$\widehat{A}$-GENUS AND COLLAPSING

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Abstract. If M is a compact spin manifold, we give relationships between the vanishing of $\widehat{A}(M)$ and the possibility that M can collapse with curvature bounded below.

1. Introduction

The purpose of this paper is to extend the following simple lemma.

Lemma 1. *If M is a connected closed Riemannian spin manifold of nonnegative sectional curvature with $\dim(M) > 0$ then $\widehat{A}(M) = 0$.*

Proof. Let K denote the sectional curvature of M and let R denote its scalar curvature. Suppose that $\widehat{A}(M) \neq 0$. Let D denote the Dirac operator on M. From the Atiyah-Singer index theorem, there is a nonzero spinor field ψ on M such that $D\psi = 0$. From Lichnerowicz's theorem,

$$0 = \int_M |D\psi|^2 \, d\mathrm{vol} = \int_M |\nabla\psi|^2 \, d\mathrm{vol} + \int_M \frac{R}{4} \, |\psi|^2 \, d\mathrm{vol}. \tag{1.1}$$

From our assumptions, $R \geq 0$. Hence $\nabla\psi = 0$. This implies that $|\psi|^2$ is a nonzero constant function on M and so we must also have $R = 0$. Then as $K \geq 0$, we must have $K = 0$. This implies, from the integral formula for $\widehat{A}(M)$ [14, p. 231], that $\widehat{A}(M) = 0$. $\square$

The spin condition is necessary in Lemma 1, as can be seen in the case of $M = \mathbb{C}P^{2k}$. The Ricci-analog of Lemma 1 is false, as can be seen in the case of $M = K3$.

Definition 1. *A connected closed manifold M is almost-nonnegatively-curved if for every $\epsilon > 0$, there is a Riemannian metric g on M such that $K(M,g) \cdot \mathrm{diam}(M,g)^2 \geq -\epsilon$.*

Special examples of almost-nonnegatively-curved manifolds are given by almost-flat manifolds; these all have vanishing $\widehat{A}$-genus, as can be seen by the integral formula. Along with Lemma 1, this raises the following question.

Question 1. *Given $n \in \mathbb{Z}^+$, is there an $\epsilon(n) > 0$ such that if M is a connected closed Riemannian spin manifold with $K(M,g) \cdot \mathrm{diam}(M,g)^2 \geq -\epsilon(n)$ then $\widehat{A}(M) = 0$?*

We answer Question 1 under the assumption of an upper curvature bound.

Proposition 1. *For any $n \in \mathbb{Z}^+$ and any $\Lambda > 0$, there is an $\epsilon(n,\Lambda) > 0$ such that if M is a connected closed n-dimensional Riemannian spin manifold with*

$$-\epsilon(n,\Lambda) \leq K(M,g) \cdot \mathrm{diam}(M,g)^2 \leq \Lambda \tag{1.2}$$

then $\widehat{A}(M) = 0$.

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The proof of Proposition 1 uses Gromov's convergence theorem [11, 13]. Using the results of [17], the upper bound on the sectional curvature in Proposition 1 can be replaced by a lower bound on the conjugacy radius $\mathrm{conj}(M, g)$.

An affirmative answer to Question 1 would imply that an almost-nonnegatively-curved spin manifold has vanishing $\widehat{A}$-genus. There is a fiber bundle construction to create new almost-nonnegatively-curved manifolds out of old ones. The following proposition shows that the vanishing of the $\widehat{A}$-genus is consistent with this construction.

Proposition 2. *Let N be a connected closed manifold of nonnegative sectional curvature. Let G be a compact Lie group which acts on N by isometries. Let P be a principal G-bundle with connected closed base B. Put $M = P \times_G N$.*
1. If B is almost-nonnegatively-curved then M is almost-nonnegatively-curved [8, Theorem 0.18].
2. If M is spin and $\dim(N) > 0$ *then* $\widehat{A}(M) = 0$.

Part 2. of Proposition 2 also follows easily from the multiplicativity of the $\widehat{A}$-genus for such fiber bundles [16]. We give a direct geometric proof which will be useful later. Proposition 2 covers a wide range of almost-nonnegatively-curved manifolds. It seems conceivable that every almost-nonnegatively-curved manifold has a finite cover which is the total space of a fiber bundle whose base is almost-flat and whose fiber has a metric of nonnegative sectional curvature.

Rescaling metrics, a manifold M is almost-nonnegatively-curved if for every $\epsilon > 0$, there is a Riemannian metric g on M such that $K(M, g) \geq -1$ and $\mathrm{diam}(M, g) \leq \epsilon$. That is, there is a sequence of metrics $\{g_i\}_{i=1}^{\infty}$ such that $K(M, g_i) \geq -1$ and the metric spaces $\{(M, g_i)\}_{i=1}^{\infty}$ converge in the Gromov-Hausdorff topology to a point. It is natural to extend Question 1 to a question about the $\widehat{A}$-genus of a spin manifold M with a sequence of metrics $\{g_i\}_{i=1}^{\infty}$ such that $K(M, g_i) \geq -1$ and the metric spaces $\{(M, g_i)\}_{i=1}^{\infty}$ converge in the Gromov-Hausdorff topology to some lower-dimensional length space, not necessarily a point. The following definition is convenient for our purposes.

Definition 2. *A connected manifold M collapses with curvature bounded below and diameter bounded above if there is a number $D > 0$ such that for any $\epsilon > 0$, there is a Riemannian metric g on M with $K(M, g) \geq -1$,* $\mathrm{diam}(M, g) \leq D$ *and* $\mathrm{vol}(M, g) \leq \epsilon$.

We remark that in the noncollapsing case there is a finiteness result [12]. Namely, given $D, v > 0$ and $n > 3$, there is a finite number of homeomorphism classes of connected manifolds M^n admitting a Riemannian metric g satisfying $K(M, g) \geq -1$, $\mathrm{diam}(M, g) \leq D$ and $\mathrm{vol}(M, g) \geq v$.

Question 2. *Given $n \in \mathbb{Z}^+$ and $D > 0$, is there a $v(n, D) > 0$ such that if M is a connected closed n-dimensional Riemannian spin manifold with $K(M, g) \geq -1$,* $\mathrm{diam}(M, g) \leq D$ *and* $\mathrm{vol}(M, g) \leq v(n, D)$ *then* $\widehat{A}(M) = 0$*?*

An affirmative answer to Question 2 would imply that a spin manifold which collapses with curvature bounded below and diameter bounded above has vanishing $\widehat{A}$-genus. In the next proposition we show that this is indeed the case for a large class of collapsing examples.

Proposition 3. *Let Z and N be connected closed Riemannian manifolds. Suppose that N has nonnegative sectional curvature. Let G be a compact Lie group which acts on Z and N*

by isometries. Suppose that for a generic point z in Z, the stabilizer group G_z does not act transitively on N. Suppose that the diagonal action of G on $Z \times N$ has the property that all of its orbits are principal orbits. Let M be the quotient manifold $Z \times_G N$. Then
1. M collapses with curvature bounded below and diameter bounded above. The collapsing sequence converges in the Gromov-Hausdorff topology to the length space Z/G.
2. If M is spin then $\widehat{A}(M) = 0$.

In Proposition 3, M is the total space of a possibly-singular fibration whose base is Z/G. The fiber over a coset $z'G \in Z/G$ is $G_{z'} \backslash N$. The hypotheses imply that the generic fiber has positive dimension. Some special cases of Proposition 3 are :
1. If G acts freely on Z. Then Proposition 3 is equivalent to Proposition 2.
2. If $N = G$ is a connected compact Lie group which acts nontrivially on Z. Then $M = Z \times_G G = Z$ and the second part of Proposition 3 is equivalent to the Atiyah-Hirzebruch theorem [2].

To put the results of this paper in perspective, let us mention known necessary conditions for a connected closed manifold M to be almost-nonnegatively-curved :
1. The fundamental group $\pi_1(M)$ must be virtually nilpotent [8, Theorem 0.1].
2. If $\pi_1(M)$ is infinite then the Euler characteristic of M must vanish [8, Corollary 0.12].
3. M must be dominated by a CW-complex with the number of cells bounded above by a function of $\dim(M)$ [10, 18].
4. If M is spin then $|\widehat{A}(M)| \leq 2^{\frac{dim(M)}{2}-1}$. (This is a necessary condition for M to have almost-nonnegative-Ricci curvature [9].)

I thank Peter Petersen for his interest in these questions.

2. Proof of Proposition 1

For background material on spin geometry, we refer to [14]. Before giving the proof of Proposition 1, we must discuss how to compare spinors on diffeomorphic Riemannian manifolds which are not necessarily isometric. This is an elementary point which has caused confusion in the literature.

Let M be a smooth connected closed n-dimensional oriented manifold. Let PM be a principal $Spin(n)$-bundle on M. Let S_n be the complex spinor module of $Spin(n)$. Then we can form the associated Hermitian vector bundle $S = PM \times_{Spin(n)} S_n$ on M. The corresponding spinor fields are defined to be the sections of S, or equivalently, the $Spin(n)$-equivariant maps from PM to S_n. So far we have made no reference to a Riemannian metric on M.

Let $p : FM \to M$ be the oriented frame bundle of M, a principal $GL^+(n, \mathbb{R})$-bundle on M. Given $\gamma \in GL^+(n, \mathbb{R})$, let $R_\gamma \in \mathrm{Diff}(FM)$ denote the right action of γ on FM. There is a canonical $\mathbb{R}^n$-valued 1-form θ on FM such that if $f = \{f_i\}_{i=1}^n$ is an oriented frame at $m \in M$ and $v \in T_f FM$ then $dp(v) = \sum_{i=1}^n \theta^i(v) f_i$. It has the properties that
1. If V is a vertical vector field on FM then $\theta(V) = 0$.
2. For all $\gamma \in GL^+(n, \mathbb{R})$, $R_\gamma^* \theta = \gamma^{-1} \cdot \theta$.
3. For all $f \in FM$, $\theta : T_f FM \to \mathbb{R}^n$ is onto.

Giving a Riemannian metric g on M is equivalent to giving a reduction $i : OM \to FM$ of the oriented frame bundle from a principal $GL^+(n, \mathbb{R})$-bundle to a principal $SO(n)$-bundle OM. As a topological fiber bundle, OM is unique. We obtain an $\mathbb{R}^n$-valued 1-form $\tau = i^*\theta$ on OM with the properties that
1. If V is a vertical vector field on OM then $\tau(V) = 0$.
2. For all $\gamma \in SO(n)$, $R_\gamma^* \tau = \gamma^{-1} \cdot \tau$.
3. For all $f \in OM$, $\tau : T_f OM \to \mathbb{R}^n$ is onto.

Conversely, given the topological $SO(n)$-bundle $\pi : OM \to M$ and an $\mathbb{R}^n$-valued 1-form τ on OM satisfying properties 1.-3. immediately above, one recovers the metric g. Namely, for $v, w \in T_m M$, choose $m' \in \pi^{-1}(m)$ and $v', w' \in T_{m'} OM$ such that $d\pi(v') = v$ and $d\pi(w') = w$. Then $g(v, w) = \langle \tau(v'), \tau(w') \rangle$.

Let $h : Spin(n) \to SO(n)$ be the double-covering homomorphism. Giving a spin structure on M means giving a principal $Spin(n)$-bundle PM on M such that $OM = PM \times_{Spin(n)} SO(n)$. The 1-form τ lifts to an $\mathbb{R}^n$-valued 1-form τ' on PM with the properties that
1. If V is a vertical vector field on PM then $\tau'(V) = 0$.
2. For all $\gamma \in Spin(n)$, $R_\gamma^* \tau' = h(\gamma^{-1}) \cdot \tau'$.
3. For all $f \in PM$, $\tau' : T_f PM \to \mathbb{R}^n$ is onto.

Thus a Riemannian spin manifold consists of
1. The principal $Spin(n)$-manifold PM on M and
2. An $\mathbb{R}^n$-valued 1-form τ' on PM satisfying properties 1.-3. immediately above.
We can think of PM, as a topological fiber bundle, as being metric-independent. Thus the notion of a spinor field on M is also metric-independent. The metric only enters in defining the $\mathbb{R}^n$-valued 1-form τ' on PM. In this way we can compare spinor fields on two different Riemannian manifolds with the same underlying smooth structure.

Proof of Proposition 1 :

Suppose that the proposition is not true. Then there is some $n \in \mathbb{Z}^+$, some $\Lambda > 0$ and a sequence $\{\epsilon_i\}_{i=1}^\infty$ of positive numbers such that
1. $\lim_{i \to \infty} \epsilon_i = 0$.
2. For each i, there is a connected closed n-dimensional spin manifold M_i with a Riemannian metric g_i such that $-\epsilon_i \leq K(M_i, g_i) \cdot \operatorname{diam}(M_i, g_i)^2 \leq \Lambda$ and $\widehat{A}(M_i) \neq 0$.

By rescaling, we can assume that $\operatorname{diam}(M_i, g_i) = 1$. If i is large enough then $|K(M_i, g_i)| \leq \Lambda$. We can write

$$\widehat{A}(M_i) = \int_{M_i} P(K(M_i, g_i)) \, d\mathrm{vol}(M_i) \tag{2.1}$$

for some explicit homogeneous polynomial P in the curvature tensor [14, p. 231]. Thus there is an explicit number $v(n, \Lambda) > 0$ such that $\operatorname{vol}(M_i, g_i) \geq v(n, \Lambda)$, as otherwise we could conclude from the integral formula that $\widehat{A}(M_i) = 0$. By Gromov's convergence theorem and its elaborations [11, 13], there are
1. A smooth manifold M equipped with a metric g_∞ which is $C^{1,\alpha}$-smooth for all $0 < \alpha < 1$ and

2. A subsequence of $\{M_i\}_{i=1}^\infty$, which we will relabel to again call $\{M_i\}_{i=1}^\infty$, and a sequence of diffeomorphisms $F_i : M \to M_i$ such that $\lim_{i\to\infty} F_i^* g_i = g_\infty$ in the $C^{1,\alpha}$-topology for all $0 < \alpha < 1$.

Replacing (M_i, g_i) by $(M, F_i^* g_i)$, we may assume that the metrics $\{g_i\}_{i=1}^\infty$ all live on the same manifold M (with $\widehat{A}(M) \neq 0$) and converge to g_∞ in the $C^{1,\alpha}$-topology. In particular, the Christoffel symbols of g_∞ are locally $C^{0,\alpha}$ on M. In fact, we may assume that for all $p \in [1, \infty)$, the sequence $\{g_i\}_{i=1}^\infty$ converges to g_∞ in the Sobolev space $L^{2,p}$ of covariant 2-tensors on M whose first two derivatives are L^p; a somewhat similar case is treated in [1, §2]. Let K_i denote the curvature tensor of g_i and let K_∞ denote the curvature tensor of g_∞, an L^p-tensor for all $p \geq 1$. Then $\lim_{i\to\infty} K_i = K_\infty$ in L^p for all $p \geq 1$. In particular, $K_\infty \geq 0$ in the sense of sectional curvatures. Let R_i denote the scalar curvature of g_i and let $R_\infty \geq 0$ denote the scalar curvature of g_∞.

The manifold M is spin and we fix a spin structure on it. As discussed above, we may take spinor fields to be sections of the Hermitian vector bundle $S = PM \times_{Spin(n)} S_n$, regardless of the Riemannian metric g_i. Let $d\mathrm{vol}_i \in \Omega^n(M)$ be the volume form coming from g_i and let $d\mathrm{vol}_\infty \in \Omega^n(M)$ be the volume form coming from g_∞. Let ∇_i be the connection on S coming from g_i and let ∇_∞ be the connection on S coming from g_∞. Then as $i \to \infty$, the tensor $\nabla_i - \nabla_\infty \in \mathrm{End}(S, S \otimes T^*M)$ converges to zero in the $C^{0,\alpha}$-topology. Let D_i denote the Dirac operator on S coming from g_i. Let H^0 be the Hilbert space of L^2-spinors on M with norm

$$\parallel \psi \parallel_{H^0}^2 = \int_M |\psi|^2 \, d\mathrm{vol}_\infty. \tag{2.2}$$

Let H^1 be the Sobolev space of spinors on M with norm

$$\parallel \psi \parallel_{H^1}^2 = \int_M \left(|\nabla_\infty \psi|^2 + |\psi|^2 \right) \, d\mathrm{vol}_\infty. \tag{2.3}$$

As $\widehat{A}(M) \neq 0$, the Atiyah-Singer index theorem implies that there is a nonzero spinor field ψ_i on M such that $D_i \psi_i = 0$. We may assume that $\int_M |\psi_i|^2 \, d\mathrm{vol}_i = 1$. From the Lichnerowicz formula,

$$0 = \int_M |D_i \psi_i|^2 \, d\mathrm{vol}_i = \int_M \left(|\nabla_i \psi_i|^2 + \frac{R_i}{4} |\psi_i|^2 \right) \, d\mathrm{vol}_i. \tag{2.4}$$

By our assumptions,

$$\int_M \frac{R_i}{4} |\psi_i|^2 \, d\mathrm{vol}_i \geq - \frac{n(n-1)\epsilon_i}{4}. \tag{2.5}$$

Hence

$$0 \leq \int_M |\nabla_i \psi_i|^2 \, d\mathrm{vol}_i = - \int_M \frac{R_i}{4} |\psi_i|^2 \, d\mathrm{vol}_i \leq \frac{n(n-1)\epsilon_i}{4}. \tag{2.6}$$

Thus $\lim_{i\to\infty} \parallel \psi_i \parallel_{H^1} = 1$. Taking a subsequence, we may assume that $\{\psi_i\}_{i=1}^\infty$ converges weakly to some $\psi_\infty \in H^1$. By compactness, $\{\psi_i\}_{i=1}^\infty$ converges strongly to ψ_∞ in H^0. Thus $\parallel \psi_\infty \parallel_{H^0} = 1$. Furthermore, for general reasons,

$$\parallel \psi_\infty \parallel_{H^1} \leq \lim_{i\to\infty} \parallel \psi_i \parallel_{H^1} = 1. \tag{2.7}$$

Hence

$$1 = \int_M |\psi_\infty|^2 \, d\mathrm{vol}_\infty \ \le \ \int_M \left(|\nabla_\infty \psi_\infty|^2 + |\psi_\infty|^2 \right) \, d\mathrm{vol}_\infty \le 1. \tag{2.8}$$

Thus $\nabla_\infty \psi_\infty = 0$. In particular, $|\psi_\infty|^2$ is a nonzero constant function on M. Also, from (2.6), $\{\psi_i\}_{i=1}^\infty$ converges strongly to ψ_∞ in H^1.

As the $\widehat{A}$-genus is only nonzero in dimensions divisible by 4, we may assume that $n > 2$. Then H^1 embeds continuously in $L^{\frac{2n}{n-2}}$. Hence $\lim_{i \to \infty} |\psi_i|^2 = |\psi_\infty|^2$ in $L^{\frac{n}{n-2}}$. As $\lim_{i \to \infty} R_i = R_\infty$ in $L^{\frac{n}{2}}$, (2.6) implies that

$$\int_M \frac{R_\infty}{4} \, |\psi_\infty|^2 \, d\mathrm{vol}_\infty = \lim_{i \to \infty} \int_M \frac{R_i}{4} \, |\psi_i|^2 \, d\mathrm{vol}_i = 0. \tag{2.9}$$

As $R_\infty \ge 0$, we conclude that $R_\infty = 0$. Hence $K_\infty = 0$. As $\lim_{i \to \infty} K_i = K_\infty$ in $L^{\frac{n}{2}}$, we obtain

$$\widehat{A}(M) = \lim_{i \to \infty} \int_M P(K_i) \, d\mathrm{vol}_i = 0. \tag{2.10}$$

This is a contradiction. □

Let us make some comments about Ricci curvature. As mentioned in the introduction, if M is a connected closed n-dimensional spin manifold with $|\widehat{A}(M)| > 2^{\frac{n}{2}-1}$ then M cannot have almost-nonnegative-Ricci curvature [9]. One can ask what happens when $|\widehat{A}(M)|$ lies between 1 and $2^{\frac{n}{2}-1}$. It may be that any such manifold with almost-nonnegative-Ricci curvature is necessarily very special. The method of proof of Proposition 1, along with the smoothing result of [17], gives the following proposition.

Proposition 4. *For any $n \in \mathbb{Z}^+$ and any $c > 0$, there is an $\epsilon(n, c) > 0$ such that if M is a connected closed n-dimensional Riemannian spin manifold with $\mathrm{Ric}(M, g) \cdot \mathrm{diam}(M, g)^2 \ge -\epsilon(n, c)$ and $\mathrm{conj}(M, g) \ge c \cdot \mathrm{diam}(M, g)$ then $\widehat{A}(M) = 0$ or M admits a $C^{1,\alpha}$-metric g_0 whose local holonomy group factorizes into products of $\{SU(m)\}_{m=2}^\infty$, $\{Sp(m)\}_{m=1}^\infty$, $Spin(7)$ and G_2.*

As was pointed out to me by Peter Petersen, the metric g_0 constructed in Proposition 4 is actually smooth, as it has vanishing L^p-Ricci curvature.

Question 3. *Given $n \in \mathbb{Z}^+$, is there an $\epsilon(n) > 0$ such that if M is a connected closed n-dimensional Riemannian spin manifold with $\mathrm{Ric}(M, g) \cdot \mathrm{diam}(M, g)^2 \ge -\epsilon(n)$ then $\widehat{A}(M) = 0$ or the frame bundle of M admits a topological reduction to a principal bundle whose local structure group factorizes into products of $\{SU(m)\}_{m=2}^\infty$, $\{Sp(m)\}_{m=1}^\infty$, $Spin(7)$ and G_2?*

For example, $M = K3\#(S^2 \times S^2)$ is a spin manifold with $\widehat{A}(M) = 2$ but without an almost complex structure having $c_1 = 0$. Does M have almost-nonnegative-Ricci curvature? An affirmative answer to Question 3 would imply that it does not.

Remark : One may think of trying to answer Question 1 by an extension of the Bochner method. However, such an approach cannot work, at least not directly. For example, a flat torus is almost-nonnegatively-curved but, with the right spin structure, does have harmonic spinors. It is just the index of its Dirac operator which vanishes. Also, a nonflat nilmanifold has locally homogeneous metrics of constant negative scalar curvature, for which the use of Lichnerowicz's formula is problematic.

3. Proof of Proposition 2

Part 1. of Proposition 2 is proven in [8, §2]. More precisely, if there is a metric h on B with $K(B,h) \cdot \mathrm{diam}(B,h)^2 > -\epsilon$ then there is a metric g on M with $K(M,g) \cdot \mathrm{diam}(M,g)^2 > -\epsilon$.

We now prove part 2. Suppose first that N is not flat. Recall that the $\widehat{A}$-genus is multiplicative under finite coverings. Hence by taking a double cover if necessary, we may assume that B is orientable. Fix an orientation of B. Choose a metric on B and a connection on P. There is an induced metric on M.

Let $\pi : M \to B$ be the projection map. Put $b = \dim(B)$. Let $\{U_i\}_{i=1}^K$ be a finite covering of B by open sets such that for any $k \in \mathbb{Z}^+$ and any $i_1, \ldots, i_k \in \{1, \ldots, K\}$, the intersection $U_{i_1} \cap \ldots \cap U_{i_k}$ is empty or is diffeomorphic to $\mathbb{R}^b$. Each nonempty intersection $U_{i_1} \cap \ldots \cap U_{i_k}$ acquires an orientation from B and then has a unique spin structure.

The preimage $\pi^{-1}(U_i) \subset M$ of U_i is diffeomorphic to $N \times U_i$ and has a spin structure coming from that of M. The spin structures on U_i and $\pi^{-1}(U_i)$ give a spin structure on the vertical tangent bundle TN over $\pi^{-1}(U_i)$. As each fiber N has nonnegative scalar curvature which is positive somewhere, it follows as in Lemma 1 that $\mathrm{Ker}(D_N) = 0$. From [5], there is a canonically-defined differential form $\widetilde{\eta}_i \in \Omega^*(U_i)$ such that on U_i,

$$d\widetilde{\eta}_i = \int_N \widehat{A}\left(\nabla^{TN}\right). \tag{3.1}$$

The canonical nature of the constructions implies that on an intersection $U_{i_1} \cap U_{i_2}$, we have $\widetilde{\eta}_{i_1} = \widetilde{\eta}_{i_2}$. Hence we obtain a globally-defined differential form $\widetilde{\eta} \in \Omega^*(B)$ such that

$$d\widetilde{\eta} = \int_N \widehat{A}\left(\nabla^{TN}\right). \tag{3.2}$$

Then

$$\widehat{A}(M) = \int_B \widehat{A}\left(\nabla^{TB}\right) \wedge \int_N \widehat{A}\left(\nabla^{TN}\right) = \int_B \widehat{A}\left(\nabla^{TB}\right) \wedge d\widetilde{\eta} = \int_B d\left(\widehat{A}\left(\nabla^{TB}\right) \wedge \widetilde{\eta}\right) = 0. \tag{3.3}$$

Now suppose that N is flat. Then $N = T^k/F$ for some $k > 0$, where T^k has a flat metric, F is a finite group of isometries of T^k and T^k is a minimal such covering. Let $\rho : G \to \mathrm{Isom}(N)$ describe the action of G on N. Let $\mathrm{Isom}(T^k)^F$ denote the isometries of T^k which commute with F. There is a homomorphism $\Theta : \mathrm{Isom}(T^k)^F \to \mathrm{Isom}(N)$. The induced map on Lie algebras $\theta : \mathrm{isom}(T^k)^F \to \mathrm{isom}(N)$ is an isomorphism, as $\mathrm{isom}(N)$ is the Lie algebra of Killing vector fields on N, each of which can be lifted to an F-invariant Killing vector field on T^k. As $\mathrm{Ker}(\Theta) = \mathrm{center}(F)$, Θ restricts to an isomorphism between $\mathrm{Isom}(T^k)_0^F$ and $\mathrm{Isom}(N)_0$, the connected components of the identity.

Put

$$G' = \left\{(g_1, g_2) \in \mathrm{Isom}(T^k)_0^F \times G : \Theta(g_1) = \rho(g_2)\right\}. \tag{3.4}$$

There is a finite covering $P \times_{G'} T^k \to M$. As $\mathrm{Isom}(T^k)_0^F$ acts on T^k by translations, it commutes with the action of T^k on itself by translations and so there is a nontrivial T^k-action on $P \times_{G'} T^k$. By the Atiyah-Hirzebruch theorem [2], the $\widehat{A}$-genus of $P \times_{G'} T^k$ vanishes. Thus $\widehat{A}(M) = 0$.

Remark : Under the hypotheses of Proposition 2, it may not be true that the vertical tangent bundle TN, a real vector bundle on M, has a spin structure; I thank Stephan Stolz for showing me such an example. This is why we do the pasting procedure to define $\widetilde{\eta}$.

4. Proof of Proposition 3

Let $\overline{Z}$ be the union of the principal orbits for the action of G on Z. Put $\overline{B} = \overline{Z}/G$, a smooth manifold and put $\overline{M} = \overline{Z} \times_G N$, a dense open subset of M. There is a Riemannian submersion $\pi : \overline{M} \to \overline{B}$ whose fiber over $zG \in \overline{B}$ is $G_z \backslash N$.

To describe the geometry of $\overline{M}$ more explicitly, fix $z \in \overline{Z}$. Let $N(G_z)$ denote the normalizer of G_z in G. Then $\overline{Z}$ is a fiber bundle over $\overline{B}$ with structure group contained in $K = G_z \backslash N(G_z)$ [6, Theorem 3.3]. That is, there is a principal K-bundle $\overline{P}$ over $\overline{B}$ such that $\overline{Z} = \overline{P} \times_K (G_z \backslash G)$. Furthermore, $\overline{Z} \to \overline{B}$ is a Riemannian submersion whose horizontal distribution comes from a connection on $\overline{P}$. We note that although all of the G-orbits on $\overline{Z}$ are diffeomorphic to $G_z \backslash G$, their Riemannian metrics may vary from fiber to fiber. Topologically, we can write $\overline{M} = \overline{P} \times_K (G_z \backslash N)$. The horizontal distribution on the Riemannian submersion $\pi : \overline{M} \to \overline{B}$ again comes from the connection on $\overline{P}$. Metrically, the fibers of π can be more accurately written as $(G_z \backslash G) \times_G N$, with the orbit $G_z \backslash G$ obtaining its metric from its embedding in $\overline{Z}$.

Proof of 1. Let g_Z and g_N be the Riemannian metrics on Z and N. Let $K_0 > 0$ be such that $K(Z, g_Z) \geq -K_0$. For $j \geq 1$, consider the Riemannian metric $h_j = g_Z + j^{-2} g_N$ on $Z \times N$. Clearly $K(Z \times N, h_j) \geq -K_0$ and $\mathrm{diam}(Z \times N, h_j) \leq \mathrm{diam}(Z, g_Z) + j^{-1} \mathrm{diam}(N, g_N)$. Let $(M, g_j) = (Z \times N, h_j)/G$ be the quotient metric on M. From the O'Neill formula [4, Chapter 9], $K(M, g_j) \geq -K_0$. Clearly $\mathrm{diam}(M, g_j) \leq \mathrm{diam}(Z \times N, h_j)$.

Let $\frac{N}{j}$ denote $(N, j^{-2} g_N)$. Let $\overline{g_j}$ denote the restriction of g_j to $\overline{M}$. Then $(\overline{M}, \overline{g_j})$ is obtained from the Riemannian submersion π by changing the fiber from $(G_z \backslash G) \times_G N$ to $(G_z \backslash G) \times_G \frac{N}{j}$. Let us concentrate on a given fiber $(G_z \backslash G) \times_G N$. As $G_z \backslash N$ is a smooth manifold, there is a number $v_{min} > 0$ such that every G_z-orbit on (N, g) has volume at least v_{min}. Then

$$\mathrm{vol}\left((G_z \backslash G) \times_G N\right) \leq \frac{\mathrm{vol}(G_z \backslash G) \cdot \mathrm{vol}(N)}{\mathrm{vol}(G_z \backslash G) \cdot v_{min}} = \frac{\mathrm{vol}(N)}{v_{min}}. \tag{4.1}$$

Replacing N by $\frac{N}{j}$ gives

$$\mathrm{vol}\left((G_z \backslash G) \times_G \frac{N}{j}\right) \leq j^{-dim(G_z \backslash N)} \frac{\mathrm{vol}(N)}{v_{min}}. \tag{4.2}$$

By assumption, $\dim(G_z \backslash N) > 0$. Thus

$$\lim_{j \to \infty} \mathrm{vol}\left((G_z \backslash G) \times_G \frac{N}{j}\right) = 0. \tag{4.3}$$

It follows that

$$\lim_{j \to \infty} \mathrm{vol}(M, g_j) = \lim_{j \to \infty} \mathrm{vol}(\overline{M}, \overline{g_j}) = 0. \tag{4.4}$$

We can think of the projection map $\pi : M \to Z/G$ as a singular fibration whose fiber over $z'G \in Z/G$ is $G_{z'} \backslash N$. From the same arguments as above, we see that $\lim_{j\to\infty}(M, g_j) = Z/G$ in the Gromov-Hausdorff topology.

Proof of 2. Without loss of generality, we may assume that $\dim(M)$ is even. As before, we fix $z \in \overline{Z}$. Put $B = Z/G$,

$$Z^{sing} = \{z' \in Z : \dim(G_{z'}) > \dim(G_z)\}, \tag{4.5}$$

$B^{sing} = Z^{sing}/G$ and $M^{sing} = Z^{sing} \times_G N$. Given $\epsilon > 0$, let $B^{sing}(\epsilon)$ be the ϵ-neighborhood of B^{sing} in B, let $Z^{sing}(\epsilon)$ be its preimage in Z and put $M^{sing}(\epsilon) = Z^{sing}(\epsilon) \times_G N$. Put $M_1 = \overline{M^{sing}(\epsilon)}$, $M_2 = M - M^{sing}(\epsilon)$, $W = \partial M_1 = \partial M_2$ and $B_2 = B - B^{sing}(\epsilon)$. We note that B_2 is a smooth orbifold and that M_2 is a fiber bundle over B_2.

By the O'Neill formula, $G_z \backslash N$ has a metric of nonnegative sectional curvature. Given this fact, it follows from the Cheeger-Gromoll splitting theorem [7] that the condition that $G_z \backslash N$ be flat is topological in nature, namely that $\pi_1(G_z \backslash N)$ have a free abelian subgroup of rank $\dim(G_z \backslash N)$. We divide the proof of 2. into two cases.

Case 1. $G_z \backslash N$ **is not flat.**

We first prove a general result about the index of the Dirac operator on a compact spin manifold-with-boundary.

Lemma 2. *Let X be a compact even-dimensional Riemannian spin manifold with boundary ∂X. Let D_X be the Dirac operator on X with Atiyah-Patodi-Singer boundary conditions* [3]*. Let $H_{\partial X} \in C^\infty(\partial X)$ be the mean curvature function. (With our conventions, if X is the unit ball in $\mathbb{R}^n$, $n > 1$, then $H_{\partial X} > 0$.) Suppose that $R_X \geq 0$ and $H_{\partial X} \geq 0$. Suppose that R_X is positive somewhere or $H_{\partial X}$ is positive somewhere. Then* $\mathrm{ind}(D_X) = 0$.

Proof. Let $\{e_j\}_{j=1}^n$ denote a local orthonormal frame on X. Let γ^j denote Clifford multiplication by e_j. With our conventions,

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta^{ij}. \tag{4.6}$$

The Dirac operator on X has the local form

$$D_X = -i \sum_{j=1}^n \gamma^j \, \nabla^X_{e_j}. \tag{4.7}$$

Along ∂X, we take e_n to be an inward-pointing unit normal vector. With respect to the decomposition $S = S^+ \oplus S^-$, we can write $\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}$ for $1 \leq j \leq n-1$ and $\gamma^n = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, where $\{\sigma_j\}_{j=1}^{n-1}$ are generators for the Clifford algebra on $\mathbb{R}^{n-1}$. The Dirac operator on ∂X has the local form

$$D_{\partial X} = -i \sum_{j=1}^{n-1} \sigma^j \, \nabla^{\partial X}_{e_j}. \tag{4.8}$$

Let ψ be a spinor field on X. Let $\psi = \psi^+ + \psi^-$ be its decomposition with respect to the $\mathbb{Z}_2$-grading on spinors. Let $\psi_{\partial X}$ be its restriction to ∂X. Let $P^{\geq 0}$ be the projection onto the subspace of spinors on ∂X spanned by eigenvectors of $D_{\partial X}$ of nonnegative eigenvalue, and similarly for $P^{<0}$. The Atiyah-Patodi-Singer boundary conditions are

$$P^{\geq 0} \psi^+_{\partial X} = P^{<0} \psi^-_{\partial X} = 0. \tag{4.9}$$

These boundary conditions are usually considered when X is a product near the boundary, but one obtains an elliptic self-adjoint boundary condition for D_X regardless of whether or not X is a product near the boundary.

Suppose that $D_X \psi = 0$. The Lichnerowicz equation

$$0 = D_X^2 \psi = \left(\nabla^X \right)^* \nabla^X \psi + \frac{R_X}{4} \, \psi \tag{4.10}$$

is valid on the interior of X. Define a vector field $J = \sum_{j=1}^n J^j e_j$ on X by $J^j = \langle \psi, \nabla^X_{e_j} \psi \rangle$. Then

$$\begin{aligned} 0 &= \int_X \langle \psi, \left(\nabla^X \right)^* \nabla^X \psi \rangle \, d\mathrm{vol} + \int_X \frac{R_X}{4} \, |\psi|^2 \, d\mathrm{vol} &(4.11) \\ &= \int_X \langle \nabla^X \psi, \nabla^X \psi \rangle \, d\mathrm{vol} - \int_X \mathrm{div}(J) \, d\mathrm{vol} + \int_X \frac{R_X}{4} \, |\psi|^2 \, d\mathrm{vol} \\ &= \int_X \langle \nabla^X \psi, \nabla^X \psi \rangle \, d\mathrm{vol} + \int_X \frac{R_X}{4} \, |\psi|^2 \, d\mathrm{vol} + \int_{\partial X} J^n \, d\mathrm{vol} \end{aligned}$$

$$(4.12)$$

Now $D_X \psi = 0$ implies that $\nabla^X_{e_n} \psi = - \sum_{j=1}^{n-1} \gamma^n \gamma^j \, \nabla^X_{e_j} \psi$. Hence

$$0 = \int_X \langle \nabla^X \psi, \nabla^X \psi \rangle \, d\mathrm{vol} + \int_X \frac{R_X}{4} \, |\psi|^2 \, d\mathrm{vol} - \int_{\partial X} \langle \psi, \sum_{j=1}^{n-1} \gamma^n \gamma^j \, \nabla^X_{e_j} \psi \rangle \, d\mathrm{vol}. \tag{4.13}$$

A computation gives that on ∂X,

$$\sum_{j=1}^{n-1} \gamma^n \gamma^j \, \nabla^X_{e_j} = \sum_{j=1}^{n-1} \gamma^n \gamma^j \, \nabla^{\partial X}_{e_j} - (n-1) \, \frac{H_{\partial X}}{2}. \tag{4.14}$$

Then

$$\begin{aligned} 0 = & \int_X \langle \nabla^X \psi, \nabla^X \psi \rangle \, d\mathrm{vol} + \int_X \frac{R_X}{4} \, |\psi|^2 \, d\mathrm{vol} &(4.15) \\ & - \int_{\partial X} \langle \psi, \sum_{j=1}^{n-1} \gamma^n \gamma^j \, \nabla^{\partial X}_{e_j} \psi \rangle \, d\mathrm{vol} + (n-1) \int_{\partial X} \frac{H_{\partial X}}{2} \, |\psi|^2 \, d\mathrm{vol}. \end{aligned}$$

The Atiyah-Patodi-Singer boundary conditions (4.9) imply that

$$- \int_{\partial X} \langle \psi, \sum_{j=1}^{n-1} \gamma^n \gamma^j \, \nabla^{\partial X}_{e_j} \psi \rangle \, d\mathrm{vol} \geq 0. \tag{4.16}$$

As $R_X \geq 0$ and $H_{\partial X} \geq 0$, we obtain from (4.15) that $\nabla^X \psi = 0$. In particular, $|\psi|^2$ is locally constant on X. Equation (4.15), along with the fact that R_X or $H_{\partial X}$ is positive somewhere, implies that $|\psi|^2 = 0$. Thus there are no nonzero solutions to $D_X\psi = 0$ and so $\mathrm{ind}(D_X) = 0$. □

Our strategy to prove the proposition in Case 1 is the following. If D_W is invertible then the Atiyah-Patodi-Singer index theorem (and its generalization to the case of nonproduct boundary) implies that

$$\mathrm{ind}(D_M) = \mathrm{ind}(D_{M_1}) + \mathrm{ind}(D_{M_2}). \tag{4.17}$$

We will show that after shrinking the fiber metrics, we can apply Lemma 2 to show that $\mathrm{ind}(D_{M_1}) = 0$. Then we will use index theory techniques to show that $\mathrm{ind}(D_{M_2}) = 0$.

Lemma 3. *Define the metric g_j on M as in the proof of part 1. Then for large j, there is an $\epsilon > 0$ such that the submanifold M_1 of (M, g_j) has $R_{M_1} > 0$ and $H_W > 0$.*

Proof. This follows from computations as in [15, Sect. 7-10]. We omit the details but give an illustrative example which has all of the features of the general case. Suppose that $z' \in Z^{sing}$ is a G-fixed point. By the equivariant tubular neighborhood theorem, there is a G-vector space V which is G-diffeomorphic to an ϵ-neighborhood of z' in Z [6, Theorem VI.2.2]. (In particular, for generic $v \in V$, the G-stabilizer of v is conjugate to G_z.) Then $U_M = V \times_G N$ is a neighborhood of $\pi^{-1}(z'G) \subset M$. Consider the case when N is a homogeneous space G/H.

If N has positive scalar curvature then it is easy to see that from the O'Neill formula that for large j, (U_M, g_j) has positive scalar curvature. Suppose, to take the other extreme, that N is flat. Taking a finite cover, we may assume that $N = T^k$ and that G acts on T^k through a homomorphism $\rho' : G \to T^k$. As U_M is a smooth manifold, $\mathrm{Ker}(\rho')$ must act trivially on V. Put $\widetilde{G} = G/\mathrm{Ker}(\rho')$. There are induced homomorphisms $\widetilde{\rho} : \widetilde{G} \to \mathrm{Aut}(V)$ and $\widetilde{\rho}' : \widetilde{G} \to T^k$, with $\widetilde{\rho}'$ being an inclusion. Then $U_M = V \times_{\widetilde{G}} T^k$. Taking a finite cover, we may assume that $\widetilde{G}$ is connected. If V is flat then one can check that for large j, (U_M, g_j) has a positive scalar curvature function whose value at $[0, t] \in V \times_{\widetilde{G}} T^k$ is $O(j^2)$. This can be seen intuitively by the fact that $\widetilde{\rho}$ reduces into trivial $\mathbb{R}$-factors and at least one nontrivial $\mathbb{R}^2$-factor. If $V = \mathbb{R}^2$ and $\widetilde{G} = T^k = S^1$ then $\mathbb{R}^2 \times_{S^1} S^1$ has a torpedo shape which becomes more curved at the tip as the S^1-factor shrinks.

In the general case, the torpedo effect ensures that if j is large enough and ϵ is small enough then (U_M, g_j) will have positive scalar curvature. In fact, for large j we can take $\epsilon = O(j^{-(\frac{1}{2}+\alpha)})$ for any $\alpha > 0$. As $B^{sing} \subset B$ has codimension at least two, the mean curvature of ∂M_1 is positive for large j. Doing a similar procedure for a finite collection of $z' \in Z^{sing}$, we can deal with all of the strata of Z^{sing}. The lemma follows. □

As $G_z \backslash N$ is not flat, each fiber of the fiber bundle $M_2 \to B_2$ has a nonnegative scalar curvature function which is positive somewhere. Then by the Lichnerowicz formula, the Dirac operator on each fiber is invertible. For large j, the geometry of a fiber $(G_z \backslash G) \times_G \frac{N}{j}$ is asymptotically that of $\frac{G_z \backslash N}{j}$. If j is large and $0 < \alpha < \frac{1}{2}$ then it follows as in [5, Proposition 4.41] that D_W is also invertible. By Lemmas 2 and 3, $\mathrm{ind}(D_{M_1}) = 0$.

We now show that $\mathrm{ind}(D_{M_2}) = 0$. Let $[0, \delta) \times W \subset M_2$ be a neighborhood of W such that if $u \in [0, \delta)$ is the coordinate function then ∂_u is a unit length vector field whose flow generates unit-speed geodesics which are normal to W. We can write the metric near W as $du^2 + h(u)$, where $h(u)$ is a metric on W. Let $F : [0, \infty) \to [0, 1]$ be a smooth nondecreasing function such that $F(x)$ is identically zero for x near zero and identically one if $x \geq \frac{1}{2}$. For $v \in [0,1]$, define $f_v : [0,\infty) \rightarrow [0,\infty)$ by

$$f_v(u) = \begin{cases} u & \text{if } v = 0, \\ u\, F(\frac{u}{\delta v}) & \text{if } v \in (0,1]. \end{cases} \tag{4.18}$$

Let $M_2(v)$ be the manifold M_2 with the metric $du^2 + h(f_v(u))$ on $[0,\delta) \times W$. Then $M_2(0)$ is the same as M_2 with the original metric and $M_2(1)$ has a product metric near its boundary. For all $v \in [0,1]$, $\partial M_2(v)$ is isometric to W. Then the Dirac operators on $M_2(v)$, with Atiyah-Patodi-Singer boundary conditions, form a continuous family of Fredholm operators and so have constant index with respect to v. Thus for computational purposes, we may assume that M_2 is a product near the boundary.

By the Atiyah-Patodi-Singer index theorem [3],

$$\mathrm{ind}(D_{M_2}) = \int_{M_2} \widehat{A}(\nabla^{TM_2}) - \frac{1}{2}\, \eta_W. \tag{4.19}$$

From [5, Theorems 4.35 and 4.95], we have an equality in $\Omega^*(B_2)$:

$$d\widetilde{\eta}_{M_2} = \int_{G_z\backslash N} \widehat{A}\left(\nabla^{T(G_z\backslash N)}\right). \tag{4.20}$$

(Strictly speaking, we have to generalize the results of [5] from smooth fiber bundles to fiber bundles with orbifold base. Such a generalization is straightforward. We will not give the details here.)

Also,

$$\begin{aligned}\lim_{j\rightarrow\infty} \int_{M_2} \widehat{A}(\nabla^{TM_2}) &= \int_{B_2} \int_{G_z\backslash N} \widehat{A}\left(\nabla^{T(G_z\backslash N)}\right) \wedge \widehat{A}\left(\nabla^{TB_2}\right) = \int_{B_2} d\widetilde{\eta}_{M_2} \wedge \widehat{A}\left(\nabla^{TB_2}\right) \\ &= \int_{\partial B_2} \widetilde{\eta}_W \wedge \widehat{A}\left(\nabla^{T\partial B_2}\right). \end{aligned} \tag{4.21}$$

On the other hand, by [5, Theorems 4.35 and 4.95],

$$\lim_{j\rightarrow\infty} \frac{1}{2}\, \eta_W = \int_{\partial B_2} \widetilde{\eta}_W \wedge \widehat{A}\left(\nabla^{T\partial B_2}\right). \tag{4.22}$$

Combining equations (4.19)-(4.22) gives that $\mathrm{ind}(D_{M_2}) = 0$. This proves the proposition in Case 1.

Case 2. $G_z\backslash N$ is flat.

We will show that there is a finite cover of M with a nontrivial S^1-action. It then follows from the Atiyah-Hirzebruch theorem [2] that $\widehat{A}(M) = 0$.

The various fibers of the fiber bundle $\overline{M} \rightarrow \overline{B}$ are all flat. They are not necessarily all isometric. However, they are all affine-equivalent.

Write $G_z\backslash N = F\backslash T^k$, where $k > 0$, $F \subset \mathrm{Aff}(T^k)$ is a finite group of affine diffeomorphisms of T^k and T^k is a minimal such covering. Let $\rho : K \rightarrow \mathrm{Aff}(G_z\backslash N)$ describe the action of K on $G_z\backslash N$. Let $\mathrm{Aff}(T^k)^F$ denote the affine diffeomorphisms of T^k which commute with F and let $\mathrm{Aff}(T^k)_0^F$ denote the connected component of the identity. There is a homomorphism $\Theta : \mathrm{Aff}(T^k)^F \to \mathrm{Aff}(G_z \backslash N)$. Put

$$K' = \left\{ (k_1, k_2) \in \mathrm{Aff}(T^k)_0^F \times K : \Theta(k_1) = \rho(k_2) \right\}. \tag{4.23}$$

Put $\overline{M}' = \overline{P} \times_{K'} T^k$, a finite cover of $\overline{M}$. Then $\overline{M}'$ is a fiber bundle over $\overline{B}$ with fiber $K \times_{K'} T^k$, a finite disjoint union of tori. There is a nontrivial T^k-action on $\overline{M}'$. Similarly, we want to show that there is a finite cover M' of M with a nontrivial T^k-action. The problem is that $\overline{M}'$ may not extend to a finite cover of M. However, we will show that the disjoint union of a certain number of copies of it does extend.

Choose $z' \in Z - \overline{Z}$. Let $G_{z'} \subseteq G$ be its stabilizer subgroup. Then $G_z \subset G_{z'}$ and $G_{z'} \backslash N$ is a smooth manifold. There is a Riemannian submersion $G_z \backslash N \to G_{z'} \backslash N$ with fiber $G_z \backslash G_{z'}$. As $G_z \backslash N$ is flat and both $G_{z'} \backslash N$ and $G_z \backslash G_{z'}$ are nonnegatively curved, one sees from the homotopy groups that $G_{z'} \backslash N$ and $G_z \backslash G_{z'}$ must be flat. As $G_z \backslash G_{z'}$ is a globally homogeneous space, it must be a disjoint union of tori of dimension $\dim(G_{z'}) - \dim(G_z)$.

By the equivariant tubular neighborhood theorem, there is a finite-dimensional real vector space V, a representation $\rho : G_{z'} \to \mathrm{Aut}(V)$ and a neighborhood U_Z of the G-orbit of z' such that U_Z is G-diffeomorphic to $V \times_{G_{z'}} G$. Then $U_M = U_Z \times_G N = V \times_{G_{z'}} N$ is a neighborhood of $\pi^{-1}(z'G) \subset M$.

As $G_{z'} \backslash N$ is flat, we can write it as $F' \backslash T^{k'}$, where F' is a finite group of affine diffeomorphisms of $T^{k'}$ and $T^{k'}$ is a minimal such covering. Let $s : T^{k'} \to F' \backslash T^{k'}$ be the projection map. Consider the fiber bundle $F \backslash T^k \xrightarrow{r} F' \backslash T^{k'}$ with fiber $G_z \backslash G_{z'}$. Put

$$C = \{(t, t') \in (F \backslash T^k) \times T^{k'} : r(t) = s(t')\}. \tag{4.24}$$

Equivalently, $C = r^* T^{k'}$, as shown in the diagram

$$\begin{array}{ccc} C & \rightarrow & T^{k'} \\ \downarrow & & \downarrow s \\ F \backslash T^k & \xrightarrow{r} & F' \backslash T^{k'}. \end{array} \tag{4.25}$$

We claim that C is a disjoint union of k-dimensional tori. To see this, put $\Gamma = \pi_1(G_z \backslash N)$ and $\Gamma' = \pi_1(G_{z'} \backslash N)$. Then we have a diagram of exact sequences :

$$\begin{array}{ccccccccccc} & & & & & & 1 & & & & \\ & & & & & & \downarrow & & & & \\ & & & & & & \mathbb{Z}^{k'} & & & & \\ & & & & & & \downarrow \beta & & & & \\ 1 & \rightarrow & \pi_1(G_z \backslash G_{z'}) & \rightarrow & \Gamma & \xrightarrow{\alpha} & \Gamma' & \rightarrow & \pi_0(G_z \backslash G_{z'}) & \rightarrow & 1. \\ & & & & & & \downarrow & & & & \\ & & & & & & F' & & & & \\ & & & & & & \downarrow & & & & \\ & & & & & & 1 & & & & \end{array} \tag{4.26}$$

Now

$$\pi_1(C) = \{(\gamma, \zeta) \in \Gamma \times \mathbb{Z}^{k'} : \alpha(\gamma) = \beta(\zeta)\}. \tag{4.27}$$

Projecting $\pi_1(C)$ on Γ or $\mathbb{Z}^{k'}$, it follows from (4.26) that there are exact sequences

$$1 \to \pi_1(C) \to \Gamma \tag{4.28}$$

and

$$1 \to \pi_1(G_z \backslash G_{z'}) \to \pi_1(C) \to \mathbb{Z}^{k'}. \tag{4.29}$$

Put $A = \mathrm{Im}(\pi_1(C) \to \mathbb{Z}^{k'})$, a free abelian subgroup of finite index. Then (4.29) is equivalent to the short exact sequence

$$1 \to \pi_1(G_z \backslash G_{z'}) \to \pi_1(C) \to A \to 1. \tag{4.30}$$

From (4.28), $\pi_1(C)$ is isomorphic to a subgroup of Γ. One can see that it is of finite index in Γ and so has polynomial growth of degree k. This implies that the sequence (4.30) splits and that $\pi_1(C) = \pi_1(G_z \backslash G_{z'}) \times A \cong \mathbb{Z}^k$. As C is flat, it must be the disjoint union of m copies of T^k for some $m > 0$.

Let $\delta : N \to G_{z'} \backslash N$ be the projection map. Put

$$R = \{(n, t') \in N \times T^{k'} : \delta(n) = s(t')\}. \tag{4.31}$$

That is, $R = \delta^* T^{k'}$, as shown in the diagram

$$\begin{array}{ccc} R & \to & T^{k'} \\ \downarrow & & \downarrow s \\ N & \xrightarrow{\delta} & G_{z'} \backslash N, \end{array} \tag{4.32}$$

whose vertical arrows are finite coverings. There is an action of $G_{z'}$ on R by $g(n,t') = (gn, t')$, with $G_z \backslash R = C$ (compare (4.25)).

Let $[K : K']$ denote the index of K' in K. Let $[K : K']\, R$ denote the disjoint union of $[K : K']$ copies of R. Consider the finite covering $V \times_{G_{z'}} [K : K']\, R \to U_M$. We claim that this extends the covering $m\overline{M}' \to \overline{M}$ over U_M. To see this, note that we have a diagram of fiber bundles

$$\begin{array}{ccccc} G_z \backslash G_{z'} & \to & V \times_{G_z} [K : K']\, R & \to & V \times_{G_{z'}} [K : K']\, R \\ & & \downarrow & & \downarrow \\ G_z \backslash G_{z'} & \to & V \times_{G_z} N & \to & V \times_{G_{z'}} N. \end{array} \tag{4.33}$$

Let $\overline{V}$ be the set of points in V whose stabilizer group is conjugate to G_z, a dense open subset of V. Put $\overline{U}_M = \overline{M} \cap U_M = \overline{V} \times_{G_{z'}} N$, a dense open subset of U_M. Let $\overline{U}'_M$ be the pre-image of $\overline{U}_M$ under the covering $\overline{M}' \to \overline{M}$. Then (4.33) restricts to

$$\begin{array}{ccccc} G_z \backslash G_{z'} & \to & \overline{V} \times_{G_z} [K : K']\, R & \to & \overline{V} \times_{G_{z'}} [K : K']\, R \\ & & \downarrow & & \downarrow \\ G_z \backslash G_{z'} & \to & \overline{V} \times_{G_z} N & \to & \overline{V} \times_{G_{z'}} N. \end{array} \tag{4.34}$$

That is, we have a diagram of fiber bundles

$$\begin{array}{ccccc} & & m[K : K']\, T^k & & m[K : K']\, T^k \\ & & \downarrow & & \downarrow \\ G_z \backslash G_{z'} & \to & \overline{V} \times [K : K']\, C & \to & \overline{V} \times_{G_{z'}} [K : K']\, R \\ & & \downarrow & & \downarrow \\ G_z \backslash G_{z'} & \to & \overline{V} \times (G_z \backslash N) & \to & \overline{U}_M. \end{array} \tag{4.35}$$

By the constructions, it follows that the right-hand-column of (4.35) is the same as

$$\begin{array}{c} mK \times_{K'} T^k \\ \downarrow \\ m\overline{U}'_M \\ \downarrow \\ \overline{U}_M. \end{array} \tag{4.36}$$

Thus $V \times_{G_{z'}} [K : K'] \, R \to U_M$ does extend the covering $m\overline{M}' \to \overline{M}$ over U_M. Furthermore, the obvious T^k-action on $m\overline{U}'_M$ comes from the T^k-action on $\overline{V} \times [K : K'] \, C$, which extends to the T^k-action on $V \times_{G_z} [K : K'] \, R = V \times [K : K'] \, C$, which pushes down to a T^k-action on $V \times_{G_{z'}} [K : K'] \, R$. Of course, the T^k-action on $V \times_{G_{z'}} [K : K'] \, R$ may not be free.

Repeating the process for a finite number of z''s whose G-orbits exhaust the singular G-strata of Z, we end up with a finite covering $M' \to M$. The preimage of $\overline{M}$ in M' is the disjoint union of a finite number of copies of $\overline{M}'$ and so has a nontrivial T^k-action. From the nature of the above extension procedure, we know that it extends to a T^k-action on M'. The proposition follows.

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