. Suppose not. Then $20\mathfrak{r}_p \le d(p, q) \le 19\mathfrak{r}_q$, so $\frac{\mathfrak{r}_p}{\mathfrak{r}_q} \le \frac{19}{20}$. On the other hand, since $|\mathfrak{r}_q - \mathfrak{r}_p| \le \Lambda d(p, q)$, we also know that $\mathfrak{r}_q - \mathfrak{r}_p \le \Lambda d(p, q) \le 19\Lambda \mathfrak{r}_q$, so $\frac{\mathfrak{r}_p}{\mathfrak{r}_q} \ge 1 - 19\Lambda$. If Λ is sufficiently small then this is a contradiction.

Thus there are $i, j \in I_{\text{2-stratum}}$ such that $p \in |\eta_i|^{-1}[0, 8)$, $q \in |\eta_j|^{-1}[0, 8)$, $\zeta_i(p) = 1 = \zeta_j(q)$ and $\zeta_i(q) = 0 = \zeta_j(p)$. Then

$$
\begin{aligned}
(12.24) \qquad |x - y| &= |\mathcal{E}^0(p) - \mathcal{E}^0(q)| \ge \max(\, \mathfrak{r}_{p_i} |\zeta_i(p) - \zeta_i(q)|\,,\, \mathfrak{r}_{p_j} |\zeta_j(p) - \zeta_j(q)|\,) \\
&= \max(\mathfrak{r}_{p_i}\,,\, \mathfrak{r}_{p_j}) \ge \frac{1}{2} \max(\mathfrak{r}_p\,,\, \mathfrak{r}_q) = \frac{\max(r_1(x)\,,\, r_1(y))}{2\,\Sigma_1}\,.
\end{aligned}
$$

So $|r_1(x) - r_1(y)| \le |x - y|$ provided $\Sigma_1 \le \frac{1}{4}$. Thus condition (1) of Definition B.2 will be satisfied.

We now verify condition (2) of Definition B.2. Given $x \in S_1$, let $i \in I_{\text{2-stratum}}$ and $p \in M$ be such that $\mathcal{E}^0(p) = x$ and $|\eta_i(p)| \le 7$. Taking $\Sigma = \frac{1}{100}$ in Sublemma 12.21, we have $\operatorname{Im}(\mathcal{E}^0) \cap B\left(x, \frac{R_i}{100}\right) \subset \operatorname{Im}\left(\mathcal{E}^0\Big|_{|\eta_i|^{-1}[0,7.2)}\right)$. Thus we can restrict attention to the action of $\mathcal{E}^0$ on $|\eta_i|^{-1}[0, 7.2)$. Now $\operatorname{Im}\left(\mathcal{F}^0\Big|_{B(0,7.2)}\right)$ is the restriction to $B(0, 7.2)$ of the graph of a function $G^0_i : H'_i \to (H'_i)^\perp$, since $T_{ii} = \mathrm{Id}$ and $\zeta_i\Big|_{B(0,7.2)} = 1$. Furthermore, in view of the universality of the functions $\{\Phi_j\}_{j\in J}$ and the bound on the cardinality of J, there are uniform C^1-estimates on G_i^0. Hence we can find Σ_1 (as a function of Γ_1 and $\mathcal{M}$) to ensure that $\left(\frac{1}{r_1(x)}\operatorname{Im}\left(\mathcal{F}^0\Big|_{B(0,7.2)}\right), x\right)$ is $\frac{\Gamma_1}{2}$-close in the pointed Hausdorff topology to $x+\operatorname{Im}(d\mathcal{F}_p^0)$. Finally, if the parameter ϵ_2 of Sublemma 12.17 is sufficiently small then we can ensure that $\left(\frac{1}{r_1(x)}\operatorname{Im}\left(\mathcal{E}^0\right), x\right)$ is Γ_1-close in the pointed Hausdorff topology to $x+\operatorname{Im}(D\mathcal{F}_p^0)$. Thus condition (2) of Definition B.2 will be satisfied.

To finish the proof of Lemma 12.7, equation (12.8) is clearly satisfied if the parameter ϵ_2 of Sublemma 12.17 is sufficiently small. Equation (12.9) is equivalent to upper and lower bounds on the eigenvalues of the matrix $(\pi_{A_x^0}\circ D\mathcal{E}_p^0)(\pi_{A_x^0}\circ D\mathcal{E}_p^0)^*$, which acts on the two-dimensional space A_x^0. In view of Sublemma 12.17 and the definition of A_x, it is sufficient to show upper and lower bounds on the eigenvalues of $D\mathcal{F}^0_{\eta_i(p)}(D\mathcal{F}^0_{\eta_i(p)})^*$ acting on A_x^0. In terms of the function G_i^0, these are the same as the eigenvalues of $I_2+((DG_i^0)_{\eta_i(p)})^*(DG_i^0)_{\eta_i(p)}$, acting on $\mathbb{R}^2$. The eigenvalues are clearly bounded below by one. In view of the C^1-bounds on G_i^0, there is an upper bound on the eigenvalues in terms of $\dim(H)$, which in turn is bounded above in terms of $\mathcal{M}$. This shows equation (12.9).

Finally, given $i\in I_{\text{2-stratum}}$, we can write $\frac{1}{R_i}\mathcal{F}^0$ on $\overline{B(0,8)}\subset\mathbb{R}^2$ in the form $\frac{1}{R_i}\mathcal{F}^0=\left(I,\frac{1}{R_i}\widehat{\mathcal{E}}_i^0\right)$ with respect to the orthogonal decomposition $H=H_i'\oplus(H_i')^\perp$. (Recall that $\mathcal{F}^0$ is defined in reference to the given value of i.) We use this to define $\widehat{\mathcal{E}}_i^0$. Equation (12.11) is a consequence of Sublemma 12.17. The last statement of Lemma 12.7 follows from the definition of A_x^0.

12.4. **Structure of $\mathcal{E}^0$ near the edge stratum.** Recall that $Q_2=H_{\text{0-stratum}}\oplus H_{\text{slim}}\oplus H_{\text{edge}}$, and $\pi_2:H\to Q_2$ is the orthogonal projection.

Put

$$\tilde{A}_2=\bigcup_{i\in I_{\text{edge}}}\{|\eta_i|\le 8\Delta,\ \eta_{E'}\le 8\Delta\},\quad A_2=\bigcup_{i\in I_{\text{edge}}}\{|\eta_i|\le 7\Delta,\ \eta_{E'}\le 7\Delta\}\tag{12.25}$$

and

$$\tilde{S}_2=(\pi_2\circ\mathcal{E}^0)(\tilde{A}_2),\quad S_2=(\pi_2\circ\mathcal{E}^0)(A_2).\tag{12.26}$$

Let $\Sigma_2,\Gamma_2>0$ be new parameters. Define $r_2:\tilde{S}_2\to(0,\infty)$ by putting $r_2(x)=\Sigma_2\,\mathfrak{r}_p$ for some $p\in(\pi_2\circ\mathcal{E}^0)^{-1}(x)\cap\tilde{A}_2$.

The analog of Lemma 12.7 for the region near edge points is:

Lemma 12.27. *There is a constant $\Omega_2=\Omega_2(\mathcal{M})$ so that under the constraints $\Sigma_2<\overline{\Sigma}_2(\Gamma_2,\mathcal{M})$, $\beta_E<\overline{\beta}_E(\Gamma_2,\Sigma_2,\beta_2,\Delta,\mathcal{M})$, $\sigma_E<\overline{\sigma}_E(\Gamma_2,\Sigma_2,\beta_2,\Delta,\mathcal{M})$, $\varsigma_{\text{edge}}<\overline{\varsigma}_{\text{edge}}(\Gamma_2,\Sigma_2,\beta_2,\Delta,\mathcal{M})$, $\beta_1<\overline{\beta}_1(\Gamma_2,\Sigma_2,\beta_2,\Delta,\mathcal{M})$, $\varsigma_{\text{slim}}<\overline{\varsigma}_{\text{slim}}(\Gamma_2,\Sigma_2,\beta_2,\Delta,\mathcal{M})$ $\varsigma_{\text{0-stratum}}<\overline{\varsigma}_{\text{0-stratum}}(\Gamma_2,\Sigma_2,\beta_2,\Delta,\mathcal{M})$ and $\Lambda<\overline{\Lambda}(\Gamma_2,\Sigma_2,\beta_2,\Delta,\mathcal{M})$, the following holds.*

(1) *The triple $(\tilde{S}_2,S_2,r_2)$ is a $(2,\Gamma_2)$ cloudy 1-manifold.*

(2) *The affine subspaces* $\{A_x\}_{x \in S_2}$ *inherent in the definition of the cloudy* 1*-manifold can be chosen to have the following property. Pick* $p \in A_2$ *and put* $x = (\pi_2 \circ \mathcal{E}^0)(p) \in S_2$*. Let* $A^0_x \subset Q_2$ *be the linear subspace parallel to* A_x *(i.e.* $A_x = A^0_x + x$*) and let* $\pi_{A^0_x} : H \to A^0_x$ *denote orthogonal projection onto* A^0_x*. Then*

$$\|D(\pi_2 \circ \mathcal{E}^0)_p - \pi_{A^0_x} \circ D(\pi_2 \circ \mathcal{E}^0)_p\| < \Gamma_2 \tag{12.28}$$

and

$$\Omega_2^{-1}\|v\| \leq \|(\pi_{A^0_x} \circ D(\pi_2 \circ \mathcal{E}^0))(v)\| \leq \Omega_2 \|v\| \tag{12.29}$$

for every $v \in T_pM$ *which is orthogonal to* $\ker(\pi_{A^0_x} \circ D(\pi_2 \circ \mathcal{E}^0)_p)$*.*

(3) *Given* $i \in I_{\mathrm{edge}}$*, there is a smooth map* $\widehat{\mathcal{E}}^0_i : (\overline{B(0, 8\Delta)} \subset \mathbb{R}) \to (H'_i)^\perp \cap Q_2$ *such that*

$$\|D\widehat{\mathcal{E}}^0_i\| \leq \Omega_2 R_i \tag{12.30}$$

and on the subset $\{|\eta_i| \leq 8\Delta, \eta_{E'} \leq 8\Delta\}$*, we have*

$$\left\| \frac{1}{R_i}\pi_2 \circ \mathcal{E}^0 - \left(\eta_i, \frac{1}{R_i}\widehat{\mathcal{E}}^0_i \circ \eta_i\right)\right\|_{C^1} < \Gamma_2. \tag{12.31}$$

Furthermore, if $x \in S_2$ *then there are some* $i \in I_{\mathrm{edge}}$ *and* $p \in \{|\eta_i| \leq 7\Delta, \eta_{E'} \leq 7\Delta\}$ *such that* $x = (\pi_2 \circ \mathcal{E}^0)(p)$ *and* $A^0_x = \mathrm{Im}\left(I, \frac{1}{R_i}(D\widehat{\mathcal{E}}^0_i)_{\eta_i(p)}\right)$*.*

We omit the proof as it is similar to the proof of Lemma 12.7.

12.5. **Structure of $\mathcal{E}^0$ near the slim 1-stratum.** Recall that $Q_3 = H_{0\text{-stratum}} \oplus H_{\mathrm{slim}}$, and $\pi_3 : H \to Q_3$ is the orthogonal projection.

Put

$$\tilde{A}_3 = \bigcup_{i \in I_{\mathrm{edge}}} \{|\eta_i| \leq 8 \cdot 10^5 \Delta\}, \quad A_3 = \bigcup_{i \in I_{\mathrm{edge}}} \{|\eta_i| \leq 7 \cdot 10^5 \Delta\} \tag{12.32}$$

and

$$\tilde{S}_3 = (\pi_3 \circ \mathcal{E}^0)(\tilde{A}_3), \quad S_3 = (\pi_3 \circ \mathcal{E}^0)(A_3). \tag{12.33}$$

Let $\Sigma_3, \Gamma_3 > 0$ be new parameters. Define $r_3 : \tilde{S}_3 \to (0, \infty)$ by putting $r_3(x) = \Sigma_3 \, \mathfrak{r}_p$ for some $p \in (\pi_3 \circ \mathcal{E}^0)^{-1}(x) \cap \tilde{A}_3$.

The analog of Lemma 12.7 for the slim 1-stratum points is:

Lemma 12.34. *There is a constant* $\Omega_3 = \Omega_3(\mathcal{M})$ *so that under the constraints* $\Sigma_3 < \overline{\Sigma}_3(\Gamma_3, \mathcal{M})$, $\beta_E < \overline{\beta}_E(\Gamma_3, \Sigma_3, \beta_2, \Delta, \mathcal{M})$, $\sigma_E < \overline{\sigma}_E(\Gamma_3, \Sigma_3, \beta_2, \Delta, \mathcal{M})$, $\varsigma_{\mathrm{edge}} < \overline{\varsigma}_{\mathrm{edge}}(\Gamma_3, \Sigma_3, \beta_2, \Delta, \mathcal{M})$, $\beta_1 < \overline{\beta}_1(\Gamma_3, \Sigma_3, \beta_2, \Delta, \mathcal{M})$, $\varsigma_{\mathrm{slim}} < \overline{\varsigma}_{\mathrm{slim}}(\Gamma_3, \Sigma_3, \beta_2, \Delta, \mathcal{M})$ $\varsigma_{0\text{-stratum}} < \overline{\varsigma}_{0\text{-stratum}}(\Gamma_3, \Sigma_3, \beta_2, \Delta, \mathcal{M})$ *and* $\Lambda < \overline{\Lambda}(\Gamma_3, \Sigma_3, \beta_2, \Delta, \mathcal{M})$*, the following holds.*

(1) *The triple* $(\tilde{S}_3, S_3, r_3)$ *is a* $(2, \Gamma_3)$ *cloudy* 1*-manifold.*

(2) *The affine subspaces* $\{A_x\}_{x \in S_3}$ *inherent in the definition of the cloudy* 1*-manifold can be chosen to have the following property. Pick* $p \in A_3$ *and put* $x = (\pi_3 \circ \mathcal{E}^0)(p) \in S_3$*. Let* $A_x^0 \subset Q_3$ *be the linear subspace parallel to* A_x *(i.e.* $A_x = A_x^0 + x$*) and let* $\pi_{A_x^0} : H \to A_x^0$ *denote orthogonal projection onto* A_x^0*. Then*

$$\| D(\pi_3 \circ \mathcal{E}^0)_p - \pi_{A_x^0} \circ D(\pi_3 \circ \mathcal{E}^0)_p \| < \Gamma_3 \tag{12.35}$$

and

$$\Omega_3^{-1} \| v \| \leq \| (\pi_{A_x^0} \circ D(\pi_3 \circ \mathcal{E}^0)(v) \| \leq \Omega_3 \| v \| \tag{12.36}$$

for every $v \in T_pM$ *which is orthogonal to* $\ker(\pi_{A_x^0} \circ D(\pi_3 \circ \mathcal{E}^0)_p)$*.*

(3) *Given* $i \in I_{\mathrm{slim}}$*, there is a smooth map* $\widehat{\mathcal{E}}_i^0 : (\overline{B(0, 8 \cdot 10^5 \Delta)} \subset \mathbb{R}) \to (H_i')^\perp \cap Q_3$ *such that*

$$\| D\widehat{\mathcal{E}}_i^0 \| \leq \Omega_3 R_i \tag{12.37}$$

and on the subset $\{|\eta_i| \leq 8 \cdot 10^5 \Delta\}$*, we have*

$$\left\| \frac{1}{R_i} \pi_3 \circ \mathcal{E}^0 - \left(\eta_i, \frac{1}{R_i} \widehat{\mathcal{E}}_i^0 \circ \eta_i \right) \right\|_{C^1} < \Gamma_3. \tag{12.38}$$

Furthermore, if $x \in S_3$ *then there are some* $i \in I_{\mathrm{slim}}$ *and* $p \in \{|\eta_i| \leq 7 \cdot 10^5 \Delta\}$ *such that* $x = (\pi_3 \circ \mathcal{E}^0)(p)$ *and* $A_x^0 = \mathrm{Im}\left(I, \frac{1}{R_i} (D\widehat{\mathcal{E}}_i^0)_{\eta_i(p)} \right)$*.*

We omit the proof as it is similar to the proof of Lemma 12.7.

12.6. **Structure of $\mathcal{E}^0$ near the 0-stratum.** The only information we will need near the 0-stratum is:

Lemma 12.39. *For* $i \in I_{0\text{-stratum}}$*, the only nonzero component of the map* $\pi_4 \circ \mathcal{E}^0 : M \to Q_4 = H_{0\text{-stratum}}$ *in the region* $\{\eta_i \in [\frac{3}{10}, \frac{8}{10}]\}$ *is* $\mathcal{E}_i^0$*, where it coincides with* $(R_i \eta_i, R_i)$*.*

13. Adjusting the map to Euclidean space

The main result of this section is the following proposition, which asserts that it is possible to adjust $\mathcal{E}^0$ slightly, to get a new map $\mathcal{E}$ which is a submersion in different parts of M. In Section 14 this structure will yield compatible fibrations of different parts of M.

Let $c_{\mathrm{adjust}} > 0$ be a parameter.

Proposition 13.1. *Under the constraints imposed in this and prior sections, there is a smooth map* $\mathcal{E} : M \to H$ *with the following properties:*

(1) *For every* $p \in M$*,*

$$\| \mathcal{E}(p) - \mathcal{E}^0(p) \| < c_{\mathrm{adjust}} \, \mathfrak{r}(p) \quad \text{and} \quad \| D\mathcal{E}_p - D\mathcal{E}_p^0 \| < c_{\mathrm{adjust}} \, . \tag{13.2}$$

(2) *For $j \in \{1,2,3\}$ the restriction of $\pi_j \circ \mathcal{E} : M \to Q_j$ to the region $U_j \subset M$ is a submersion to a k_j-manifold $W_j \subset Q_j$, where*

$$\begin{aligned} U_1 &= \bigcup_{i \in I_{\text{2-stratum}}} \{|\eta_i| < 5\}\,, \\ U_2 &= \bigcup_{i \in I_{\text{edge}}} \{|\eta_i| < 5\Delta\,,\ \eta_{E'} < 5\Delta\}\,, \\ U_3 &= \bigcup_{i \in I_{\text{slim}}} \{|\eta_i| < 5 \cdot 10^5 \Delta\} \end{aligned} \tag{13.3}$$

and $k_1 = 2$, $k_2 = k_3 = 1$.

We will use the following additional parameters in this section: $c_{\text{2-stratum}}, c_{\text{edge}}, c_{\text{slim}} > 0$ and $\Xi_i > 0$ for $i \in \{1,2,3\}$.

13.1. **Overview of the proof of Proposition 13.1.** In certain regions of M, the map $\mathcal{E}^0$ defined in the previous section, as well as its composition with projection onto certain summands of H, behaves like a "rough fibration". As indicated in the overview in Section 1.5, the next step is to modify the map $\mathcal{E}^0$ so as to promote these rough fibrations to honest fibrations, in such a way that they are compatible on their overlap. We will do this by producing a sequence of maps $\mathcal{E}^j : M \to H$, for $j \in \{1,2,3\}$, which are successive adjustments of the map $\mathcal{E}^0$.

To construct the map $\mathcal{E}^j$ from $\mathcal{E}^{j-1}$, $j \in \{1,2,3\}$, we will use the following procedure. We consider the orthogonal splitting $H = Q_j \oplus Q_j^\perp$ of H, and let $\pi_j = \pi_{1,j} : H \to Q_j$, $\pi_j^\perp : H \to Q_j^\perp$ be the orthogonal projections. In Section 12 we considered a pair of subsets $(\widetilde{A}_j, A_j)$ in M whose image $(\widetilde{S}_j, S_j)$ under the composition $\pi_j \circ \mathcal{E}^{j-1}$ is a cloudy k_j-manifold in Q_j, in the sense of Definition B.1 of Appendix B. We think of the restriction of $\mathcal{E}^{j-1}$ to A_j as defining a "rough submersion" over the cloudy k_j-manifold $(\widetilde{S}_j, S_j)$. By Lemma B.2, there is a k_j-dimensional manifold $W_j \subset Q_j$ near $(\widetilde{S}_j, S_j)$ and a projection map P_j onto W_j, defined in a neighborhood $\widehat{W}_j$ of W_j. Hence we have a well-defined map

$$H \supset \widehat{W}_j \times Q_j^\perp \overset{(\,P_j \circ \pi_j\,,\,\pi_j^\perp\,)}{\longrightarrow} Q_j \oplus Q_j^\perp = H\,. \tag{13.4}$$

Then using a partition of unity, we blend the composition $(P_j \circ \pi_j, \pi_j^\perp) \circ \mathcal{E}^{j-1}$ with $\mathcal{E}^{j-1} : M \to H$ to obtain $\mathcal{E}^j : M \to H$. In fact, $\mathcal{E}^j$ will be the postcomposition of $\mathcal{E}^{j-1}$ with a map from H to itself.

We draw attention to two key features of the construction. First, in passing from $\mathcal{E}^{j-1}$ to $\mathcal{E}^j$, we do not change it much. More precisely, at a point $p \in M$, we have $|\mathcal{E}^{j-1}(p)) - \mathcal{E}^j(p)| <$ const. $\mathfrak{r}_p$ and $|D\mathcal{E}^{j-1}_p - D\mathcal{E}^j_p| <$ const. for some small constants. Second, the passage from $\mathcal{E}^j$ to $\mathcal{E}^{j-1}$ respects the submersions defined by $\mathcal{E}^{j-1}$.

13.2. **Adjusting the map near the 2-stratum.** Our first adjustment step involves the 2-stratum.

We take $Q_1 = H$, $Q_1^\perp = \{0\}$, and we let $\widetilde{A}_1$, A_1, $\widetilde{S}_1$, S_1 and $r_1 : \widetilde{S}_1 \to (0,\infty)$ be as in Section 12.3.

Thus $(\tilde{S}_1, S_1, r_1)$ is a $(2, \Gamma_1)$ cloudy 2-manifold by Lemma 12.7. By Lemma B.2, there is a 2-manifold $W_1^0 \subset H$ so that the conclusion of Lemma B.2 holds, where the parameter ϵ in the lemma is given by $\Xi_1 = \Xi_1(\Gamma_1)$. (We remark that W_1^0 will not be the same as the W_1 of Proposition 13.1, due to subsequent adjustments.) In particular, there is a well-defined nearest point projection

$$P_1 : N_{r_1}(S_1) = \widehat{W}_1 \to W_1^0\,, \tag{13.5}$$

where we are using the notation for variable thickness neighborhoods from Section 3.

We now define a certain cutoff function.

Lemma 13.6. *There is a smooth function $\psi_1 : H \to [0,1]$ with the following properties:*

(1)

$$\psi_1 \circ \mathcal{E}^0 \equiv 1 \textit{ in } \bigcup_{i \in I_{\text{2-stratum}}} \{|\eta_i| < 6\} \textit{ and} \tag{13.7}$$
$$\psi_1 \circ \mathcal{E}^0 \equiv 0 \textit{ outside } \bigcup_{i \in I_{\text{2-stratum}}} \{|\eta_i| < 7\}\,.$$

(2) $\operatorname{supp}(\psi_1) \cap \operatorname{im}(\mathcal{E}^0) \subset \widehat{W}_1$.

(3) *There is a constant $\Omega_1' = \Omega_1'(\mathcal{M})$ such that*

$$|(d\psi_1)_x| < \Omega_1'\, x_{\mathfrak{r}}^{-1} \tag{13.8}$$

for all $x \in \operatorname{im}(\mathcal{E}^0)$.

Proof. Let $\psi_1 : H \to [0,1]$ be given by

$$\psi_1(x) = 1 - \Phi_{\frac{1}{2},1}\left(\sum_{\{i \in I_{\text{2-stratum}} \mid x_i'' > 0\}} \Phi_{6,6.5}\left(\frac{|x_i'|}{x_i''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right)\right)\,. \tag{13.9}$$

For each $i \in I_{\text{2-stratum}}$, the function $x \mapsto \Phi_{6,6.5}\left(\frac{|x_i'|}{x_i''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right)$ is well-defined and smooth in the set $\{x_i'' > 0\}$, with support contained in the set $\{x_i'' \geq \frac{1}{2} R_i\}$; so extending it by zero defines a smooth function on H. Hence ψ_1 is smooth.

To prove part (1), suppose that $i \in I_{\text{2-stratum}}$ and $|\eta_i(p)| < 6$. Then $\zeta_i(p) = 1$ and $|\eta_i| < 6$. Putting $x = \mathcal{E}^0(p)$, we have

$$x_i = (x_i', x_i'') = \mathcal{E}_i^0(p) = (R_i \zeta_i(p) \eta_i(p), R_i \zeta_i(p))\,, \tag{13.10}$$

so $x_i'' = R_i$ and $\frac{|x_i'|}{x_i''} \in [0, 6)$. Hence

$$\Phi_{6,6.5}\left(\frac{|x_i'|}{x_i''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right) \;=\; 1, \tag{13.11}$$

so $\psi_1(x) = 1$.

Suppose now that $|\eta_i(p)| \geq 7$ for every $i \in I_{\text{2-stratum}}$. Putting $x = \mathcal{E}^0(p)$, for each $i \in I_{\text{2-stratum}}$ we claim that

$$\Phi_{6,6.5}\left(\frac{|x_i'|}{x_i''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right) = 0; \tag{13.12}$$

otherwise we would have $|x_i'| < 6.5\, x_i''$ and $x_i'' \geq R_i/2$, which contradicts our assumption on p. It follows that $\psi_1(x) = 0$. This proves part (1).

To prove part (2), suppose $x = \mathcal{E}^0(p)$ and $\psi_1(x) > 0$. Then from part (1), $|\eta_i(p)| < 7$ for some $i \in I_{\text{2-stratum}}$. Therefore, $p \in A_1$ and $x \in \mathcal{E}^0(A_1) = S_1 \subset \widehat{W}_1$, so part (2) follows.

To prove part (3), suppose that $x = \mathcal{E}^0(p)$. If $x_i'' > 0$ then $\zeta_i(p) > 0$, so the number of such indices $i \in I_{\text{2-stratum}}$ is bounded by the multiplicity of the 2-stratum cover; for the remaining indices $j \in I_{\text{2-stratum}}$, the quantity $1 - \Phi_{\frac{1}{2},1}\left(\frac{x_j''}{R_j}\right)$ vanishes near x. Thus by the chain rule, it suffices to bound the differential of

$$\Phi_{6,6.5}\left(\frac{|x_i'|}{x_i''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right) \tag{13.13}$$

for each $i \in I_{\text{2-stratum}}$ for which $x_i'' > 0$. But the differential is nonzero only when $\frac{|x_i|}{x_i''} \leq 6.5$ and $\frac{x_i''}{R_i} \geq \frac{1}{2}$. In this case, R_i will be comparable to $x_{\mathfrak{r}}$ and the estimate (13.8) follows easily. □

Define $\Psi_1 : H \to H$ by $\Psi_1(x) = x$ if $x \notin \widehat{W}_1$ and

$$\Psi_1(x) = \psi_1(x) P_1(x) + (1 - \psi_1(x))x \tag{13.14}$$

otherwise. Put $\mathcal{E}^1 = \Psi_1 \circ \mathcal{E}^0$.

Lemma 13.15. *Under the constraints $\Sigma_1 < \overline{\Sigma}_1(\Omega_1, c_{\text{2-stratum}})$, $\Gamma_1 < \overline{\Gamma}_1(\Omega_1, c_{\text{2-stratum}})$ and $\Xi_1 < \overline{\Xi}_1(c_{\text{2-stratum}})$, we have:*

1. *$\mathcal{E}^1$ is smooth.*
2. *For all $p \in M$,*

$$\|\mathcal{E}^1(p) - \mathcal{E}^0(p)\| < c_{\text{2-stratum}}\, \mathfrak{r}(p) \quad \textit{and} \quad \|D\mathcal{E}^1_p - D\mathcal{E}^0_p\| < c_{\text{2-stratum}}\,. \tag{13.16}$$

3. *The restriction of $\mathcal{E}^1$ to $\bigcup_{i \in I_{\text{2-stratum}}} \{|\eta_i| < 6\}$ is a submersion to W_1^0.*

Proof. That $\mathcal{E}^1$ is smooth follows from part (2) of Lemma 13.6.

Given $p \in M$, put $x = \mathcal{E}^0(p)$. We have

$$\mathcal{E}^1(p) - \mathcal{E}^0(p) = \psi_1(x)\left(P_1(x) - x\right). \tag{13.17}$$

Now $|\psi_1(x)| \leq 1$. From Lemma B.2(1), $|P_1(x) - x| \leq \Xi_1 r_1(x)$. From Sublemma 12.21, we can assume that $r_1(x) \leq 10\mathfrak{r}_p$. This gives the first equation in (13.16).

Next,

$$\begin{aligned} D\mathcal{E}^1_p - D\mathcal{E}^0_p =& (D\psi_1)_x\ (P_1(x) - x)\ +\ \psi_1(x)\left((DP_1)_x \circ D\mathcal{E}^0_p - D\mathcal{E}^0_p\right) \\ =& (D\psi_1)_x\ (P_1(x) - x)\ +\ \psi_1(x)\left((DP_1)_x - \pi_{A^0_x}\right) \circ D\mathcal{E}^0_p + \\ & \psi_1(x)\left(\pi_{A^0_x} \circ D\mathcal{E}^0_p - D\mathcal{E}^0_p\right). \end{aligned} \tag{13.18}$$

Equation (13.8) gives a bound on $|(D\psi_1)_x|$. Lemma B.2(1) gives a bound on $|P_1(x) - x|$. Lemma B.2(7) gives a bound on $|(DP_1)_x - \pi_{A^0_x}|$. Lemma 12.5 gives a bound on $|D\mathcal{E}^0_p|$. Equation (12.8) gives a bound on $|\pi_{A^0_x} \circ D\mathcal{E}^0_p - D\mathcal{E}^0_p|$. The second equation in (13.16) follows from these estimates.

Finally, the restriction of $\mathcal{E}^1$ to $\bigcup_{i \in I_{\text{2-stratum}}} \{|\eta_i| < 6\}$ equals $P_1 \circ \mathcal{E}^0$. For $p \in \bigcup_{i \in I_{\text{2-stratum}}} \{|\eta_i| < 6\}$, put $x = \mathcal{E}^0(p)$. Then

$$D(P_1 \circ \mathcal{E}^0)_p = \pi_{A^0_x} \circ d\mathcal{E}^0_p + \left((DP_1)_x - \pi_{A^0_x}\right) \circ D\mathcal{E}^0_p. \tag{13.19}$$

Using (12.9) and Lemma B.2(7), if Ξ_1 is sufficiently small then $D(P_1 \circ \mathcal{E}^0)_p$ maps onto $(TW^0_1)_{P_1(x)}$. This proves the lemma. □

13.3. **Adjusting the map near the edge points.** Our second adjustment step involves the region near the edge points.

Recall that $Q_2 = H_{\text{0-stratum}} \oplus H_{\text{slim}} \oplus H_{\text{edge}}$ and $\pi_2 : H \to Q_2$ is orthogonal projection. We let $\widetilde{A}_2$, A_2, $\widetilde{S}_2$, S_2 and $r_2 : \widetilde{S}_2 \to (0, \infty)$ be as in Section 12.4.

Thus $(\tilde{S}_2, S_2, r_2)$ is a $(2, \Gamma_2)$ cloudy 1-manifold by Lemma 12.27. By Lemma B.2, there is a 1-manifold $W^0_2 \subset Q_2$ so that the conclusion of Lemma B.2 holds, where the parameter ϵ in the lemma is given by $\Xi_2 = \Xi_2(\Gamma_2)$. (We remark that W^0_2 will not be the same as the W_2 of Proposition 13.1, due to subsequent adjustments.) In particular, there is a well-defined nearest point projection

$$P_2 : N_{r_2}(S_2) = \widehat{W_2} \to W^0_2\,, \tag{13.20}$$

where we are using the notation for variable thickness neighborhoods from Section 3.

Lemma 13.21. *Under the constraint $c_{\text{2-stratum}} < \bar{c}_{\text{2-stratum}}$, there is a smooth function $\psi_2 : \{x_{\mathfrak{r}} > 0\} \to [0, 1]$ with the following properties:*

(1)

$$\psi_2 \circ \mathcal{E}^1 \equiv 1 \text{ in } \bigcup_{i \in I_{\text{edge}}} \{|\eta_i| < 6\Delta, \eta_{E'} < 6\Delta\} \text{ and} \tag{13.22}$$
$$\psi_2 \circ \mathcal{E}^1 \equiv 0 \text{ outside } \bigcup_{i \in I_{\text{edge}}} \{|\eta_i| < 7\Delta, \eta_{E'} < 7\Delta\}\,.$$

(2) $\operatorname{supp}(\psi_2) \cap \operatorname{im}(\mathcal{E}^1) \subset \widehat{W_2} \times Q_2^{\perp}$.

(3) *There is a constant $\Omega'_2 = \Omega'_2(\mathcal{M})$ such that*

$$|(D\psi_2)_x| < \Omega'_2\, x_{\mathfrak{r}}^{-1} \tag{13.23}$$

for all $x \in \operatorname{im}(\mathcal{E}^1)$.

Proof. If the parameter $c_{\text{2-stratum}}$ is sufficiently small then $\mathcal{E}^1(p) \in \bigcup_{i \in I_{\text{edge}}} \{|\eta_i| < 6\Delta, \eta_{E'} < 6\Delta\}$ implies that $\mathcal{E}^0(p) \in \bigcup_{i \in I_{\text{edge}}} \{|\eta_i| < 6.1\Delta, \eta_{E'} < 6.1\Delta\}$, and $\mathcal{E}^1(p) \notin \bigcup_{i \in I_{\text{edge}}} \{|\eta_i| < 7\Delta, \eta_{E'} < 7\Delta\}$ implies that $\mathcal{E}^0(p) \notin \bigcup_{i \in I_{\text{edge}}} \{|\eta_i| < 6.9\Delta, \eta_{E'} < 6.9\Delta\}$.

In analogy to (9.28), put

$$z_{\mathrm{edge}} = 1 - \Phi_{\frac{1}{2},1}\left(\sum_{i \in I_{\mathrm{edge}}} \frac{x_i''}{R_i}\right). \tag{13.24}$$

Define $\psi_2 : \{x_{\mathfrak{r}} > 0\} \to [0,1]$ by

$$\begin{aligned}\psi_2(x) =& 1 - \Phi_{\frac{1}{2},1}\left(\sum_{\{i \in I_{\mathrm{edge}} \mid x_i''>0\}} \Phi_{6.1\Delta,6.5\Delta}\left(\frac{|x_i'|}{x_i''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right) \cdot \right.\\ & \left.\left[\left(1 - \Phi_{\frac{1}{4},\frac{1}{2}}\left(\frac{x_{E'}''}{x_{\mathfrak{r}}}\right)\right)\Phi_{6.1\Delta,6.5\Delta}\left(\frac{|x_{E'}'|}{x_{E'}''}\right) + 10\left(\frac{x_i''}{R_i} z_{\mathrm{edge}} - \frac{x_{E'}''}{x_{\mathfrak{r}}}\right)\right]\right).\end{aligned} \tag{13.25}$$

It is easy to see that ψ_2 is smooth.

To prove part (1), it is enough to show that

$$\begin{aligned}&\psi_2 \circ \mathcal{E}^0 \equiv 1 \text{ in } \bigcup_{i \in I_{\mathrm{edge}}} \{|\eta_i| < 6.1\Delta, \eta_{E'} < 6.1\Delta\} \text{ and}\\ &\psi_2 \circ \mathcal{E}^0 \equiv 0 \text{ outside } \bigcup_{i \in I_{\mathrm{edge}}} \{|\eta_i| < 6.9\Delta, \eta_{E'} < 6.9\Delta\}.\end{aligned} \tag{13.26}$$

Suppose that $i \in I_{\mathrm{edge}}$, $|\eta_i(p)| < 6.1\Delta$ and $\eta_{E'}(p) < 6.1\Delta$. Put $x = \mathcal{E}^0(p)$. Recall that $x_i'' = R_i\zeta_i(p)$, where ζ_i is given in (9.19) with $p \rightsquigarrow p_i$, and $x_{E'}'' = \mathfrak{r}_p\zeta_{E'}(p)$, where $\zeta_{E'}$ is the expresssion in (9.29). Hence

$$\begin{aligned}&\frac{x_i''}{R_i} = \zeta_i(p) = 1,\\ &1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right) = 1,\\ &\Phi_{6.1\Delta,6.5\Delta}\left(\frac{|x_i'|}{x_i''}\right) = \Phi_{6.1\Delta,6.5\Delta}\left(|\eta_i(p)|\right) = 1.\end{aligned} \tag{13.27}$$

If $\frac{x_{E'}''}{x_{\mathfrak{r}}} = \zeta_{E'}(p) \geq \frac{1}{2}$ then

$$\begin{aligned}&1 - \Phi_{\frac{1}{4},\frac{1}{2}}\left(\frac{x_{E'}''}{x_{\mathfrak{r}}}\right) = 1,\\ &\Phi_{6.1\Delta,6.5\Delta}\left(\frac{|x_{E'}'|}{x_{E'}''}\right) = \Phi_{6.1\Delta,6.5\Delta}\left(|\eta_{E'}|(p)\right) = 1,\\ &\frac{x_i''}{R_i} z_{\mathrm{edge}} - \frac{x_{E'}''}{x_{\mathfrak{r}}} = \zeta_i(p)\zeta_{\mathrm{edge}}(p) - \zeta_{E'}(p) = \zeta_{\mathrm{edge}}(p) - \zeta_{E'}(p) \geq 0.\end{aligned} \tag{13.28}$$

If $\frac{x_{E'}''}{x_{\mathfrak{r}}} = \zeta_{E'}(p) < \frac{1}{2}$ then $\left(1 - \Phi_{\frac{1}{4},\frac{1}{2}}\left(\frac{x_{E'}''}{x_{\mathfrak{r}}}\right)\right)\Phi_{6.1\Delta,6.5\Delta}\left(\frac{|x_{E'}'|}{x_{E'}''}\right) \geq 0$ and

$$\frac{x_i''}{R_i} z_{\mathrm{edge}} - \frac{x_{E'}''}{x_{\mathfrak{r}}} = \zeta_i(p)\zeta_{\mathrm{edge}}(p) - \zeta_{E'}(p) = 1 - \zeta_{E'}(p) \geq \frac{1}{2}. \tag{13.29}$$

In either case, the argument of $\Phi_{\frac{1}{2},1}$ in (13.25) is bounded below by one and so $\psi_2(x) = 1$.

Now suppose that for all $i \in I_{\text{edge}}$, either $\zeta_i(p) = 0$, or $\zeta_i(p) > 0$ and $|\eta_i(p)| \geq 6.9\Delta$, or $\zeta_i(p) > 0$ and $|\eta_i(p)| < 6.9\Delta$ and $\eta_{E'}(p) \geq 6.9\Delta$. If $\zeta_i(p) = 0$, or $\zeta_i(p) > 0$ and $|\eta_i(p)| \geq 6.9\Delta$, then

$$\Phi_{6.1\Delta, 6.5\Delta}\left(\frac{|x_i'|}{x_i''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right) = \Phi_{6.1\Delta,6.5\Delta}\left(|\eta_i|(p)\right) \cdot \Big(1 - \Phi_{\frac{1}{2},1}\left(\zeta_i(p)\right)\Big) = 0. \tag{13.30}$$

If $|\eta_i(p)| < 6.9\Delta$ and $\eta_{E'}(p) \geq 6.9\Delta$ then

$$\left(1 - \Phi_{\frac{1}{4},\frac{1}{2}}\left(\frac{x''_{E'}}{x_{\mathfrak{r}}}\right)\right) \Phi_{6.1\Delta,6.5\Delta}\left(\frac{|x'_{E'}|}{x''_{E'}}\right) = \Big(1 - \Phi_{\frac{1}{4},\frac{1}{2}}\left(\zeta_{E'}(p)\right)\Big) \cdot \Phi_{6.1\Delta,6.5\Delta}\left(|\eta_{E'}|(p)\right) = 0. \tag{13.31}$$

and

$$\begin{aligned} \frac{x_i''}{R_i} z_{\text{edge}} - \frac{x''_{E'}}{x_{\mathfrak{r}}} &= \zeta_i(p)\zeta_{\text{edge}}(p) - \zeta_{E'}(p) \\ &= \Phi_{8\Delta,9\Delta}(\eta_{E'}(p)) \cdot \zeta_{\text{edge}}(p) - \Phi_{\frac{2}{10}\Delta,\frac{3}{10}\Delta,8\Delta,9\Delta}(\eta_{E'}(p)) \cdot \zeta_{\text{edge}}(p) = 0. \end{aligned} \tag{13.32}$$

Hence $\psi_2(x) = 0$.

This proves part (1) of the lemma.

The proof of the rest of the lemma is similar to that of Lemma 13.6. ☐

We can assume that $\widehat{W}_2 \subset \{x_{\mathfrak{r}} > 0\}$. Define $\Psi_2 : \{x_{\mathfrak{r}} > 0\} \to \{x_{\mathfrak{r}} > 0\}$ by $\Psi_2(x) = x$ if $\pi_2(x) \notin \widehat{W}_2$ and

$$\Psi_2(x) = (\psi_2(x)P_2(\pi_2(x)) + (1 - \psi_2(x))\pi_2(x), \pi_2^\perp(x)) \tag{13.33}$$

otherwise. Put $\mathcal{E}^2 = \Psi_2 \circ \mathcal{E}^1$.

Lemma 13.34. *Under the constraints $\Sigma_2 < \overline{\Sigma}_2(\Omega_2, c_{\text{edge}})$, $\Gamma_2 < \overline{\Gamma}_2(\Omega_2, c_{\text{edge}})$, $\Xi_2 < \overline{\Xi}_2(c_{\text{edge}})$ and $c_{\text{2-stratum}} < \overline{c}_{\text{2-stratum}}(c_{\text{edge}})$, we have:*

1. *$\mathcal{E}^2$ is smooth.*
2. *For all $p \in M$,*

$$\|\mathcal{E}^2(p) - \mathcal{E}^0(p)\| < c_{\text{edge}}\, \mathfrak{r}(p) \quad \text{and} \quad \|D\mathcal{E}^2_p - D\mathcal{E}^0_p\| < c_{\text{edge}}\,. \tag{13.35}$$

3. *The restriction of $\pi_2 \circ \mathcal{E}^2$ to $\bigcup_{i \in I_{\text{edge}}}\{|\eta_i| < 6\Delta, \eta_{E'} < 6\Delta\}$ is a submersion to W_2^0.*

Proof. As in the proof of Lemma 13.15, $\mathcal{E}^2$ is smooth and we can ensure that

$$\|\mathcal{E}^2(p) - \mathcal{E}^1(p)\| < \frac{1}{2}c_{\text{edge}}\, \mathfrak{r}(p) \quad \text{and} \quad \|D\mathcal{E}^2_p - D\mathcal{E}^1_p\| < \frac{1}{2}c_{\text{edge}}\,. \tag{13.36}$$

Along with (13.16), part (2) of the lemma follows.

The proof of part (3) is similar to that of Lemma 13.15(3). We omit the details. ☐

13.4. **Adjusting the map near the slim 1-stratum.** Our third adjustment step involves the slim stratum.

Recall that $Q_3 = H_{0\text{-stratum}} \oplus H_{\mathrm{slim}}$ and $\pi_3 : H \to Q_3$ is orthogonal projection. We let $\widetilde{A}_3$, A_3, $\widetilde{S}_3$, S_3 and $r_3 : \widetilde{S}_3 \to (0, \infty)$ be as in Section 12.5.

Thus $(\tilde{S}_3, S_3, r_3)$ is a $(2, \Gamma_3)$ cloudy 1-manifold by Lemma 12.34. By Lemma B.2, there is a 1-manifold $W_3^0 \subset Q_3$ so that the conclusion of Lemma B.2 holds, where the parameter ϵ in the lemma is given by $\Xi_3 = \Xi_3(\Gamma_3)$. In particular, there is a well-defined nearest point projection

$$P_3 : N_{r_3}(S_3) = \widehat{W}_3 \to W_3^0\,, \tag{13.37}$$

where we are using the notation for variable thickness neighborhoods from Section 3.

Lemma 13.38. *Under the constraint $c_{\mathrm{edge}} < \overline{c}_{\mathrm{edge}}$, there is a smooth function $\psi_3 : H \to [0, 1]$ with the following properties:*

(1)

$$\psi_3 \circ \mathcal{E}^2 \equiv 1 \textit{ in } \bigcup_{i \in I_{\mathrm{slim}}} \{|\eta_i| < 6 \cdot 10^5 \Delta\} \textit{ and} \tag{13.39}$$

$$\psi_3 \circ \mathcal{E}^2 \equiv 0 \textit{ outside } \bigcup_{i \in I_{\mathrm{slim}}} \{|\eta_i| < 7 \cdot 10^5 \Delta\}\,.$$

(2) $\mathrm{supp}(\psi_3) \cap \mathrm{im}(\mathcal{E}^2) \subset \widehat{W}_3 \times Q_3^\perp$.

(3) *There is a constant $\Omega_3' = \Omega_3'(\mathcal{M})$ such that*

$$|(D\psi_3)_x| < \Omega_3'\, x_{\mathfrak{r}}^{-1} \tag{13.40}$$

for all $x \in \mathrm{im}(\mathcal{E}^2)$.

Proof. Let $\psi_3 : H \to [0, 1]$ be given by

$$\psi_3(x) = 1 - \Phi_{\frac{1}{2},1}\left(\sum_{\{i \in I_{\mathrm{slim}} \mid x_i'' > 0\}} \Phi_{6.1 \cdot 10^5 \Delta, 6.5 \cdot 10^5 \Delta}\left(\frac{|x_i'|}{x_i''}\right) \cdot \left(1 - \Phi_{\frac{1}{2},1}\left(\frac{x_i''}{R_i}\right)\right)\right)\,. \tag{13.41}$$

The rest of the proof is similar to that of Lemma 13.21. We omit the details. $\square$

Define $\Psi_3 : H \to H$ by $\Psi_3(x) = x$ if $\pi_3(x) \notin \widehat{W}_3$ and

$$\Psi_3(x) = (\psi_3(x) P_3(\pi_3(x)) + (1 - \psi_3(x))\pi_3(x), \pi_3^\perp(x)) \tag{13.42}$$

otherwise. Put $\mathcal{E}^3 = \Psi_3 \circ \mathcal{E}^2$.

Lemma 13.43. *Under the constraints $\Sigma_3 < \overline{\Sigma}_3(\Omega_3, c_{\mathrm{slim}})$, $\Gamma_3 < \overline{\Gamma}_3(\Omega_3, c_{\mathrm{slim}})$, $\Xi_3 < \overline{\Xi}_3(c_{\mathrm{slim}})$ and $c_{\mathrm{edge}} < \overline{c}_{\mathrm{edge}}(c_{\mathrm{slim}})$, we have:*

(1) *$\mathcal{E}^3$ is smooth.*
(2) *For all $p \in M$,*

$$\|\mathcal{E}^3(p) - \mathcal{E}^0(p)\| < c_{\mathrm{slim}}\, \mathfrak{r}(p) \quad \textit{and} \quad \|D\mathcal{E}^3_p - D\mathcal{E}^0_p\| < c_{\mathrm{slim}}\,. \tag{13.44}$$

(3) *The restriction of $\pi_3 \circ \mathcal{E}^3$ to $\bigcup_{i \in I_{\mathrm{slim}}} \{|\eta_i| < 6 \cdot 10^5 \Delta\}$ is a submersion to W_3^0.*

Proof. The proof is similar to that of Lemma 13.34. We omit the details. □

13.5. **Proof of Proposition 13.1.** Note from (13.42) that Ψ_3 can be factored as $\Psi_3^{Q_2} \times I_{Q_2^\perp}$ for some $\Psi_3^{Q_2} : Q_2 \to Q_2$. In particular, $\pi_2 \circ \Psi_3 = \Psi_3^{Q_2} \circ \pi_2$.

Put $\mathcal{E} = \mathcal{E}^3$, $c_{\text{adjust}} = c_{\text{slim}}$ and

(13.45)
$$\begin{aligned} W_1 &= (\Psi_3 \circ \Psi_2)(W_1^0) \cap \bigcup_{i \in I_{\text{2-stratum}}} \{ y \in H \ : \ y_i'' > .9 R_i, \ |y_i'| < 5.5 R_i \}, \\ W_2 &= \Psi_3^{Q_2}(W_2^0) \cap \bigcup_{i \in I_{\text{edge}}} \{ y \in Q_2 \ : \ y_i'' > .9 R_i, \ |y_i'| < 5.5 \Delta R_i, \ y_{\mathfrak{r}} > 0, \ y_{E'} < 5.5 \Delta y_{\mathfrak{r}} \}, \\ W_3 &= W_3^0 \cap \bigcup_{i \in I_{\text{slim}}} \{ y \in Q_3 \ : \ y_i'' > .9 R_i, \ |y_i'| < 5.5 \cdot 10^5 \Delta R_i \}. \end{aligned}$$

The smoothness of $\mathcal{E}$ follows from part (1) of Lemma 13.43. Part (1) of Proposition 13.1 follows from part (2) of Lemma 13.43.

Lemma 13.46. *W_i is a k_i-manifold.*

Proof. We will show that W_1 is a 2-manifold; the proofs for W_2 and W_3 are similar.

Choose $x \in W_1$. For some $i \in I_{\text{2-stratum}}$, we have $x_i'' > .9 R_i$ and $|x_i'| < 5.5 R_i$. Putting

$$V_i = W_1 \cap \{ y \in H \ : \ y_i'' > .9 R_i, \ |y_i'| < 5.5 R_i \} \tag{13.47}$$

gives a neighborhood of x in W_1. As $(\pi_{H_i'}, \pi_{H_i''}) \circ (\Psi_3 \circ \Psi_2) = (\pi_{H_i'}, \pi_{H_i''})$, it follows that V_i is the image, under $\Psi_3 \circ \Psi_2$, of the 2-manifold

$$V_i^0 = W_1^0 \cap \{ y \in H \ : \ y_i'' > .9 R_i, \ |y_i'| < 5.5 R_i \}. \tag{13.48}$$

If we can show that $\pi_{H_i'}$ maps V_i^0 diffeomorphically to its image in H_i' then V_i^0 will be a graph over a domain in H_i', and the same will be true for V_i.

In view of (13.16) and the definition of $\mathcal{E}^0$, if $c_{\text{2-stratum}}$ is sufficiently small then we are ensured that $V_i^0 = \mathcal{E}^1(\{|\eta_i| < 7\}) \cap \{ y \in H \ : \ |y_i'| < 5.5 R_i \}$. From Lemma 12.7(3), Lemma B.2(3) and Lemma B.2(5), if Ξ_1 is sufficiently small then we are ensured that $\pi_{H_i'}$ restricts to a proper surjective local diffeomorphism from V_i^0 to $B(0, 5.5 R_i) \subset H_i'$. Hence V_i^0 is a proper covering space of $B(0, 5.5 R_i) \subset H_i'$ and so consists of a finite number of connected components, each mapping diffeomorphically under π_i' to $B(0, 5.5 R_i) \subset H_i'$. It remains to show that there is only one connected component.

If V_i^0 has more than one connected component then there are $y_1, y_2 \in V_i^0 \cap \pi_{H_i'}^{-1}(0)$ with $y_1 \neq y_2$. We can write $y_1 = \mathcal{E}^1(p_1)$ and $y_2 = \mathcal{E}^1(p_2)$ for some $p_1, p_2 \in \{|\eta_i| < 7\}$. We claim that there is a smooth path γ in M from p_1 to p_2 so that $\mathcal{E}^1 \circ \gamma$ lies within $B\left(y_1, \frac{1}{10} R_i\right)$. To see this, we first note that if Γ_1 and $c_{\text{2-stratum}}$ are sufficiently small then Lemma 12.7(3) and (13.16) ensure that $|\eta_i(p_1)| << 1$ and $|\eta_i(p_2)| << 1$, as otherwise we would contradict the assumption that $(y_1)_i' = (y_2)_i' = 0$. Let $\widehat{\gamma}$ be a straight line from $\eta_i(p_1)$ to $\eta_i(p_2)$. Relative to the fiber bundle structure defined by η_i (see Lemma 8.4), let γ_1 be a lift of $\widehat{\gamma}$, with initial point p_1. Let γ_2 be a curve in the S^1-fiber containing p_2, going from the endpoint of γ_1 to p_2. Let γ be a smooth concatenation of γ_1 and γ_2. Then $\eta_i \circ \gamma$ lies in a ball whose diameter is much smaller than one. If Γ_1 and $c_{\text{2-stratum}}$ are sufficiently small then Lemma 12.7(3) and (13.16) ensure that $\mathcal{E}^1 \circ \gamma$ lies in a ball whose diameter is much smaller than R_i.

On the other hand, since p_1 and p_2 lie in different connected components of V_i^0, any curve in W_1^0 from p_1 to p_2 must go from p_1 to $\{y \in H \,:\, |y_i'| = R_i\}$. This is a contradiction.

Thus V_i^0 is connected and W_1 is a manifold. □

Recall the definition of U_1 from Proposition 13.1. By Lemma 13.15(3), the restriction of $\mathcal{E}^1$ to U_1 is a submersion from U_1 to W_1^0. From Lemma 12.7(3) and (13.44), if Γ_1 and c_{slim} are sufficiently small then $\mathcal{E} = \Psi_3 \circ \Psi_2 \circ \mathcal{E}^1$ maps U_1 to $W_1 \subset (\Psi_3 \circ \Psi_2)(W_1^0)$. To see that it is a submersion, suppose that $|\eta_i(p)| < 5$ for some $i \in I_{\text{2-stratum}}$. Put $x^0 = \mathcal{E}^0(p)$ and $x = \mathcal{E}(p)$. Note that $x_i' = (x_0)_i'$. From Lemma 12.7(3) and Lemma B.2(3), if Ξ_1 is sufficiently small then we are ensured that $(D\pi_{H_i'})_{x^0} \circ D\mathcal{E}_p^0$ maps onto $T_{(x^0)_i'} H_i' \cong \mathbb{R}^2$. Then $(D\pi_{H_i'})_x \circ D\mathcal{E}_p = (D\pi_{H_i'})_x \circ D(\Psi_3 \circ \Psi_2)_{x^0} \circ D\mathcal{E}_p^0 = (D\pi_{H_i'})_{x_0} \circ D\mathcal{E}_p^0$ maps onto $T_{x_i'} H_i' \cong \mathbb{R}^2$. Thus $D\mathcal{E}_p$ must map T_pM onto T_xW_1, showing that $\mathcal{E}$ is a submersion near p.

Next, by Lemma 13.34(3), the restriction of $\pi_2 \circ \mathcal{E}^2$ to U_2 is a submersion from U_2 to W_2^0. Lemma 12.27(3) and (13.44) imply that if Γ_2 and c_{slim} are sufficiently small then $\pi_2 \circ \mathcal{E} = \pi_2 \circ \Psi_3 \circ \mathcal{E}^2 = \Psi_3^{Q_3} \circ \pi_2 \circ \mathcal{E}^2$ maps U_2 to $W_2 \subset \Psi_3^{Q_3}(W_2^0)$. By a similar argument to the preceding paragraph, the restriction of $\pi_2 \circ \mathcal{E}$ to U_2 is a submersion to W_2.

Finally, by Lemma 13.43(3), the restriction of $\pi_3 \circ \mathcal{E} = \pi_3 \circ \mathcal{E}^3$ to U_3 is a submersion to $W_3 = W_3^0$. This proves Proposition 13.1.

14. Extracting a good decomposition of M

In this section we will use the map $\mathcal{E}$ to find a decomposition of M into fibered pieces which are compatible along the intersections:

Proposition 14.1. *There is a decomposition*

$$M = M^{\text{0-stratum}} \cup M^{\text{slim}} \cup M^{\text{edge}} \cup M^{\text{2-stratum}} \tag{14.2}$$

into compact domains with disjoint interiors, where each connected component of M^{slim}, M^{edge}, or $M^{\text{2-stratum}}$ may be endowed with a fibration structure, such that:

1. *$M^{\text{0-stratum}}$ and M^{slim} are domains with smooth boundary, while M^{edge} and $M^{\text{2-stratum}}$ are smooth manifolds with corners, each point of which has a neighborhood diffeomorphic to $\mathbb{R}^{3-k} \times [0,\infty)^k$ for some $k \le 2$.*
2. *Connected components of $M^{\text{0-stratum}}$ are diffeomorphic to one of the following: $S^1 \times S^2$, $S^1 \times_{\mathbb{Z}_2} S^2 = \mathbb{R}P^3 \# \mathbb{R}P^3$, T^3/Γ (where Γ is a finite subgroup of $\operatorname{Isom}^+(T^3)$ which acts freely on T^3), S^3/Γ (where Γ is a finite subgroup of $\operatorname{Isom}^+(S^3)$ which acts freely on S^3), a solid torus $S^1 \times D^2$, a twisted line bundle $S^2 \times_{Z_2} I$ over $\mathbb{R}P^2$, or a twisted line bundle $T^2 \times_{Z_2} I$ over a Klein bottle.*
3. *The components of M^{slim} have a fibration with S^2-fibers or T^2-fibers.*
4. *Components of M^{edge} are diffeomorphic (as manifolds with corners) to a solid torus $S^1 \times D^2$ or $I \times D^2$, and have a fibration with D^2 fibers.*
5. *$M^{\text{2-stratum}}$ is a smooth domain with corners with a smooth S^1-fibration; in particular the S^1-fibration is compatible with any corners.*

(6) *Each fiber of the fibration $M^{\mathrm{edge}} \to B^{\mathrm{edge}}$, lying over a boundary point of the base B^{edge}, is contained in the boundary of $M^{\text{0-stratum}}$ or the boundary of M^{slim}.*
(7) *The part of $\partial M^{\mathrm{edge}}$ which carries an induced S^1-fibration is contained in $M^{\text{2-stratum}}$, and the S^1-fibration induced from M^{edge} agrees with the one inherited from $M^{\text{2-stratum}}$.*

To prove the proposition, we show that the submersions identified in Proposition 13.1 become fibrations, when restricted to appropriate subsets. Using this, we remove fibered regions around successive strata in the following order: 0-stratum, slim stratum, the edge region and the 2-stratum. The compatibility of the fibrations is automatic from the compatibility of the various projection maps π_j, for $j \in \{1, 2, 3, 4\}$.

14.1. **The definition of $M^{\text{0-stratum}}$.** For each $i \in I_{\text{0-stratum}}$, put

$$M_i^{\text{0-stratum}} = B(p_i, .35R_i) \cup \mathcal{E}^{-1}\left\{x \in H \;:\; x_i'' \geq .9R_i\,,\; \frac{x_i'}{x_i''} \leq \frac{4}{10}\right\}. \tag{14.3}$$

Lemma 14.4. *Under the constraints $\varsigma_{\text{0-stratum}} < \overline{\varsigma}_{\text{0-stratum}}$ and $c_{\mathrm{adjust}} < \overline{c}_{\mathrm{adjust}}$, $\{M_i^{\text{0-stratum}}\}_{i \in I_{\text{0-stratum}}}$ is a disjoint collection and each $M_i^{\text{0-stratum}}$ is a compact manifold with boundary, which is diffeomorphic to one of the possibilities in Proposition 14.1(2).*

Proof. Note that

$$(\mathcal{E}^0)^{-1}\left\{x \in H \;:\; x_i'' \geq .9R_i\,,\; \frac{x_i'}{x_i''} \leq \frac{4}{10}\right\} = \{p \in M \;:\; \zeta_i(p) \geq .9\,,\; \eta_i(p) \leq .4\}. \tag{14.5}$$

In particular, if $\varsigma_{\text{0-stratum}}$ is sufficiently small then this set contains $A(p_i, .31R_i, .39R_i)$ and is contained in $A(p_i, .29R_i, .41R_i)$. Then if c_{adjust} is sufficiently small, $\mathcal{E}^{-1}\left\{x \in H \mid x_i'' \geq .9R_i\,,\; \frac{x_i'}{x_i''} \leq \frac{4}{10}\right\}$ contains $A(p_i, .32R_i, .38R_i)$ and is contained in $A(p_i, .28R_i, .42R_i)$.

In particular, $B(p_i, .38R_i) \subset M_i^{\text{0-stratum}} \subset B(p_i, .42R_i)$. It now follows from Lemma 11.5 that $\{M_i^{\text{0-stratum}}\}_{i \in I_{\text{0-stratum}}}$ are disjoint.

To characterize the topology of $M_i^{\text{0-stratum}}$, if c_{adjust} is sufficiently small then we can find a smooth function $f^0 : M \to \mathbb{R}$ such that

1. If $p \in A(p_i, .3R_i, .5R_i)$ and $x = \mathcal{E}(p)$ then $f^0(p) = \frac{x_i'}{x_i''}$.
2. If $p \in B(p_i, .35R_i)$ then $f^0(p) \leq .39$.
3. If $p \notin B(p_i, .5R_i)$ then $f^0(p) \geq .41$.

Put $f^1 = \eta_i$ and define $F : M \times [0,1] \to \mathbb{R}$ by $F(p,t) = (1-t)f^0(p) + tf^1(p)$. Put $f^t(p) = F(p,t)$ and $X = (-\infty, .4]$. If c_{adjust} and $\varsigma_{\text{0-stratum}}$ are sufficiently small then Lemma 11.1 implies that for each $t \in [0,1]$, f^t is transverse to $\partial X = \{.4\}$. By Lemma C.1, $M_i^{\text{0-stratum}} = (f^0)^{-1}(X)$ is diffeomorphic to $(f^1)^{-1}(X)$. By Lemma 11.3, the latter is diffeomorphic to one of the possibilities in Proposition 14.1(2). This proves the lemma. $\square$

We let $M^{\text{0-stratum}} = \bigcup_{i \in I_{\text{0-stratum}}} M_i^{\text{0-stratum}}$, and put $M_1 = M \setminus \mathrm{int}(M^{\text{0-stratum}})$. Thus $M^{\text{0-stratum}}$ and M_1 are smooth compact manifolds with boundary.

14.2. **The definition of M^{slim}.** We first truncate W_3. Put

$$W_3' = W_3 \cap \bigcup_{i \in I_{\mathrm{slim}}} \left\{ x \in Q_3 \mid x_i'' > .9 R_i \, , \, \left| \frac{x_i'}{x_i''} \right| < 4 \cdot 10^5 \Delta \right\} \tag{14.6}$$

and define $U_3' = (\pi_3 \circ \mathcal{E})^{-1}(W_3')$.

Lemma 14.7. *Under the constraints $\varsigma_{\mathrm{slim}} < \overline{\varsigma}_{\mathrm{slim}}(\Delta)$ and $c_{\mathrm{adjust}} < \overline{c}_{\mathrm{adjust}}$, we have*

(1) $\bigcup_{i \in I_{\mathrm{slim}}} \{ |\eta_i| \le 3.5 \cdot 10^5 \Delta \} \subset U_3' \subset U_3$, *where U_3 is as in Proposition 13.1.*
(2) *The restriction of $\pi_3 \circ \mathcal{E}$ to U_3' gives a proper submersion to W_3'. In particular, it is a fibration.*
(3) *The fibers of $\pi_3 \circ \mathcal{E} \, : \, U_3' \to W_3'$ are diffeomorphic to S^2 or T^2.*
(4) *M_1 intersects U_3' in a submanifold with boundary which is a union of fibers of $\pi_3 \circ \mathcal{E}$: $U_3' \to W_3'$.*

Proof. For a given $i \in I_{\mathrm{slim}}$, suppose that $p \in M$ satisfies $|\eta_i(p)| \le 3.5 \cdot 10^5 \Delta$. Putting $y = (\pi_3 \circ \mathcal{E}^0)(p) \in Q_3$, we have $y_i'' = R_i$ and $\left| \frac{y_i'}{y_i''} \right| \le 3.5 \cdot 10^5 \Delta$. Hence if c_{adjust} is small enough then since $\Delta >> 1$, we are ensured that, putting $x = (\pi_3 \circ \mathcal{E})(p) \in Q_3$, we have $x_i'' > .9 R_i$ and $\left| \frac{x_i'}{x_i''} \right| < 4 \cdot 10^5 \Delta$. As $p \in U_3$, Proposition 13.1 implies that $x \in W_3$. Hence $\bigcup_{i \in I_{\mathrm{slim}}} \{ |\eta_i| \le 3.5 \cdot 10^5 \Delta \} \subset U_3'$.

Now suppose that $p \in U_3'$. Putting $x = (\pi_3 \circ \mathcal{E})(p)$, for some $i \in I_{\mathrm{slim}}$ we have $x_i'' > .9 R_i$ and $\left| \frac{x_i'}{x_i''} \right| < 4 \cdot 10^5 \Delta$. If c_{adjust} is small enough then we are ensured that, putting $y = (\pi_3 \circ \mathcal{E}^0)(p)$, we have $y_i'' \ge .8 R_i$ and $\left| \frac{y_i'}{y_i''} \right| \le 4.5 \cdot 10^5 \Delta$. Hence $|\eta_i(p)| \le 4.5 \cdot 10^5 \Delta$. This shows that $U_3' \subset U_3$, proving part (1) of the lemma.

By Proposition 13.1, $\pi_3 \circ \mathcal{E}$ is a submersion from U_3 to W_3. Hence it restricts to a surjective submersion on U_3'.

Suppose that K is a compact subset of W_3'. Then $(\pi_3 \circ \mathcal{E})^{-1}(K)$ is a closed subset of M which is contained in $\overline{U}_3 = \bigcup_{i \in I_{\mathrm{slim}}} \{ |\eta_i| \le 5 \cdot 10^5 \Delta \}$. As $\{p_i\}_{i \in I_{\mathrm{slim}}}$ are in the slim 1-stratum, it follows from the definition of adapted coordinates that $\{ |\eta_i| \le 5 \cdot 10^5 \Delta \}$ is a compact subset of M; cf. the proof of Lemma 10.3. Thus the restriction of $\pi_3 \circ \mathcal{E}$ to U_3' is a proper submersion. This proves part (2) of the lemma.

To prove part (3) of the lemma, given $x \in W_3'$, suppose that $p \in U_3'$ satisfies $(\pi_3 \circ \mathcal{E})(p) = x$. Choose $i \in I_{\mathrm{slim}}$ so that $|\eta_i(p)| \le 4.5 \cdot 10^5 \Delta$. If c_{adjust} is sufficiently small then by looking at the components in H_i, one sees that for any $p' \in U_3'$ satisfying $(\pi_3 \circ \mathcal{E})(p') = x$, we have $p' \in \{ |\eta_i| < 5 \cdot 10^5 \Delta \}$. Thus to determine the topology of the fiber, we can just consider the restriction of $\pi_3 \circ \mathcal{E}$ to $\{ |\eta_i| < 5 \cdot 10^5 \Delta \}$.

Let $\pi_{H_i'} : Q_3 \to H_i'$ be orthogonal projection and put $X = \pi_{H_i'}(x) \in H_i'$. As the restriction of $\pi_{H_i'} \circ \pi_3 \circ \mathcal{E}^0$ to $\{ |\eta_i| < 5 \cdot 10^5 \Delta \}$ equals η_i, it follows that $\pi_{H_i'} \circ \pi_3 \circ \mathcal{E}^0$ is transverse there to X. By Lemma 10.3, $\{ |\eta_i| < 5 \cdot 10^5 \Delta \} \cap (\pi_{H_i'} \circ \pi_3 \circ \mathcal{E}^0)^{-1}(X)$ is diffeomorphic to S^2 or T^2.

Consider the restriction of $(\pi_{H_i'} \circ \pi_3 \circ \mathcal{E})$ to $\{ |\eta_i| < 5 \cdot 10^5 \Delta \}$. Proposition 13.1 and Lemma C.3 imply that if c_{adjust} is sufficiently small then the fiber $\{ |\eta_i| < 5 \cdot 10^5 \Delta \} \cap (\pi_{H_i'} \circ \pi_3 \circ \mathcal{E})^{-1}(X)$

is diffeomorphic to S^2 or T^2. In particular, it is connected. Now $(\pi_{H'_i} \circ \pi_3 \circ \mathcal{E})^{-1}(X)$ is the preimage, under $\pi_3 \circ \mathcal{E} \, : \, U'_3 \to W'_3$, of the preimage of X under $\pi_{H'_i} \, : \, W'_3 \to H'_i$. From connectedness of the fiber, the preimage of X under $\pi_{H'_i} : W'_3 \to H'_i$ must just be x. Hence $(\pi_3 \circ \mathcal{E})^{-1}(x)$ is diffeomorphic to S^2 or T^2. This proves part (3) of the lemma.

To prove part (4) of the lemma, given $j \in I_{\text{0-stratum}}$, suppose that $p \in \partial M_j^{\text{0-stratum}}$. If $x = \mathcal{E}(p)$ then $x''_j \geq .9 R_j$ and $x'_j = .4 x''_j$. Suppose that $p \in U'_3$. If $q \in U'_3$ is a point in the same fiber of $\pi_3 \circ \mathcal{E} : U'_3 \to W'_3$ as p, put $y = \mathcal{E}(q) \in H$. As $\pi_3(x) = \pi_3(y)$, we have $y''_j \geq .9 R_j$ and $y'_j = .4 y''_j$. Thus $q \in \partial M_j^{\text{0-stratum}}$. Hence $\partial M_j^{\text{0-stratum}}$ is a union of fibers of $\pi_3 \circ \mathcal{E} : U'_3 \to W'_3$. In fact, since $\partial M_j^{\text{0-stratum}}$ is a connected 2-manifold, it is a single fiber of $\pi_3 \circ \mathcal{E}$. This proves part (4) of the lemma. $\square$

Let $W''_3 \subset W'_3$ be a compact 1-dimensional manifold with boundary such that $(\pi_3 \circ \mathcal{E})^{-1}(W''_3)$ contains $\bigcup_{i \in I_{\text{slim}}} \{|\eta_i| \leq 3.5 \cdot 10^5 \Delta\}$, and put $M^{\text{slim}} = M_1 \cap (\pi_3 \circ \mathcal{E})^{-1}(W''_3)$. We endow M^{slim} with the fibration induced by $\pi_3 \circ \mathcal{E}$.

Put $M_2 = M_1 \setminus \operatorname{int}(M^{\text{slim}})$.

14.3. **The definition of M^{edge}.** We first truncate W_2. Put

$$W'_2 = W_2 \cap \bigcup_{i \in I_{\text{edge}}} \{x \in Q_2 \mid x''_i \geq .9 R_i \, , \, \left| \frac{x'_i}{x''_i} \right| < 4\Delta\} \tag{14.8}$$

and

$$U'_2 = (\pi_2 \circ \mathcal{E})^{-1}(W'_2) \cap \left(\{\eta_{E'} \leq .35\Delta\} \cup \mathcal{E}^{-1}\{x \in H \mid x_{\mathfrak{r}} > 0 \, , \, \frac{x_{E'}}{x_{\mathfrak{r}}} \leq 4\Delta\} \right) . \tag{14.9}$$

Lemma 14.10. *Under the constraints $\Lambda < \overline{\Lambda}(\Delta)$, $\varsigma_{\text{edge}} < \overline{\varsigma}_{\text{edge}}(\Delta)$ and $c_{\text{adjust}} < \overline{c}_{\text{adjust}}$, we have*

1. $\bigcup_{i \in I_{\text{edge}}} \{|\eta_i| \leq 3.5\Delta \, , \, |\eta_{E'}| \leq 3.5\Delta\} \subset U'_2 \subset U_2$, *where U_2 is as in Proposition 13.1.*
2. *The restriction of $\pi_2 \circ \mathcal{E}$ to U'_2 gives a proper submersion to W'_2. In particular, it is a fibration.*
3. *The fibers of $\pi_2 \circ \mathcal{E} \, : \, U'_2 \to W'_2$ are diffeomorphic to D^2.*
4. *M_2 intersects U'_2 in a submanifold with corners which is a union of fibers of $\pi_2 \circ \mathcal{E}$: $U'_2 \to W'_2$.*

Proof. The proof is similar to that of Lemmas 14.4 and 14.7. We omit the details. $\square$

Lemma 14.11. *Under the constraint $c_{\text{adjust}} < \overline{c}_{\text{adjust}}$, $M_2 \cap U'_2$ is compact.*

Proof. Suppose that $M_2 \cap U'_2$ is not compact. As M is compact, there is a sequence $\{q^k\}_{k=1}^\infty \subset M_2 \cap U'_2$ with a limit $q \in M$, for which $q \notin M_2 \cap U'_2$. Put $y = \mathcal{E}(q)$.

Since M_2 is closed we have $q \in M_2$ and so $q \notin U'_2$. Since $y_{\mathfrak{r}} > 0$ (assuming c_{adjust} is sufficiently small) we also have $q \in \{\eta_{E'} \leq .35\Delta\} \cup \mathcal{E}^{-1}\{x \in H \mid x_{\mathfrak{r}} > 0 \, , \, \frac{x_{E'}}{x_{\mathfrak{r}}} \leq 4\Delta\}$.

We know that $\pi_2(y) \in \overline{W'_2}$. As $q \notin U'_2$, it must be that $\pi_2(y) \notin W'_2$. Then for some $i \in I_{\text{edge}}$, we have $y''_i \geq .9 R_i$ and $\left| \frac{y'_i}{y''_i} \right| = 4\Delta$. Now p_i cannot be a slim 1-stratum point, as otherwise the preceding truncation step would force $B(p_i, 1000\Delta R_i) \cap M_2 = \emptyset$, which contradicts the facts that $q \in M_2$ and $d(p_i, q) < 10\Delta R_i$.

Lemma 9.26 now implies that there is a $j \in I_{\text{edge}}$ such that $|\eta_j(q)| < 2\Delta$. If c_{adjust} is sufficiently small then we are ensured that $y''_j \geq .9 R_j$ and $\left|\frac{y'_j}{y''_j}\right| < 3\Delta$. Thus $\pi_2(y) \in W'_2$ and so $q \in U'_2$, which is a contradiction. □

We put $M^{\text{edge}} = U'_2 \cap M_2$ and $W''_2 = (\pi_2 \circ \mathcal{E})(M^{\text{edge}})$. We endow M^{edge} with the fibration induced by $\pi_2 \circ \mathcal{E}$.

Put $M_3 = M_2 \setminus \operatorname{int}(M^{\text{edge}})$.

14.4. **The definition of $M^{\text{2-stratum}}$.** We first truncate W_1. Put

$$W'_1 = W_1 \cap \bigcup_{i \in I_{\text{2-stratum}}} \left\{ x \in H \mid x''_i > .9\,,\ \left|\frac{x'_i}{x''_i}\right| < 4 \right\} \tag{14.12}$$

and define $U'_1 = \mathcal{E}^{-1}(W'_1)$.

Lemma 14.13. *Under the constraints $\varsigma_{\text{2-stratum}} < \overline{\varsigma}_{\text{2-stratum}}$ and $c_{\text{adjust}} < \overline{c}_{\text{adjust}}$, we have*

1. $\bigcup_{i \in I_{\text{2-stratum}}} \{|\eta_i| \leq 3.5\} \subset U'_1 \subset U_1$, *where U_1 is as in Proposition 13.1.*
2. *The restriction of $\mathcal{E}$ to U'_1 gives a proper submersion to W'_1. In particular, it is a fibration.*
3. *The fibers of $M^{\text{2-stratum}}$ are circles.*
4. *M_3 is contained in U'_1, and is a submanifold with corners which is a union of fibers of $\mathcal{E}\big|_{U'_1} : U'_1 \to W'_1$.*

Proof. The proof is similar to that of Lemma 14.7. We omit the details. □

We put $M^{\text{2-stratum}} = M_3$, and endow it with the fibration $\mathcal{E}\big|_{M^{\text{2-stratum}}} : M^{\text{2-stratum}} \to \mathcal{E}(M^{\text{2-stratum}})$.

14.5. **The proof of Proposition 14.1.** Proposition 14.1 now follows from combining the results in this section.

15. Proof of Theorem 16.1 for closed manifolds

Recall that we are trying to get a contradiction to Standing Assumption 5.2. As before, we let M denote M^α for large α. Then M satisfies the conclusion of Proposition 14.1. To get a contradiction, we will show that M is a graph manifold.

We recall the definition of a graph manifold from Definition 1.2. It is obvious that boundary components of graph manifolds are tori. It is also obvious that if we glue two graph manifolds along boundary components then the result is a graph manifold, provided that it is orientable. In addition, the connected sum of two graph manifolds is a graph manifold. For more information about graph manifolds, we refer to [Mat03, Chapter 2.4]

15.1. M **is a graph manifold.** Each connected component of $M^{\text{0-stratum}}$ has boundary either $\emptyset$, S^2 or T^2. If there is a connected component of $M^{\text{0-stratum}}$ with empty boundary then M is diffeomorphic to $S^1 \times S^2$, $S^1 \times_{\mathbb{Z}_2} S^2 = \mathbb{R}P^3 \# \mathbb{R}P^3$, T^3/Γ (where Γ is a finite subgroup of $\mathrm{Isom}^+(T^3)$ which acts freely on T^3) or S^3/Γ (where Γ is a finite subgroup of $\mathrm{Isom}^+(S^3)$ which acts freely on S^3). In any case M is a graph manifold. So we can assume that each connected component of $M^{\text{0-stratum}}$ has nonempty boundary.

Each connected component of M^{slim} fibers over S^1 or I. If it fibers over S^1 then M is diffeomorphic to $S^1 \times S^2$ or the total space of a T^2-bundle over S^1. In either case, M is a graph manifold. Hence we can assume that each connected component of M^{slim} is diffeomorphic to $I \times S^2$ or $I \times T^2$.

Lemma 15.1. *Let $M_i^{\text{0-stratum}}$ be a connected component of $M^{\text{0-stratum}}$. If $M_i^{\text{0-stratum}} \cap M^{\text{slim}} \neq \emptyset$ then $\partial M_i^{\text{0-stratum}}$ is a boundary component of a connected component of M^{slim}. If $M_i^{\text{0-stratum}} \cap M^{\text{slim}} = \emptyset$ then we can write $\partial M_i^{\text{0-stratum}} = A_i \cup B_i$ where*

(1) *$A_i = M_i^{\text{0-stratum}} \cap M^{\text{edge}}$ is a disjoint union of 2-disks,*
(2) *$B_i = M_i^{\text{0-stratum}} \cap M^{\text{2-stratum}}$ is the total space of a circle bundle and*
(3) *$A_i \cap B_i = \partial A_i \cap \partial B_i$ is a union of circle fibers.*

Furthermore, if $\partial M_i^{\text{0-stratum}}$ is a 2-torus then $A_i = \emptyset$, while if $\partial M_i^{\text{0-stratum}}$ is a 2-sphere then A_i consists of exactly two 2-disks.

Proof. Proposition 14.1 implies all but the last sentence of the lemma. The statement about A_i follows from an Euler characteristic argument. □

Lemma 15.2. *Let M_i^{slim} be a connected component of M^{slim}. Let Y_i be one of the connected components of $\partial M_i^{\text{slim}}$. If $Y_i \cap M^{\text{0-stratum}} \neq \emptyset$ then $Y_i = \partial M_i^{\text{0-stratum}}$ for some connected component $M_i^{\text{0-stratum}}$ of $M^{\text{0-stratum}}$.*

If $Y_i \cap M^{\text{0-stratum}} = \emptyset$ then we can write $Y_i = A_i \cup B_i$ where

(1) *$A_i = Y_i \cap M^{\text{edge}}$ is a disjoint union of 2-disks,*
(2) *$B_i = Y_i \cap M^{\text{2-stratum}}$ is the total space of a circle bundle and*
(3) *$A_i \cap B_i = \partial A_i \cap \partial B_i$ is a union of circle fibers.*

Furthermore, if Y_i is a 2-torus then $A_i = \emptyset$, while if Y_i is a 2-sphere then A_i consists of exactly two 2-disks.

Proof. The proof is similar to that of Lemma 15.1. We omit the details. □

Hereafter we can assume that there is a disjoint union $M^{\text{0-stratum}} = M_{S^2}^{\text{0-stratum}} \cup M_{T^2}^{\text{0-stratum}}$, based on what the boundaries of the connected components are. Similarly, each fiber of M^{slim} is S^2 or T^2, so there is a disjoint union $M^{\text{slim}} = M_{S^2}^{\text{slim}} \cup M_{T^2}^{\text{slim}}$.

It follows from Lemmas 15.1 and 15.2 that each connected component of $M_{T^2}^{\text{0-stratum}} \cup M_{T^2}^{\text{slim}}$ is diffeomorphic to
1. A connected component of $M_{T^2}^{\text{0-stratum}}$,
2. The gluing of two connected components of $M_{T^2}^{\text{0-stratum}}$ along a 2-torus, or
3. $I \times T^2$.

In case 1, the connected component is diffeomorphic to $S^1 \times D^2$ or the total space of a twisted interval bundle over a Klein bottle. In any case, we can say that $M^{\text{0-stratum}}_{T^2} \cup M^{\text{slim}}_{T^2}$ is a graph manifold. Put $X_1 = M - \text{int}(M^{\text{0-stratum}}_{T^2} \cup M^{\text{slim}}_{T^2})$. To show that M is a graph manifold, it suffices to show that X_1 is a graph manifold. Note that $X_1 = M^{\text{0-stratum}}_{S^2} \cup M^{\text{slim}}_{S^2} \cup M^{\text{edge}} \cup M^{\text{2-stratum}}$.

Suppose that $M^{\text{0-stratum}}_i$ is a connected component of $M^{\text{0-stratum}}_{S^2}$. From Proposition 14.1, $M^{\text{0-stratum}}_i$ is diffeomorphic to D^3 or $\mathbb{R}P^3 \# D^3$. If $M^{\text{0-stratum}}_i$ is diffeomorphic to $\mathbb{R}P^3 \# D^3$, let Z_i be the result of replacing $M^{\text{0-stratum}}_i$ in X_1 by D^3. Then X_1 is diffeomorphic to $\mathbb{R}P^3 \# Z_i$. As $\mathbb{R}P^3$ is a graph manifold, if Z_i is a graph manifold then X_1 is a graph manifold. Hence without loss of generality, we can assume that each connected component of $M^{\text{0-stratum}}_{S^2}$ is diffeomorphic to a 3-disk.

From Lemmas 15.1 and 15.2, each connected component of $M^{\text{0-stratum}}_{S^2} \cup M^{\text{slim}}_{S^2}$ is diffeomorphic to
1. D^3,
2. $I \times S^2$ or
3. S^3, the result of attaching two connected components of $M^{\text{0-stratum}}_{S^2}$ by a connected component $I \times S^2$ of $M^{\text{slim}}_{S^2}$.

In case 3, X_1 is diffeomorphic to a graph manifold. In case 2, if Z is a connected component of $M^{\text{0-stratum}}_{S^2} \cup M^{\text{slim}}_{S^2}$ which is diffeomorphic to $I \times S^2$ then we can do surgery along $\{\frac{1}{2}\} \times S^2 \subset X_1$ to replace $I \times S^2 \subset X_1$ by a union of two 3-disks. Let X_2 be the result of performing the surgery. Then X_1 is recovered from X_2 by either taking a connected sum of two connected components of X_2 or by taking a connected sum of X_2 with $S^1 \times S^2$. In either case, if X_2 is a graph manifold then X_1 is a graph manifold. Hence without loss of generality, we can assume that each connected component of $(M^{\text{0-stratum}}_{S^2} \cup M^{\text{slim}}_{S^2}) \subset X_1$ is diffeomorphic to D^3.

Some connected components of M^{edge} may fiber over S^1. If Z is such a connected component then it is diffeomorphic to $S^1 \times D^2$. If $X_1 - \text{int}(Z)$ is a graph manifold then X_1 is a graph manifold. Hence without loss of generality, we can assume that each connected component of M^{edge} is diffeomorphic to $I \times D^2$.

Let G be a graph (i.e. 1-dimensional CW-complex) whose vertices correspond to connected components of $M^{\text{0-stratum}}_{S^2} \cup M^{\text{slim}}_{S^2}$, and whose edges correspond to connected components of M^{edge} joining such "vertex" components. From Lemmas 15.1 and 15.2, each vertex of G has degree two. Again from Lemmas 15.1 and 15.2, we can label the connected components of $M^{\text{0-stratum}}_{S^2} \cup M^{\text{slim}}_{S^2} \cup M^{\text{edge}}$ by connected components of G. It follows that each connected component of $M^{\text{0-stratum}}_{S^2} \cup M^{\text{slim}}_{S^2} \cup M^{\text{edge}}$ is diffeomorphic to $S^1 \times D^2$.

We have now shown that X_1 is the result of gluing a disjoint collection of $S^1 \times D^2$'s to $M^{\text{2-stratum}}$, with each gluing being performed between the boundary of a $S^1 \times D^2$ factor and a toral boundary component of $M^{\text{2-stratum}}$. As $M^{\text{2-stratum}}$ is the total space of a circle bundle, it is a graph manifold. Thus X_1 is a graph manifold. Hence we have shown:

Proposition 15.3. *Under the constraints imposed in the earlier sections, M is a graph manifold.*

15.2. **Satisfying the constraints.** We now verify that it is possible to simultaneously satisfy all the constraints that appeared in the construction.

We indicate a partial ordering of the parameters which is respected by all the constraints appearing in the paper. This means that every constraint on a given parameter is an upper (or lower) bound given as a function of other parameters which are strictly smaller in the partial order. Consequently, all constraints can be satisfied simultaneously, since we may choose values for parameters starting with those parameters which are minimal with respect to the partial order, and proceeding upward.

$$\begin{aligned}(15.4)\quad & \{\mathcal{M}, \beta_3\} \prec \{c_{\mathrm{slim}}, \Omega_i, \Omega_i'\} \prec \Gamma_3 \prec \{\Sigma_3, \Xi_3\} \prec c_{\mathrm{edge}} \prec \Gamma_2 \prec \{\Sigma_2, \Xi_2\} \prec c_{\text{2-stratum}} \prec \\ & \Gamma_1 \prec \{\Sigma_1, \Xi_1\} \prec \varsigma_{\text{2-stratum}} \prec \beta_2 \prec \Delta \prec \{\varsigma_{\mathrm{edge}}, \varsigma_{E'}, \varsigma_{\mathrm{slim}}\} \prec \varsigma_{\text{0-stratum}} \prec \{\beta_{E'}, \sigma_{E'}\} \prec \\ & \sigma_E \prec \{\sigma, \Lambda\} \prec \bar{w} \prec w' \prec \beta_E \prec \beta_1 \prec \{\Upsilon_0, \delta_0\} \prec \Upsilon_0'.\end{aligned}$$

This proves Theorem 1.3.

16. Manifolds with boundary

In this section we consider manifolds with boundary. Since our principal application is to the geometrization conjecture, we will only deal with manifolds whose boundary components have a nearly cuspidal collar. We recall that a *hyperbolic cusp* is a complete manifold with boundary diffeomorphic to $T^2 \times [0, \infty)$, which is isometric to the quotient of a horoball by an isometric $\mathbb{Z}^2$-action. More explicitly, a cusp is isometric to a quotient of the upper half space $\mathbb{R}^2 \times [0, \infty) \subset \mathbb{R}^3$, with the metric $dz^2 + e^{-z}(dx^2 + dy^2)$, by a rank-2 group of horizontal translations. (For application to the geometrization conjecture, we take the cusp to have constant sectional curvature $-\frac{1}{4}$).

Theorem 16.1. *Let $K \geq 10$ be a fixed integer. Fix a function $A : (0, \infty) \to (0, \infty)$. Then there is some $w_0 \in (0, c_3)$ such that the following holds.*

Suppose that (M, g) is a compact connected orientable Riemannian 3-manifold with boundary. Assume in addition that

1. *The diameters of the connected components of ∂M are bounded above by w_0.*
2. *For each component X of ∂M, there is a hyperbolic cusp $\mathcal{H}_X$ with boundary $\partial \mathcal{H}_X$, along with a C^{K+1}-embedding of pairs $e : (N_{100}(\partial \mathcal{H}_X), \partial \mathcal{H}_X) \to (M, X)$ which is w_0-close to an isometry.*
3. *For every $p \in M$ with $d(p, \partial M) \geq 10$, we have $\operatorname{vol}(B(p, R_p)) \leq w_0 R_p^3$.*
4. *For every $p \in M$, $w' \in [w_0, c_3)$, $k \in [0, K]$, and $r \leq R_p$ such that $\operatorname{vol}(B(p, r)) \geq w' r^3$, the inequality*

$$|\nabla^k \operatorname{Rm}| \leq A(w')\, r^{-(k+2)} \tag{16.2}$$

   holds in the ball $B(p, r)$.

Then M is a graph manifold.

In order to prove Theorem 16.1, we make the following assumption.

Standing Assumption 16.3. *Let $K \geq 10$ be a fixed integer and let $A' : (0,\infty) \times (0,\infty) \to (0,\infty)$ be a function.*

We assume that $\{(M^\alpha, g^\alpha)\}_{\alpha=1}^\infty$ is a sequence of connected closed Riemannian 3-manifolds such that

1. *The diameters of the connected components of ∂M^α are bounded above by $\frac{1}{\alpha}$.*
2. *For each component X^α of ∂M^α, there is a hyperbolic cusp $\mathcal{H}_{X^\alpha}$ along with a C^{K+1}-embedding of pairs $e : (N_{100}(\partial \mathcal{H}_{X^\alpha}), \partial \mathcal{H}_{X^\alpha}) \to (M^\alpha, X^\alpha)$ which is $\frac{1}{\alpha}$-close to an isometry.*
3. *For all $p \in M^\alpha$ with $d(p, \partial M^\alpha) \geq 10$, the ratio $\frac{R_p}{r_p(1/\alpha)}$ of the curvature scale at p to the $\frac{1}{\alpha}$-volume scale at p is bounded below by α.*
4. *For all $p \in M^\alpha$ and $w' \in [\frac{1}{\alpha}, c_3)$, let $r_p(w')$ denote the w'-volume scale at p. Then for each integer $k \in [0, K]$ and each $C \in (0, \alpha)$, we have $|\nabla^k \operatorname{Rm}| \leq A'(C, w') r_p(w')^{-(k+2)}$ on $B(p, C r_p(w'))$.*
5. *Each M^α fails to be a graph manifold.*

As in Lemma 5.1, to prove Theorem 16.1 it suffices to get a contradiction from Standing Assumption 16.3. As before, we let M denote the manifold M^α for large α. The argument to get a contradiction from Standing Assumption 16.3 is a slight modification of the argument in the closed case, the main difference being the appearance of a new family of points – those lying in a collared region near the boundary.

We will use the same set of the parameters as in the case of closed manifolds, with an additional parameter r_∂. It will be placed at the end of the partial ordering in (15.4), after Υ_0'.

Let $\{\partial_i M\}_{i \in I_\partial}$ be the collection of boundary components of M, and let $e_i : (N_{100}(\partial \mathcal{H}_i), \partial \mathcal{H}_i) \to (M, \partial_i M)$ be the embedding from Standing Assumption 16.3. Note that the restriction of e_i to $\partial \mathcal{H}_i$ is a diffeomorphism. Put $b_i = d_{\partial \mathcal{H}_i} \in C^\infty(\mathcal{H}_i)$. Let η_i be a slight smoothing of $b_i \circ e_i^{-1}$ on $(b_i \circ e_i^{-1})^{-1}(1, 99)$, as in Lemma 3.14.

Lemma 16.4. *We may assume that for all $p \in \eta_i^{-1}(5, 95)$,*

1. *The curvature scale satisfies $R_p \in (1, 3)$.*
2. *$\mathfrak{r}_p < r_\partial$.*
3. *There is a $(1, \beta_1)$-splitting of $\left(\frac{1}{\mathfrak{r}_p} M, p\right)$ for which $\frac{1}{\mathfrak{r}_p}\eta_i$ is an adapted coordinate of quality ζ_{slim}.*

Proof. In view of the quality of the embedding e_i, it suffices to check the claim on the constant-curvature space $b_i^{-1}(4, 96)$. The diameter of $\partial \mathcal{H}_i$ can be assumed to be arbitrarily small by taking α to be large enough. The Riemannian metric on $\mathcal{H}_i$ has the form $dz^2 + e^{-z} g_{T^2}$ for a flat metric g_{T^2} on T^2, with $z \in [0, 100)$. The lemma follows from elementary estimates. ∎

We now select 2-stratum balls, edge balls, slim 1-stratum balls and 0-balls as in the closed case, except with the restriction that the center points p_i all satisfy $d(p_i, \partial M) \geq 10$.

Given $i \in I_\partial$, let B_i be the connected component of $M - \eta_i^{-1}(90)$ containing $\partial_i M$.

Lemma 16.5. *If $r_\partial < \overline{r}_\partial(\Upsilon'_0)$ then M is diffeomorphic to $I \times T^2$ or $\{B_i\}_{i \in I_\partial} \cup \{B(p_i, r^0_{p_i})\}_{i \in I_{\text{0-stratum}}}$ is a disjoint collection of open sets.*

Proof. We can assume that M is not diffeomorphic to $I \times T^2$.

Suppose first that $B_i \cap B_j \neq \emptyset$ for some $i, j \in I_\partial$ with $i \neq j$. Then $\eta_i^{-1}(5, 90)$ must intersect $\eta_j^{-1}(5, 90)$. It follows easily that $M = N_{10}(B_i) \cup N_{10}(B_j)$ is diffeomorphic to $I \times T^2$, which is a contradiction. Thus $B_i \cap B_j = \emptyset$.

Next, suppose that $B_i \cap B(p_j, r^0_{p_j}) \neq \emptyset$ for some $i \in I_\partial$ and $j \in I_{\text{0-stratum}}$. If $r_\partial < \overline{r}_\partial(\Upsilon'_0)$ then by Lemma 16.4 we will have $\Upsilon'_0 \mathfrak{r}_{p_j} < \frac{1}{100}$. Hence $r^0_{p_j} < \frac{1}{100}$ and the triangle inequality implies that $p_j \in \eta_i^{-1}(5, 95)$. However, from Lemma 16.4(3), this contradicts the fact that p_j is a 0-stratum point.

Finally, if $i, j \in I_{\text{0-stratum}}$ and $i \neq j$ then $B(p_i, r^0_{p_i}) \cap B(p_j, r^0_{p_j}) = \emptyset$ from Lemma 11.5(1). $\square$

Hereafter we assume that M is not diffeomorphic to $I \times T^2$, which is already a graph manifold.

For each $i \in I_\partial$, let H_i be a copy of $\mathbb{R}^2$. Put $H_\partial = \bigoplus_{i \in I_\partial} H_i$. We also put

- $Q_1 = H \bigoplus H_\partial$,
- $Q_2 = H_{\text{0-stratum}} \bigoplus H_{\text{slim}} \bigoplus H_{\text{edge}} \bigoplus H_\partial$,
- $Q_3 = H_{\text{0-stratum}} \bigoplus H_{\text{slim}} \bigoplus H_\partial$,
- $Q_4 = H_{\text{0-stratum}} \bigoplus H_\partial$.

For $i \in I_\partial$, let $\zeta_i \in C^\infty(M)$ be the extension by zero of $\Phi_{20,30,80,90} \circ \eta_i$ to M. Define $\mathcal{E}^0_i : M \to H_i$ by $\mathcal{E}^0_i(p) = (\eta_i(p)\zeta_i(p), \zeta_i(p))$. We now go through Sections 12 and 13, treating H_∂ in parallel to $H_{\text{0-stratum}}$. Next, in analogy to (14.3), for each $i \in I_\partial$ we put

$$M^\partial_i = N_{35}(\partial_i M) \cup \mathcal{E}^{-1} \left\{ x \in H \ : \ x''_i \geq .9 \, , \ \frac{x'_i}{x''_i} \leq 40 \right\} . \tag{16.6}$$

Then M^∂_i is diffeomorphic to $I \times T^2$. We now go through the argument of Section 15, treating each M^∂_i as if it were an element of $M^{\text{0-stratum}}_{T^2}$ without a core. As in Section 15, we conclude that M is a graph manifold. This proves Theorem 16.1.

17. Application to the geometrization conjecture

We now use the terminology of [KL08] and [Per]. Let $(M, g(\cdot))$ be a Ricci flow with surgery whose initial 3-manifold is compact. We normalize the metric by putting $\widehat{g}(t) = \frac{g(t)}{t}$. Let $(M_t, \widehat{g}(t))$ be the time-t manifold. (If t is a surgery time then we take M_t to be the post-surgery manifold.) We recall that the w-thin part $M^-(w, t)$ of M_t is defined to be the set of points $p \in M_t$ so that either $R_p = \infty$ or $\mathrm{vol}(B(p, R_p)) < w R_p^3$. The w-thick part $M^+(w, t)$ of M_t is $M_t - M^-(w, t)$.

The following theorem is proved in [Per, Section 7.3]; see also [KL08, Proposition 90.1].

Theorem 17.1. [Per] *There is a finite collection* $\{(H_i, x_i)\}_{i=1}^k$ *of pointed complete finite-volume Riemannian* 3*-manifolds with constant sectional curvature* $-\frac{1}{4}$ *and, for large* t*, a decreasing function* $\beta(t)$ *tending to zero and a family of maps*

$$f_t : \coprod_{i=1}^k H_i \supset \coprod_{i=1}^k B\left(x_i, \frac{1}{\beta(t)}\right) \to M_t \tag{17.2}$$

such that

1. f_t *is* $\beta(t)$*-close to being an isometry.*
2. *The image of* f_t *contains* $M^+(\beta(t), t)$*.*
3. *The image under* f_t *of a cuspidal torus of* $\{H_i\}_{i=1}^k$ *is incompressible in* M_t*.*

Given a sequence $t^\alpha \to \infty$, let Y^α be the truncation of $\coprod_{i=1}^k H_i$ obtained by removing horoballs at distance approximately $\frac{1}{2\beta(t^\alpha)}$ from the basepoints x_i. Put $M^\alpha = M_{t^\alpha} - f_{t^\alpha}(Y_{t^\alpha})$.

Theorem 17.3. [Per] *For large* α, M^α *is a graph manifold.*

Proof. We check that the hypotheses of Theorem 16.1 are satisfied for large α. Conditions (1) and (2) of Theorem 16.1 follow from the almost-isometric embedding of $\coprod_{i=1}^k \left(B(x_i, \frac{1}{\beta(t^\alpha)}) - B(x_i, \frac{1}{2\beta(t^\alpha)})\right) \subset \coprod_{i=1}^k H_i$ in M^α.

Next, Theorem 17.1 says that for any $\bar{w} > 0$, for large α the $\bar{w}$-thick part of M_{t^α} has already been removed in forming M^α. Thus Condition (3) of Theorem 16.1 holds.

From Ricci flow arguments, for each $w' \in (0, c_3)$ there are $\overline{r}(w') > 0$ and $K_k(w') < \infty$ so that for large α the following holds: for every $p \in M^\alpha$, $w' \in (0, c_3)$, $k \in [0, K]$ and $r \leq \min(R_p, \overline{r}(w'))$, the inequality $|\nabla^k \operatorname{Rm}| \leq K_k(w')\, r^{-(k+2)}$ holds in the ball $B(p, r)$ [KL08, Lemma 92.13]. Hence to verify Condition (4) of Theorem 16.1, at least for large α, we must show that if $p \in M^\alpha$ then the conditions $r \leq R_p$ and $\operatorname{vol}(B(p,r)) \geq w' r^3$ imply that $r \leq \overline{r}(w')$.

Suppose not, i.e. we have $\overline{r}(w') < r \leq R_p$. Then $\operatorname{Rm}\Big|_{B(p,r)} \geq -\frac{1}{r^2}$. Using the fact that $\operatorname{vol}(B(p,r)) \geq w' r^3$, the Bishop-Gromov inequality gives an inequality of the form $\operatorname{vol}(B(p, \overline{r}(w'))) \geq w'' \overline{r}^3(w')$ for some $w'' = w''(w') > 0$.

We also have $\operatorname{Rm}\Big|_{B(p,\overline{r}(w'))} \geq -\frac{1}{\overline{r}^2(w')}$. Then from [Per91, Lemma 7.2] or [KL08, Lemma 88.1], for large α we can assume that the sectional curvatures on $B(p, \overline{r}(w'))$ are arbitrarily close to $-\frac{1}{4}$. In particular, $R_p \leq 5$. Then

$$\operatorname{vol}(B(p, R_p)) \geq \operatorname{vol}(B(p,r)) \geq w' r^3 = w' \left(\frac{r}{R_p}\right)^3 R_p^3 \geq w' \left(\frac{\overline{r}(w')}{5}\right)^3 R_p^3. \tag{17.4}$$

If α is sufficiently large then we conclude that $p \in f_{t^\alpha}(Y_{t^\alpha})$, which is a contradiction.

We now take $A(w')$ to be a number so that Condition (4) of Theorem 16.1 holds for all M^α. From the preceding discussion, there is a finite such number. Then for large α, all of the hypotheses of Theorem 16.1 hold. The theorem follows. ∎

Theorems 17.1 and 17.3, along with the description of how M_t changes under surgery [Per, Section 3],[KL08, Lemma 73.4], imply Thurston's geometrization conjecture.

18. Local collapsing without derivative bounds

In this section, we explain how one can remove the bounds on derivatives of curvature from the hypotheses of Theorem 1.3, to obtain:

Theorem 18.1. *There exists a $w_0 \in (0, c_3)$ such that if M is a closed, orientable, Riemannian 3-manifold satisfying*

$$\mathrm{vol}(B(p, R_p)) < w_0 R_p^3 \tag{18.2}$$

for every $p \in M$, then M is a graph manifold.

The bounds on the derivatives of curvature are only used to obtain pointed C^K-limits of sequences at the (modified) volume scale. This occurs in Lemmas 9.21 and 11.1. We explain how to adapt the statements and proofs.

Modifications in Lemma 9.21. The statement of the Lemma does not require modification. In the proof, the map ϕ will be a Gromov-Hausdorff approximation rather than a C^{K+1}-map close to an isometry, and Z will be a complete 2-dimensional nonnegatively curved Alexandrov space. As critical point theory for functions works the same way for Alexandrov spaces as for Riemannian manifolds, and 2-dimensional Alexandrov spaces are topological manifolds, the statement and proof of Lemma 3.12 remain valid for 2-dimensional Alexandrov spaces. The main difference in the proof of Lemma 9.21 is the method for verifying the fiber topology. For this, we use:

Theorem 18.3 (Linear local contractibility [GP88])**.** *For every $w \in (0, \infty)$ and every positive integer n, there exist $r_0 \in (0, \infty)$ and $C \in (1, \infty)$ with the following property. If $B(p, 1)$ is a unit ball with compact closure in a Riemannian n-manifold,* $\mathrm{Rm}\Big|_{B(p,1)} \geq -1$ *and $\mathrm{vol}(B(p, 1)) \geq w$ then the inclusion $B(p, r) \to B(p, Cr)$ is null-homotopic for every $r \in (0, r_0)$.*

This uniform contractibility may be used to promote a Gromov-Hausdorff approximation f_0 to a nearby continuous map f: one first restricts f_0 to the 0-skeleton of a fine triangulation, and then extends it inductively to higher skeleta simplex by simplex, using the controlled contractibility radius.

Lemma 18.4. *With notation from the proof of Lemma 9.21, the fiber $F = \eta_p^{-1}(\{0\})$ is homotopy equivalent to $B(\star_Z, 4\Delta) \subset Z$.*

Proof. Let $\widehat{\phi}$ be a quasi-inverse to the Gromov-Hausdorff approximation ϕ.

To produce a map $F \to B(\star_Z, 4\Delta)$ we take $\pi_Z \circ \phi\Big|_F$, promote it to a continuous map as above, and then use the absence of critical points of d_Y near $S(\star_Z, 4\Delta)$ to homotope this to a map taking values in $B(\star_Z, 4\Delta)$.

To get the map $B(\star_Z, 4\Delta) \to F$, we apply the above procedure to promote $\widehat{\phi}\Big|_F$ to a nearby continuous map $B(\star_Z, 4\Delta) \to \frac{1}{\mathfrak{r}_p}M$. Then using the fibration structures defined by η_p and $(\eta_p, \eta_{E'})$, we may perturb this to a map taking values in F.

The compositions of these maps are close to the identity maps; using a relative version of the approximation procedure one shows that these are homotopic to identity maps. □

Thus we conclude that the fiber is a contractible compact 2-manifold with boundary, so it is a 2-disk.

Modifications in Lemma 11.1. In the statement of the lemma, N_p is a 3-dimensional nonnegatively curved Alexandrov space instead of a nonnegatively curved Riemannian manifold, and "diffeomorphism" is replaced by "homeomorphism".

In the proof, the pointed C^K-convergence is replaced by pointed Gromov-Hausdorff convergence to a 3-dimensional nonnegatively curved Alexandrov space N; otherwise, we retain the notation from the proof. We need:

Theorem 18.5 (The Stability Theorem [Per91, Kap07])**.** *Suppose $\{(M_k, \star_k)\}$ is a sequence of Riemannian n-manifolds, such that the sectional curvature is bounded below by a (k-independent) function of the distance to the basepoint $\star_k$. Let X be an n-dimensional Alexandrov space with curvature bounded below, and assume that $\phi_k : (X, \star_\infty) \to (M_k, \star_k)$ is a δ_k-pointed Gromov Hausdorff approximation, where $\delta_k \to 0$. Then for every $R \in (0, \infty)$, $\epsilon \in (0, \infty)$, and every sufficiently large k, there is a pointed map $\psi_k : (B(\star_\infty, R + \epsilon), \star_\infty) \to (M_k, \star_k)$ which is a homeomorphism onto an open subset containing $B(\star_k, R)$, where $d_{C^0}\left(\psi_k, \phi_k\Big|_{B(\star_\infty, R+\epsilon)}\right) < \epsilon$.*

Using critical point theory as before, we get that the limiting Alexandrov space N is homeomorphic to the balls $B(p_\infty, R'')$ for $R'' \in (\frac{1}{2}R', 2R')$, and there are no critical points for d_{p_∞} or d_{p_j} in the respective annuli $A(p_\infty, \frac{R''}{10^3}, 10R'') \subset N$ and $A(p_j, \frac{R''}{10^3}, 10R'') \subset \frac{1}{\mathfrak{r}_{p_j}} M_j$. The Stability Theorem produces a homeomorphism ψ from the closed ball $\overline{B(p_\infty, R'')} \subset M$ to a subset close to the ball $B(p_j, R'') \subset \frac{1}{\mathfrak{r}_{p_j}} M_j$; in particular, restricting ψ to the sphere $S(p_\infty, R'')$ we obtain a Gromov-Hausdorff approximation from the surface $S(p_\infty, R'')$ to the surface $S(p_j, R'')$. Appealing to uniform contractibility (Theorem 18.3), and using homotopies guaranteed by the absence of critical points we get that $\psi\Big|_{S(p_\infty, R'')}$ is close to a homotopy equivalence. As in the proof of Theorem 11.3, we conclude that $\psi(\overline{B(p_\infty, R'')})$ is isotopic to $\overline{B(p_j, R'')}$.

Finally, we appeal to the classification of complete, noncompact, orientable, nonnegatively curved Alexandrov spaces N, when N is a noncompact topological 3-manifold, from [SY00] to conclude that the list of possible topological types is the same as in the smooth case.

Theorem 18.6 (Shioya-Yamaguchi [SY00])**.** *If X is a noncompact, orientable, 3-dimensional nonnegatively curved Alexandrov space which is a topological manifold, then X is homeomorphic to one of the following: $\mathbb{R}^3$, $S^1 \times \mathbb{R}^2$, $S^2 \times \mathbb{R}$, $T^2 \times \mathbb{R}$, or a twisted line bundle over $\mathbb{R}P^2$ or the Klein bottle.*

When N is compact, we may apply the main theorem of [Sim09] to see that the topological classification is the same as in the smooth case. Alternatively, using the splitting theorem, one may reduce to the case when N has finite fundamental group and use the elliptization conjecture (now a theorem via Ricci flow due to finite extinction time results).

Remark 18.7. Theorem 18.1 implies the collapsing result stated in the appendix of [SY05]. Note that Theorem 18.1 is strictly stronger, since the curvature scale need not be small compared to the diameter. However, we remark that the argument of [SY05] also gives the stronger result, if one uses [Sim09] or the elliptization conjecture as above.

Appendix A. Choosing ball covers

Let M be a complete Riemannian manifold and let V be a bounded subset of M. Given $p \in V$ and $r > 0$, we write $B(p,r)$ for the metric ball in M around p of radius r. Let $\mathcal{R} : V \to \mathbb{R}$ be a (not necessarily continuous) function with range in some compact positive interval. For $p \in V$, we denote $\mathcal{R}(p)$ by $\mathcal{R}_p$. Put $S_1 = V$, $\rho_1 = \sup_{p \in V} \mathcal{R}_p$ and $\rho_\infty = \inf_{p \in V} \mathcal{R}_p$. Choose a point $p_1 \in V$ so that $\mathcal{R}_{p_1} \geq \frac{1}{2}\rho_1$. Inductively, for $i \geq 1$, let S_{i+1} be the subset of V consisting of points p such that $B(p, \mathcal{R}_p)$ is disjoint from $B(p_1, \mathcal{R}_{p_1}) \cup \ldots \cup B(p_i, \mathcal{R}_{p_i})$. If $S_{i+1} = \emptyset$ then stop. If $S_{i+1} \neq \emptyset$, put $\rho_{i+1} = \sup_{p \in S_{i+1}} \mathcal{R}_p$ and choose a point $p_{i+1} \in S_{i+1}$ so that $\mathcal{R}_{p_{i+1}} \geq \frac{1}{2}\rho_{i+1}$. This process must terminate after a finite number of steps, as the ρ_1-neighborhood of V cannot contain an infinite number of disjoint balls with radius at least $\frac{1}{2}\rho_\infty$.

Lemma A.1. *$\{B(p_i, \mathcal{R}_{p_i})\}$ is a finite disjoint collection of balls such that $V \subset \bigcup_i B(p_i, 3\mathcal{R}_{p_i})$. Furthermore, given $q \in V$, there is some N so that $q \in B(p_N, 3\mathcal{R}_{p_N})$ and $\mathcal{R}_q \leq 2\mathcal{R}_{p_N}$.*

Proof. Given $q \in V$, we know that $B(q, \mathcal{R}_q)$ intersects $\bigcup_i B(p_i, \mathcal{R}_{p_i})$. Let N be the smallest number i such that $B(q, \mathcal{R}_q)$ intersects $B(p_i, \mathcal{R}_{p_i})$. Then $q \in S_N$ and so $\rho_N \geq \mathcal{R}_q$. Thus $\mathcal{R}_{p_N} \geq \frac{1}{2}\rho_N \geq \frac{1}{2}\mathcal{R}_q$. As $B(q, \mathcal{R}_q)$ intersects $B(p_N, \mathcal{R}_{p_N})$, we have $d(q, p_N) < \mathcal{R}_q + \mathcal{R}_{p_N} \leq 3\mathcal{R}_{p_N}$. □

Appendix B. Cloudy submanifolds

In this section we define the notion of a cloudy k-manifold. This is a subset of a Euclidean space with the property that near each point, it looks coarsely close to an affine subspace of the Euclidean space. The result of this appendix is that any cloudy k-manifold can be well interpolated by a smooth k-dimensional submanifold of the Euclidean space.

If H is a Euclidean space, let $\mathrm{Gr}(k, H)$ denote the Grassmannian of codimension-k subspaces of H. It is metrized by saying that for $P_1, P_2 \in \mathrm{Gr}(k, H)$, if $\pi_1, \pi_2 \in \mathrm{End}(H)$ are orthogonal projection onto P_1 and P_2, respectively, then $d(P_1, P_2)$ is the operator norm of $\pi_1 - \pi_2$. If H' is another Euclidean space then there is an isometric embedding $\mathrm{Gr}(k, H) \to \mathrm{Gr}(k, H \oplus H')$. If X is a k-dimensional submanifold of H then the normal map of X is the map $X \to \mathrm{Gr}(k, H)$ which assigns to $p \in X$ the normal space of X at p.

Definition B.1. Suppose $C, \delta \in (0, \infty)$, $k \in \mathbb{N}$, and H is a Euclidean space. A (C, δ) *cloudy k-manifold in H* is a triple $(\widetilde{S}, S, r)$, where $S \subset \widetilde{S} \subset H$ is a pair of subsets, and $r : \widetilde{S} \to (0, \infty)$ is a (possibly discontinuous) function such that:

(1) For all $x, y \in \widetilde{S}$, $|r(y) - r(x)| \leq C(|x-y| + r(x))$.
(2) For all $x \in S$, the rescaled pointed subset $(\frac{1}{r(x)}\widetilde{S}, x)$ is δ-close in the pointed Hausdorff distance to $(\frac{1}{r(x)}A_x, x)$, where A_x is a k-dimensional affine subspace of H. Here, as usual, $\frac{1}{r(x)}\widetilde{S}$ means the subset $\widetilde{S}$ equipped with the distance function of H rescaled by $\frac{1}{r(x)}$.

We will sometimes say informally that a pair $(\widetilde{S}, S)$ is a *cloudy k-manifold* if it can be completed to a triple $(\widetilde{S}, S, r)$ which is a (C,δ) cloudy k-manifold for some (C,δ). We will write $A_x^0 \subset H$ for the k-dimensional linear subspace parallel to A_x and we will write $\pi_{A_x^0}$ for orthogonal projection onto A_x^0. Let $P_{A_x} : H \to H$ be the nearest point projection to A_x, given by $P_{A_x}(y) = x + \pi_{A_x^0}(y-x)$

Lemma B.2. *For all $k, K \in \mathbb{Z}^+$, $\epsilon \in (0,\infty)$ and $C < \infty$, there is a $\delta = \delta(k,K,\epsilon,C) > 0$ with the following property. Suppose $(\widetilde{S}, S, r)$ is a (C,δ) cloudy k-manifold in a Euclidean space H, and for every $x \in S$ we denote by A_x an affine subspace as in Definition B.1. Then there is a k-dimensional smooth submanifold $W \subset H$ such that*

(1) *For all $x \in S$, the pointed Hausdorff distance from $(\frac{1}{r(x)}\widetilde{S}, x)$ to $(\frac{1}{r(x)}W, x)$ is at most ϵ.*
(2) $W \subset N_{\epsilon r}(\widetilde{S})$.
(3) *For all $x \in S$, the restriction of the normal map of W to $B(x, r(x)) \cap W$ has image contained in an ϵ-ball of $A_x^\perp$ in $\mathrm{Gr}(k, H)$.*
(4) *If I is a multi-index with $|I| \leq K$ then the I^{th} covariant derivative of the second fundamental form of W at w is bounded in norm by $\epsilon\, r(x)^{-(|I|+1)}$.*
(5) *$W \cap N_r(S)$ is properly embedded in $N_r(S)$.*
(6) *The nearest point map $P : N_r(S) \to W$ is a well-defined smooth submersion.*
(7) *If I is a multi-index with $1 \leq |I| \leq K$ then for all $x \in S$, the restriction of $P - P_{A_x}$ to $B(x, r(x))$ has I^{th} derivative bounded in norm by $\epsilon\, r(x)^{-(|I|-1)}$.*

Proof. With the notation of Lemma A.1, put $V = S$ and $\mathcal{R} = r$. Let T be the finite collection of points $\{p_i\}$ from the conclusion of Lemma A.1. Then $\{B(\widehat{x}, r_{\widehat{x}})\}_{\widehat{x}\in T}$ is a disjoint collection of balls such that for any $x \in S$, there is some $\widehat{x} \in T$ with $x \in B(\widehat{x}, 3r(\widehat{x}))$ and $r(x) \leq 2r(\widehat{x})$. Hence $B(x, r(x)) \subset B(\widehat{x}, r(x) + 3r(\widehat{x})) \subset B(\widehat{x}, 5r(\widehat{x}))$. This shows that $\bigcup_{x\in S} B(x, r(x)) \subset \bigcup_{\widehat{x}\in T} B(\widehat{x}, 5r(\widehat{x}))$.

For each $\widehat{x} \in T$, let $A_{\widehat{x}} \subset H$ be the k-dimensional affine subspace from Definition B.1, so that $\left(\frac{1}{r(\widehat{x})}\widetilde{S}, \widehat{x}\right)$ is δ-close in the pointed Hausdorff topology to $(\frac{1}{r(\widehat{x})}A_{\widehat{x}}, \widehat{x})$. Here δ is a parameter which will eventually be made small enough so the proof works. Let $A_{\widehat{x}}^0 \subset H$ be the k-dimensional linear subspace which is parallel to $A_{\widehat{x}}$. Let $p_{\widehat{x}} : H \to H$ be orthogonal projection onto the orthogonal complement of $A_{\widehat{x}}^0$.

In view of the assumptions of the lemma, a packing argument shows that for any $l < \infty$, for sufficiently small δ there is a number $m = m(k, C, l)$ so that for all $\widehat{x} \in T$, there are at most m elements of T in $B(\widehat{x}, lr(\widehat{x}))$. Fix a nonnegative function $\phi \in C^\infty(\mathbb{R})$ which is identically one on $[0,1]$ and vanishes on $[2,\infty)$. For $\widehat{x} \in T$, define $\phi_{\widehat{x}} : H \to \mathbb{R}$ by

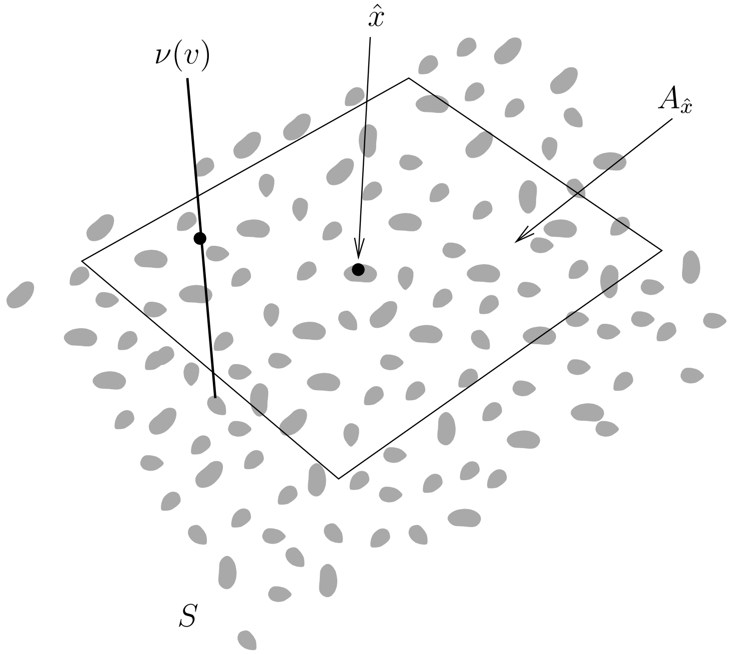

FIGURE 5

$\phi_{\widehat{x}}(v) = \phi\left(\frac{|v-\widehat{x}|}{10r(\widehat{x})}\right)$. Let $E(k,H)$ be the set of pairs consisting of a codimension-k plane in H and a point in that plane. That is, $E(k,H)$ is the total space of the universal bundle over $\mathrm{Gr}(k,H)$. Given $v \in \bigcup_{\widehat{x}\in T} B(\widehat{x}, 5r(\widehat{x}))$, put $O_v = \frac{\sum_{\widehat{x}\in T} \phi_{\widehat{x}}(v)\, p_{\widehat{x}}}{\sum_{\widehat{x}\in T} \phi_{\widehat{x}}(v)}$. Note that for small δ, there is a uniform upper bound on the number of nonzero terms in the summation, in terms of k and C; hence the rank of O_v is also bounded in terms of k and C.

If δ is sufficiently small then since the projection operators $p_{\widehat{x}}$ that occur with a nonzero coefficient in the summation are uniformly norm-close to each other, the self-adjoint operator O_v will have k eigenvalues near 0, with the rest of the spectrum being near 1. Let $\nu(v)$ be the orthogonal complement of the span of the eigenvectors corresponding to the k eigenvalues of O_v near 0. Let Q_v be orthogonal projection onto $\nu(v)$.

Recall that $\bigcup_{x\in S} B(x, r(x)) \subset \bigcup_{\widehat{x}\in T} B(\widehat{x}, 5r(\widehat{x}))$. Define $\eta : \bigcup_{\widehat{x}\in T} B(\widehat{x}, 5r(\widehat{x})) \to H$ by

$$\eta(v) = \frac{\sum_{\widehat{x}\in T} \phi_{\widehat{x}}(v)\, Q_v(v-\widehat{x})}{\sum_{\widehat{x}\in T} \phi_{\widehat{x}}(v)}. \tag{B.3}$$

Define $\pi : \bigcup_{x\in S} B(x, r(x)) \to E(k,H)$ by $\pi(v) = (\nu(v), \eta(v))$.

If δ is sufficiently small then π is uniformly transverse to the zero-section of $E(k,H)$. Hence the inverse image under π of the zero section will be a k-dimensional submanifold W. The map P is defined as in the statement of the lemma.

The conclusions of the lemma follow from a convergence argument. For example, for conclusion (3), suppose that there is a sequence $\delta_j \to 0$ and a collection of counterexamples to conclusion (3). Let $x_j \in S_j$ be the relevant point. In view of the multiplicity bounds, we can assume without loss of generality that the dimension of the Euclidean space is uniformly bounded above. Hence after passing to a subsequence, we can pass to the case when $\dim(H_j)$ is constant in j. Then $\lim_{j\to\infty}(\frac{1}{r(x_j)}S_j, x_j)$ exists in the pointed Hausdorff topology and is a k-dimensional plane (S_∞, x_∞). The map ν_∞ is a constant map and η is an orthogonal projection. Then W_∞ is a flat k-dimensional manifold, which gives a contradiction. The verifications of the other conclusions of the lemma are similar. □

Appendix C. An isotopy lemma

Lemma C.1. *Suppose that $F : Y \times [0,1] \to N$ is a smooth map between manifolds, with slices $\{f^t : Y \to N\}_{t\in[0,1]}$, and let $X \subset N$ be a submanifold with boundary ∂X. If*

- *f^t is transverse to both X and ∂X for every $t \in [0,1]$ and*
- *$F^{-1}(X)$ is compact*

then $(f^0)^{-1}(X)$ is isotopic in Y to $(f^1)^{-1}(X)$.

Proof. Suppose first that $\partial X = \emptyset$. Now $F(y,t) = f^t(y)$. For $v \in T_yY$ and $c \in \mathbb{R}$, we can write $DF(v + c\frac{\partial}{\partial t}) = Df^t(v) + c\frac{\partial f^t}{\partial t}(y)$. We know that if $F(y,t) = x \in X$ then $T_{(y,t)}(F^{-1}(X)) = (DF_{y,t})^{-1}(T_xX)$.

By assumption, for each $t \in [0,1]$, if $f_t(y) = x \in X$ then we have

$$\mathrm{Im}(Df_t)_y + T_xX = T_xN. \tag{C.2}$$

We want to show that projection onto the $[0,1]$-factor gives a submersion $F^{-1}(X) \to [0,1]$. Suppose not. Then for some $(y,t) \in F^{-1}(X)$, we have $T_{(y,t)}F^{-1}(X) \subset T_yY$. That is, putting $F(y,t) = x$, whenever $v \in T_yY$ and $c \in \mathbb{R}$ satisfy $Df^t(v) + c\frac{\partial f^t}{\partial t}(y) \in T_xX$ then we must have $c = 0$. However, for any $c \in \mathbb{R}$, equation (C.2) implies that we can solve $Df_t(-v) + w = c\frac{\partial f^t}{\partial t}(y)$ for some $v \in T_yY$ and $w \in T_xX$. This is a contradiction.

Thus we have a submersion from the compact set $F^{-1}(X)$ to $[0,1]$. This submersion must have a product structure, from which the lemma follows.

The case when $\partial X \neq \emptyset$ is similar. □

Lemma C.3. *Suppose that Y is a smooth manifold, $(X, \partial X) \subset \mathbb{R}^k$ is a smooth submanifold, $f : Y \to \mathbb{R}^k$ is transverse to both X and ∂X, and $\widehat{X} = f^{-1}(X)$ is compact. Then for any compact subset $Y' \subset Y$ whose interior contains $\widehat{X}$, there is an $\epsilon > 0$ such that if $f' : Y \to \mathbb{R}^k$ and $\|f' - f\|_{C^1(Y')} < \epsilon$ then $f'^{-1}(X)$ is isotopic to $f^{-1}(X)$.*

Proof. This follows from Lemma C.1. □

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