Evolution of three-dimensional Ricci flow

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Question

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Put a Riemannian metric on it.

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What happens?

The goal

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2. Prove convergence of the Ricci flow to an equilibrium metric.

Warmup case

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Here

1. $g \equiv g_{ij}$ is a Riemannian metric on M.
2. K is the Gaussian curvature of g.

Some explicit solutions

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$$g(t) = g_{flat}.$$

3. The expanding higher-genus surface :

$$g(t) \;=\; r^2(t)\, g_{hyp},$$

$$r^2(t) = r^2(0) + 2t.$$

Conformal invariance

From the 2-D Ricci flow equation

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we get

$$g(t) = \Phi(t)\,g(0)$$

for some positive function $\Phi(t)$.

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Here Φ satisfies the logarithmic fast diffusion equation

$$\frac{\partial \Phi}{\partial t} = \triangle_{g(0)}(\ln \Phi) - 2K_0.$$

Results of Hamilton (1988), Chow (1991)

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$$\lim_{t \to T^-} \frac{g(t)}{T - t} = 2\, g_{round}.$$

2. If M is a two-torus then the flow exists for $t \in [0, \infty)$ and

$$\lim_{t \to \infty} g(t) = g_{flat}$$

for some flat metric g_{flat}.

Hamilton/Chow

3. If M is a higher genus surface then the flow exists for $t \in [0, \infty)$. Putting

$$\widehat{g}(t) = \frac{g(t)}{t},$$

we have

$$\lim_{t \to \infty} \widehat{g}(t) \;=\; 2\, g_{hyp}$$

for some metric g_{hyp} of constant curvature -1.

Uniformization theorem

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However, one can get around this (Chen-Lu-Tian 2006).

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The "de Turck" trick gives an equivalent parabolic equation.

If M is compact then for any initial Riemannian metric $g(0)$ on M, there is a smooth Ricci flow solution on some maximal time interval $[0, T)$ with $0 < T \leq \infty$.

Constant curvature solutions

Some explicit solutions :

1. Round shrinking sphere
2. Static flat metric
3. Expanding hyperbolic metric

Product solutions

If $(M, g(\cdot))$ is a Ricci flow solution on a surface M then we get a Ricci flow solution on $M \times S^1$:

$$h(t) = g(t) + d\theta^2.$$

Here the S^1-factor is static.

First nontrivial case

Take the manifold $M \times S^1$ again, but now allow the length of the circle fiber over $m \in M$ to depend on m.

This gives a warped product metric on $M \times S^1$:

$$h = g + e^{2u} d\theta^2,$$

where u is a function on the surface M.

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Fact : if the initial metric $h(0)$ is a warped product metric then so is the Ricci flow metric $h(t)$.

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As the metric evolves, what happens? An old question in Ricci flow.

Warped product Ricci flow

The Ricci flow equation for the metric

$$h = g + e^{2u} d\theta^2$$

on $M \times S^1$ becomes the coupled equations on the surface M :

$$\begin{aligned} \frac{\partial u}{\partial t} &= \triangle_{g(t)} u, \\ \frac{\partial g_{ij}}{\partial t} &= -2\, K\, g_{ij} \, + \, 2(\partial_i u)(\partial_j u). \end{aligned}$$

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Problem : this flow is no longer conformal.

U(1) x U(1) symmetry

If $U(1) \times U(1)$ acts freely on a compact orientable three-dimensional manifold then it is topologically a 3-torus. It fibers over a circle, with the 2-torus fibers being the orbits of the $U(1) \times U(1)$ action.

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A $U(1) \times U(1)$-invariant metric takes the form

$$g = \sum_{i,j=1}^{2} G_{ij}(y)dx^i dx^j + g_{yy}(y)dy^2,$$

where

- y is a coordinate for the base circle,
- x^1 and x^2 are linear coordinates for the two-torus, and
- $\big(G_{ij}(y)\big)$ is a positive-definite symmetric 2×2 matrix.

Flow equation

The Ricci flow equation becomes

$$\begin{aligned}\frac{\partial G}{\partial t} &= g^{yy}\left(G'' - G' G^{-1} G'\right), \\ \frac{\partial g_{yy}}{\partial t} &= \frac{1}{2}\operatorname{Tr}\left(\left(G^{-1} G'\right)^2\right).\end{aligned}$$

Periodic boundary condition :

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Periodic boundary condition :

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Hamilton (1995) : Under some additional assumptions, as $t \to \infty$, the metric approaches a flat metric on T^3.

Twisted torus bundles

Twisted boundary conditions : require

$$G(y+1) = H^T G(y) H,$$

where $H \in \mathrm{SL}(2, \mathbb{Z})$. This gives a metric on the 2-torus bundle over a circle with holonomy H.

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Hamilton-Isenberg (1993) : If H has distinct real eigenvalues then under some additional assumptions, as $t \to \infty$, the metric approaches a locally homogeneous metric of Sol-type.

Natural questions

From Perelman's first Ricci flow paper :

"The natural questions that remain open are whether the normalized curvatures must stay bounded as $t \to \infty$, and whether reducible manifolds and manifolds with finite fundamental group can have metrics which evolve smoothly by the Ricci flow on the infinite interval."

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The second question was answered in the negative in Perelman's third Ricci flow paper.

We'll answer the first question in the positive in the special cases of warped products and 2-torus bundles.

Results

Joint work with Natasa Sesum (Rutgers)

Warped products

Theorem

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1. *If M is a 2-sphere then there is a finite extinction time $T < \infty$. As $t \to T^-$, the metric on $M \times S^1$ is asymptotic to the product of a shrinking round metric on M with a static metric on S^1.*

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2. *If M is a 2-torus then the flow exists for $t \in [0, \infty)$. As $t \to \infty$, the metric on $M \times S^1$ approaches a flat metric exponentially fast.*

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2. *If M is a 2-torus then the flow exists for $t \in [0, \infty)$. As $t \to \infty$, the metric on $M \times S^1$ approaches a flat metric exponentially fast.*
3. *If M is a higher genus surface then the flow exists for $t \in [0, \infty)$. The sectional curvatures on $M \times S^1$ are $O\left(t^{-1}\right)$. For large t, there is a large region of M on which the curvature of $\frac{g(t)}{t}$ is close to $-\frac{1}{2}$, and over which the circle fibers have almost constant length.*

Torus bundles

Theorem

Suppose that N is the total space of a 2-torus bundle over the circle, with holonomy $H \in \mathrm{SL}(2, \mathbb{Z})$. Let $h(\cdot)$ be a Ricci flow solution on N which is locally $U(1) \times U(1)$ invariant. Then the sectional curvatures of $(N, h(t))$ are $O\left(t^{-1}\right)$ in magnitude.

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1. *If H has finite order then as $t \to \infty$, the metric $h(t)$ approaches a flat metric exponentially fast.*

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1. *If H has finite order then as $t \to \infty$, the metric $h(t)$ approaches a flat metric exponentially fast.*
2. *If H is hyperbolic then as $t \to \infty$, the manifold $\left(N, \frac{g(t)}{t}\right)$ approaches a circle in the Gromov-Hausdorff sense. When pulled back to the universal cover, as $t \to \infty$, the Ricci flow solution approaches the homogeneous solution*

$$\left(\mathbb{R}^3, e^{-2z}\, dx^2 + e^{2z}\, dy^2 + 4\, t\, dz^2\right).$$

Evolution of three-dimensional Ricci flow

Parabolic rescaling

If $f(x, t)$ is a solution of the heat equation

$$\frac{\partial f}{\partial t} = \triangle f$$

on $\mathbb{R}^n$ then so is

$$f_s(x, t) = f(\sqrt{s}x, st).$$

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If $g(t)$ is a solution to the Ricci flow equation

$$\frac{dg}{dt} = -2 \text{ Ric}$$

then so is

$$g_s(t) = \frac{1}{s} g(st).$$

Diffeomorphism invariance

If $\phi : N \to N$ is a diffeomorphism and $g(t)$ is a Ricci flow solution on N then so is $\phi^* g(t)$.

Static solutions

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Allowing uniform expansion or contraction, we also get the Einstein metrics

$$\mathrm{Ric} = cg, \qquad c \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}.$$

Soliton solutions

If a metric satisfies the steady soliton equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g = 0$$

then we get a Ricci flow solution

$$g(t) = \phi_t^* g,$$

where $\{\phi_t\}$ is the flow generated by the vector field V.

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Similarly, there are the shrinking soliton equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g = \frac{1}{2} g$$

and the expanding soliton equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g = -\frac{1}{2} g.$$

3D solitons

Algebraic fact : if N is a three-dimensional manifold then any Einstein metric on N has constant curvature.

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Analytic fact : if N is a compact three-dimensional manifold then any Ricci soliton on N has constant curvature.

Hyperbolizable manifolds

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Theorem

(Perelman 2003) Suppose that N is a compact three-dimensional manifold that admits a hyperbolic metric g_{hyp}. Let $g(0)$ be any Riemannian metric on N. Run the Ricci flow starting with $g(0)$. Then

- *There is a finite number of surgeries.*
- *As $t \to \infty$, the rescaled metric $\frac{g(t)}{t}$ approaches $4g_{hyp}$.*

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What happens?

Locally homogeneous manifolds

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Any compact three-dimensional manifold N can be decomposed into pieces that admit finite-volume locally homogeneous metrics.

Suppose that N has a locally homogeneous metric. Run the Ricci flow. What happens?

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If N is locally homogeneous then its universal cover $\widetilde{N}$ is a globally homogeneous space G/K.

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The Ricci flow on G/K reduces to a system of ODE's. In three dimensions, the system can either be solved explicitly or its asymptotics can be computed (Isenberg-Jackson 1992).

Gromov-Hausdorff limits

Thurston type	GH limit of $\left(N, \frac{g(t)}{t}\right)$
H^3	3-manifold of constant curvature $-\frac{1}{4}$
$H^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}(2,\mathbb{R})}$	2-orbifold of constant curvature $-\frac{1}{2}$
Sol	circle or interval
Nil or $\mathbb{R}^3$	point

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In general, the compact 3-manifold N collapses to a lower-dimensional space.

Let's pass to the universal cover $\widetilde{N}$.

Expanding solitons

Proposition

(L. 2007) Suppose that N is a compact locally homogeneous three-dimensional manifold whose Ricci flow exists for $t \in [0, \infty)$. *On the universal cover* $\widetilde{N}$, *the rescaled pullback metric approaches a homogeneous expanding soliton.*

Expanding solitons

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Thurston type	Expanding soliton
H^3	$4\, t\, g_{H^3}$
$H^2 \times \mathbb{R}$ or $\widetilde{SL(2, \mathbb{R})}$	$2\, t\, g_{H^2} + g_{\mathbb{R}}$
Sol	$e^{-2z}\, dx^2 + e^{2z}\, dy^2 + 4\, t\, dz^2$
Nil	$\frac{1}{3t^{\frac{1}{3}}} \left(dx + \frac{1}{2} ydz - \frac{1}{2} zdy \right)^2 + t^{\frac{1}{3}} \left(dy^2 + dz^2 \right)$
$\mathbb{R}^3$	$g_{\mathbb{R}^3}$

Blowdown limits

The meaning of the limit in the Proposition :

Let $g(\cdot)$ be a Ricci flow solution on N that exists for $t \in [0, \infty)$.
For $s > 0$, define a blowdown solution by

$$g_s(t) = \frac{1}{s} g(st).$$

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Let $g(\cdot)$ be a Ricci flow solution on N that exists for $t \in [0, \infty)$. For $s > 0$, define a blowdown solution by

$$g_s(t) \;=\; \frac{1}{s}\, g(st).$$

Then on the universal cover $\widetilde{N}$, there are diffeomorphisms $\{\phi_s\}$ so that

$$\lim_{s \to \infty} \phi_s^* \, \widetilde{g}_s \;=\; g_{expander},$$

with smooth convergence on compact subsets.

Main tool : a convergence criterion

Theorem

(L. 2010) Suppose that $(N, g(\cdot))$ *is a Ricci flow on a compact three-dimensional manifold, that exists for t* $\in [0, \infty)$. *Suppose that the sectional curvatures are* $O\left(t^{-1}\right)$ *in magnitude, and the diameter is* $O(\sqrt{t})$. *Then the pullback of the Ricci flow to* $\widetilde{N}$ *approaches a homogeneous expanding soliton.*

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(L. 2010) Suppose that $(N, g(\cdot))$ is a Ricci flow on a compact three-dimensional manifold, that exists for $t \in [0, \infty)$. Suppose that the sectional curvatures are $O\left(t^{-1}\right)$ in magnitude, and the diameter is $O(\sqrt{t})$. Then the pullback of the Ricci flow to $\widetilde{N}$ approaches a homogeneous expanding soliton.

Remarks :

- The hypotheses are invariant under parabolic rescaling.
- The hypotheses imply that there is only one piece in the Thurston decomposition of N, so N admits a locally homogeneous metric.
- One can also describe the Gromov-Hausdorff limit of $(N, g(\cdot))$.

Ingredients of the proof

1. Monotonic quantities : modifications of Perelman's W-functional.

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2. Compactness theorem : an extension of Hamilton's compactness theorem for Ricci flows, to remove the assumption of a lower bound on the injectivity radius.

3. Contradiction arguments, using blowdown limits.

Warped products and torus bundles

In the case of warped products and 2-torus bundles, the main work is to show that the sectional curvatures are $O\left(t^{-1}\right)$ in magnitude, and that the diameter is $O(\sqrt{t})$.

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In the warped product case, the sectional curvature bound comes from a contradiction argument, using blowdown limits and the Gauss-Bonnet theorem.

In the case of torus bundles, the diameter bound comes from the evolution formula for lengths, along with the maximum principle.

What's missing so far

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2. For a torus bundle with parabolic $H \in \mathrm{SL}(2, \mathbb{Z})$, a diameter bound on the torus fibers is missing. There is a uniform volume bound for the fibers.

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3. What about the general case of a free $U(1)$ action?