Evolution of three-dimensional Ricci flow

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The question

- Two-dimensional Ricci flow
- Three-dimensional Ricci flows with symmetry

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- Main results
- Equilibrium configurations of Ricci flow
- Locally homogeneous Ricci flows
- Proofs of main results

Take a compact orientable three-dimensional manifold.

Put a Riemannian metric on it.



Take a compact orientable three-dimensional manifold.

Put a Riemannian metric on it.

Run the Ricci flow.



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Put a Riemannian metric on it.

Run the Ricci flow.

What happens?

1. Identify the equilibrium metrics.

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1. Identify the equilibrium metrics.

2. Prove convergence of the Ricci flow to an equilibrium metric.

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Say M is a compact orientable surface.

The Ricci flow equation :

$$\frac{dg}{dt} = -2 \, K \, g.$$

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Here

- 1. $g \equiv g_{ii}$ is a Riemannian metric on *M*.
- 2. *K* is the Gaussian curvature of *g*.

Some explicit solutions

1. The shrinking two-sphere :

$$g(t) = r^2(t) g_{round},$$

 $r^2(t) = r^2(0) - 2t.$

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2. The static two-torus :

$$g(t) = g_{flat}.$$

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2. The static two-torus :

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3. The expanding higher-genus surface :

$$g(t) = r^2(t) g_{hyp},$$

 $r^2(t) = r^2(0) + 2t.$

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From the 2-D Ricci flow equation

$$\frac{dg}{dt} = -2 \, K \, g,$$

we get

$$g(t) = \Phi(t) g(0)$$

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for some positive function $\Phi(t)$.

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Here Φ satisfies the logarithmic fast diffusion equation

$$\frac{\partial \Phi}{\partial t} = \bigtriangleup_{g(0)}(\ln \Phi) - 2K_0.$$

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Results of Hamilton (1988), Chow (1991)

Say g(0) is an arbitrary Riemannian metric on the surface *M*.

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Say g(0) is an arbitrary Riemannian metric on the surface *M*.

1. If *M* is a two-sphere then there is a finite extinction time $T < \infty$. Also,

$$\lim_{t\to T^-}\frac{g(t)}{T-t} = 2 g_{round}.$$

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1. If *M* is a two-sphere then there is a finite extinction time $T < \infty$. Also,

$$\lim_{t\to T^-}\frac{g(t)}{T-t} = 2 g_{round}.$$

2. If *M* is a two-torus then the flow exists for $t \in [0, \infty)$ and

$$\lim_{t\to\infty}g(t)=g_{\textit{flat}}$$

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for some flat metric g_{flat} .

3. If *M* is a higher genus surface then the flow exists for $t \in [0, \infty)$. Putting

$$\widehat{g}(t)=rac{g(t)}{t},$$

we have

$$\lim_{t\to\infty}\widehat{g}(t) = 2 \, g_{hyp}$$

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for some metric g_{hyp} of constant curvature -1.

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Not quite : there's a circularity in the argument.

In the two-sphere case, the Hamilton/Chow convergence proof uses the uniformization theorem.

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Not quite : there's a circularity in the argument.

In the two-sphere case, the Hamilton/Chow convergence proof uses the uniformization theorem.

However, one can get around this (Chen-Lu-Tian 2006).

The question

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Main results

Equilibrium configurations of Ricci flow

Locally homogeneous Ricci flows

Proofs of main results

The Ricci flow equation

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This is a weakly parabolic equation for the Riemannian metric g(t).

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The "de Turck" trick gives an equivalent parabolic equation.

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The "de Turck" trick gives an equivalent parabolic equation.

If *M* is compact then for any initial Riemannian metric g(0) on *M*, there is a smooth Ricci flow solution on some maximal time interval [0, T) with $0 < T \le \infty$.

Some explicit solutions :

- 1. Round shrinking sphere
- 2. Static flat metric
- 3. Expanding hyperbolic metric

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If $(M, g(\cdot))$ is a Ricci flow solution on a surface *M* then we get a Ricci flow solution on $M \times S^1$:

$$h(t)=g(t)+d\theta^2.$$

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Here the S^1 -factor is static.

Take the manifold $M \times S^1$ again, but now allow the length of the circle fiber over $m \in M$ to depend on m.

This gives a warped product metric on $M \times S^1$:

$$h = g + e^{2u} d\theta^2,$$

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where u is a function on the surface M.

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Fact : if the initial metric h(0) is a warped product metric then so is the Ricci flow metric h(t).

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where u is a function on the surface M.

Fact : if the initial metric h(0) is a warped product metric then so is the Ricci flow metric h(t).

As the metric evolves, what happens? An old question in Ricci flow.

The Ricci flow equation for the metric

$$h = g + e^{2u} d\theta^2$$

on $M \times S^1$ becomes the coupled equations on the surface M:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \triangle_{g(t)} u, \\ \frac{\partial g_{ij}}{\partial t} &= -2 \ K \ g_{ij} \ + \ 2(\partial_i u)(\partial_j u). \end{aligned}$$

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Problem : this flow is no longer conformal.

U(1) x U(1) symmetry

If $U(1) \times U(1)$ acts freely on a compact orientable three-dimensional manifold then it is topologically a 3-torus. It fibers over a circle, with the 2-torus fibers being the orbits of the $U(1) \times U(1)$ action.

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A $U(1) \times U(1)$ -invariant metric takes the form

$$g=\sum_{i,j=1}^2 G_{ij}(y)dx^i dx^j+g_{yy}(y)dy^2,$$

where

- ► *y* is a coordinate for the base circle,
- x^1 and x^2 are linear coordinates for the two-torus, and
- $(G_{ij}(y))$ is a positive-definite symmetric 2 × 2 matrix.

The Ricci flow equation becomes

$$\frac{\partial G}{\partial t} = g^{yy} \left(G'' - G' G^{-1} G' \right),$$
$$\frac{\partial g_{yy}}{\partial t} = \frac{1}{2} \operatorname{Tr} \left(\left(G^{-1} G' \right)^2 \right).$$

Periodic boundary condition :

$$G(y+1)=G(y).$$

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Periodic boundary condition :

$$G(y+1)=G(y).$$

Hamilton (1995) : Under some additional assumptions, as $t \to \infty$, the metric approaches a flat metric on T^3 .

Twisted boundary conditions : require

$$G(y+1)=H^TG(y)H,$$

where $H \in SL(2, \mathbb{Z})$. This gives a metric on the 2-torus bundle over a circle with holonomy *H*.

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Twisted boundary conditions : require

$$G(y+1)=H^TG(y)H,$$

where $H \in SL(2, \mathbb{Z})$. This gives a metric on the 2-torus bundle over a circle with holonomy *H*.

Hamilton-Isenberg (1993) : If *H* has distinct real eigenvalues then under some additional assumptions, as $t \to \infty$, the metric approaches a locally homogeneous metric of Sol-type.

From Perelman's first Ricci flow paper :

"The natural questions that remain open are whether the normalized curvatures must stay bounded as $t \to \infty$, and whether reducible manifolds and manifolds with finite fundamental group can have metrics which evolve smoothly by the Ricci flow on the infinite interval."

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The second question was answered in the negative in Perelman's third Ricci flow paper.

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The second question was answered in the negative in Perelman's third Ricci flow paper.

We'll answer the first question in the positive in the special cases of warped products and 2-torus bundles.

The question

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Main results

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Proofs of main results

Joint work with Natasa Sesum (Rutgers)

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Warped products

Theorem

Suppose that $h(\cdot)$ is a Ricci flow on $M \times S^1$ whose initial metric h(0) is a warped product metric over a two-dimensional compact base M.

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Warped products

Theorem

Suppose that $h(\cdot)$ is a Ricci flow on $M \times S^1$ whose initial metric h(0) is a warped product metric over a two-dimensional compact base M.

1. If M is a 2-sphere then there is a finite extinction time $T < \infty$. As $t \to T^-$, the metric on $M \times S^1$ is asymptotic to the product of a shrinking round metric on M with a static metric on S^1 .

Suppose that $h(\cdot)$ is a Ricci flow on $M \times S^1$ whose initial metric h(0) is a warped product metric over a two-dimensional compact base M.

- If M is a 2-sphere then there is a finite extinction time T < ∞. As t → T⁻, the metric on M × S¹ is asymptotic to the product of a shrinking round metric on M with a static metric on S¹.
- 2. If *M* is a 2-torus then the flow exists for $t \in [0, \infty)$. As $t \to \infty$, the metric on $M \times S^1$ approaches a flat metric exponentially fast.

Suppose that $h(\cdot)$ is a Ricci flow on $M \times S^1$ whose initial metric h(0) is a warped product metric over a two-dimensional compact base M.

- 1. If M is a 2-sphere then there is a finite extinction time $T < \infty$. As $t \to T^-$, the metric on $M \times S^1$ is asymptotic to the product of a shrinking round metric on M with a static metric on S^1 .
- 2. If *M* is a 2-torus then the flow exists for $t \in [0, \infty)$. As $t \to \infty$, the metric on $M \times S^1$ approaches a flat metric exponentially fast.
- If M is a higher genus surface then the flow exists for t ∈ [0,∞). The sectional curvatures on M × S¹ are O (t⁻¹). For large t, there is a large region of M on which the curvature of g(t)/t is close to 1/2, and over which the circle fibers have almost constant length.

Suppose that N is the total space of a 2-torus bundle over the circle, with holonomy $H \in SL(2, \mathbb{Z})$. Let $h(\cdot)$ be a Ricci flow solution on N which is locally $U(1) \times U(1)$ invariant. Then the sectional curvatures of (N, h(t)) are $O(t^{-1})$ in magnitude.

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- 1. If H has finite order then as $t \to \infty$, the metric h(t) approaches a flat metric exponentially fast.
- 2. If H is hyperbolic then as $t \to \infty$, the manifold $\left(N, \frac{g(t)}{t}\right)$ approaches a circle in the Gromov-Hausdorff sense. When pulled back to the universal cover, as $t \to \infty$, the Ricci flow solution approaches the homogeneous solution

$$\left(\mathbb{R}^3, e^{-2z} dx^2 + e^{2z} dy^2 + 4 t dz^2\right).$$

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Main results

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Parabolic rescaling

If f(x, t) is a solution of the heat equation

$$\frac{\partial f}{\partial t} = \triangle f$$

on \mathbb{R}^n then so is

$$f_{s}(x,t)=f(\sqrt{s}x,st).$$

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$$f_{s}(x,t)=f(\sqrt{s}x,st).$$

If g(t) is a solution to the Ricci flow equation

$$\frac{dg}{dt} = -2$$
 Ric

then so is

$$g_s(t)=\frac{1}{s}g(st).$$

If $\phi : N \to N$ is a diffeomorphism and g(t) is a Ricci flow solution on N then so is $\phi^*g(t)$.

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The static solutions to the Ricci flow equation

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are the Ricci-flat metrics.

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are the Ricci-flat metrics.

Allowing uniform expansion or contraction, we also get the Einstein metrics

$$\mathsf{Ric} = cg, \quad c \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}.$$

Soliton solutions

If a metric satisfies the steady soliton equation

$$\operatorname{Ric}\,+rac{1}{2}\,\mathcal{L}_V g=0$$

then we get a Ricci flow solution

$$g(t)=\phi_t^*g,$$

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Similarly, there are the shrinking soliton equation

$$\operatorname{Ric} + \frac{1}{2} \mathcal{L}_V g = \frac{1}{2} g$$

and the expanding soliton equation

$$\operatorname{Ric} + \frac{1}{2} \mathcal{L}_V g = -\frac{1}{2} g.$$

Algebraic fact : if N is a three-dimensional manifold then any Einstein metric on N has constant curvature.

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Algebraic fact : if N is a three-dimensional manifold then any Einstein metric on N has constant curvature.

Analytic fact : if N is a compact three-dimensional manifold then any Ricci soliton on N has constant curvature.

The only general result about the long-time asymptotics of three-dimensional Ricci flow :

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The only general result about the long-time asymptotics of three-dimensional Ricci flow :

Theorem

(Perelman 2003) Suppose that N is a compact three-dimensional manifold that admits a hyperbolic metric g_{hyp} . Let g(0) be any Riemannian metric on N. Run the Ricci flow starting with g(0). Then

- There is a finite number of surgeries.
- As $t \to \infty$, the rescaled metric $\frac{g(t)}{t}$ approaches $4g_{hyp}$.

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But the only quasistatic Ricci flow solutions on compact three-dimensional manifolds have constant curvature.

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What happens?

The question

- Two-dimensional Ricci flow
- Three-dimensional Ricci flows with symmetry

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Main results

Equilibrium configurations of Ricci flow

Locally homogeneous Ricci flows

Proofs of main results

A Riemannian manifold is locally homogeneous if any two points have isometric neighborhoods.

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Thurston's geometrization conjecture = Perelman's theorem :

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Any compact three-dimensional manifold *N* can be decomposed into pieces that admit finite-volume locally homogeneous metrics.

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Thurston's geometrization conjecture = Perelman's theorem :

Any compact three-dimensional manifold *N* can be decomposed into pieces that admit finite-volume locally homogeneous metrics.

Suppose that *N* has a locally homogeneous metric. Run the Ricci flow. What happens?

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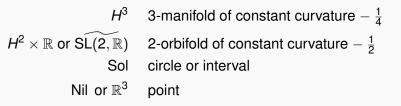
If *N* is locally homogeneous then its universal cover \widetilde{N} is a globally homogeneous space G/K.

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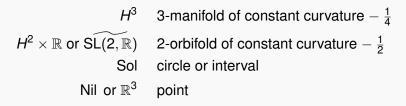
The Ricci flow on G/K reduces to a system of ODE's. In three dimensions, the system can either be solved explicitly or its asymptotics can be computed (Isenberg-Jackson 1992).

Thurston type GH limit of
$$\left(N, \frac{g(t)}{t}\right)$$



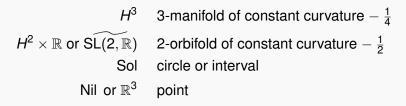
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In general, the compact 3-manifold *N* collapses to a lower-dimensional space.

Thurston type GH limit of
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In general, the compact 3-manifold *N* collapses to a lower-dimensional space.

Let's pass to the universal cover \widetilde{N} .

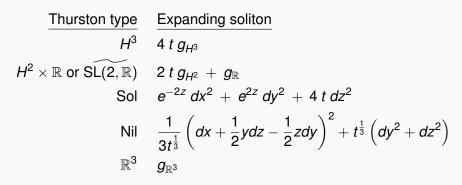
Proposition

(L. 2007) Suppose that N is a compact locally homogeneous three-dimensional manifold whose Ricci flow exists for $t \in [0, \infty)$. On the universal cover \widetilde{N} , the rescaled pullback metric approaches a homogeneous expanding soliton.

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The meaning of the limit in the Proposition :

Let $g(\cdot)$ be a Ricci flow solution on *N* that exists for $t \in [0, \infty)$. For s > 0, define a blowdown solution by

$$g_s(t) = \frac{1}{s}g(st).$$

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The meaning of the limit in the Proposition :

Let $g(\cdot)$ be a Ricci flow solution on *N* that exists for $t \in [0, \infty)$. For s > 0, define a blowdown solution by

$$g_s(t) = \frac{1}{s}g(st).$$

Then on the universal cover \widetilde{N} , there are diffeomorphisms $\{\phi_s\}$ so that

$$\lim_{s\to\infty}\phi_s^*\,\widetilde{g}_s = g_{expander},$$

with smooth convergence on compact subsets.

The question

- Two-dimensional Ricci flow
- Three-dimensional Ricci flows with symmetry

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Main results

- Equilibrium configurations of Ricci flow
- Locally homogeneous Ricci flows

Proofs of main results

Theorem

(L. 2010) Suppose that $(N, g(\cdot))$ is a Ricci flow on a compact three-dimensional manifold, that exists for $t \in [0, \infty)$. Suppose that the sectional curvatures are $O(t^{-1})$ in magnitude, and the diameter is $O(\sqrt{t})$. Then the pullback of the Ricci flow to \widetilde{N} approaches a homogeneous expanding soliton.

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Remarks :

- > The hypotheses are invariant under parabolic rescaling.
- The hypotheses imply that there is only one piece in the Thurston decomposition of *N*, so *N* admits a locally homogeneous metric.
- ► One can also describe the Gromov-Hausdorff limit of (N, g(·)).

1. Monotonic quantities : modifications of Perelman's *W*-functional.

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2. Compactness theorem : an extension of Hamilton's compactness theorem for Ricci flows, to remove the assumption of a lower bound on the injectivity radius.

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1. Monotonic quantities : modifications of Perelman's *W*-functional.

2. Compactness theorem : an extension of Hamilton's compactness theorem for Ricci flows, to remove the assumption of a lower bound on the injectivity radius.

3. Contradiction arguments, using blowdown limits.

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In the case of warped products and 2-torus bundles, the main work is to show that the sectional curvatures are $O(t^{-1})$ in magnitude, and that the diameter is $O(\sqrt{t})$.

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In the warped product case, the sectional curvature bound comes from a contradiction argument, using blowdown limits and the Gauss-Bonnet theorem. In the case of warped products and 2-torus bundles, the main work is to show that the sectional curvatures are $O(t^{-1})$ in magnitude, and that the diameter is $O(\sqrt{t})$.

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In the case of torus bundles, the diameter bound comes from the evolution formula for lengths, along with the maximum principle.

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What could go wrong :

As $t \to \infty$, the metric $\frac{g(t)}{t}$ on the surface could describe a family of hyperbolic surfaces that slowly degenerate.

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2. For a torus bundle with parabolic $H \in SL(2, \mathbb{Z})$, a diameter bound on the torus fibers is missing. There is a uniform volume bound for the fibers.

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3. What about the general case of a free U(1) action?