COMPARISON GEOMETRY OF HOLOMORPHIC BISECTIONAL CURVATURE FOR KÄHLER MANIFOLDS AND LIMIT SPACES

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Abstract

We give an analogue of triangle comparison for Kähler manifolds with a lower bound on the holomorphic bisectional curvature. We show that the condition passes to noncollapsed Gromov–Hausdorff limits. We discuss tangent cones and singular Kähler spaces.

1. Introduction

Holomorphic bisectional curvature is a Kähler analogue of Riemannian sectional curvature. We recall the definition in Section 3. There is a well-developed theory of Riemannian manifolds with lower sectional curvature bounds, including such topics as triangle comparison, Gromov–Hausdorff limits, and Alexandrov spaces. The goal of this paper is to give Kähler analogues.

To state the first main result, we define a modified distance-squared function. Given $d \geq 0$ and $K \in \mathbb{R}$, define $d_K \geq 0$ by

$$d_K^2 = \begin{cases} -\frac{4}{K} \log \cos\left(d\sqrt{\frac{K}{2}}\right) & \text{if } K > 0, \\ d^2 & \text{if } K = 0, \\ \frac{4}{-K} \log \cosh\left(d\sqrt{\frac{-K}{2}}\right) & \text{if } K < 0. \end{cases} \tag{1.1}$$

(If $K > 0$, then we restrict to $d \leq \frac{\pi}{\sqrt{2K}}$.) Let M be a complete Kähler manifold. Given $p \in M$ and $K \in \mathbb{R}$, let $d_p \in C(M)$ be the distance from p, and define $d_{K,p}$ using (1.1), replacing the d on the right-hand side by d_p.

We write $BK \geq K$ if the holomorphic bisectional curvatures of M are bounded below by $K \in \mathbb{R}$. We prove the following analogue of triangle comparison.

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THEOREM 1.2
Let M be a complete Kähler manifold. Given $K \in \mathbb{R}$, the manifold M has $BK \geq K$ if and only if it satisfies the following property. Let $i : \overline{D^2} \to M$ be an embedding of a disk into M that is holomorphic on D^2. Let Σ be the image of i. Let dA denote the area form on Σ. Let z be the local coordinate on D^2, and let $\theta \in [0, 2\pi)$ be the local coordinate on $\partial \overline{D^2}$. Then

$$d_{K,p}^2(0) \geq \frac{2}{\pi} \iint_\Sigma \log |z| dA + \frac{1}{2\pi} \int_{\partial\Sigma} d_{K,p}^2(\theta)\, d\theta, \tag{1.3}$$

where the "0" on the left-hand side denotes $i(0)$, the center of Σ.

Next, we consider noncollapsing sequences of complete pointed Kähler manifolds with $BK \geq K$. Lee and Tam [22] showed that after passing to a subsequence, there is a pointed Gromov–Hausdorff limit that is a complex manifold. Regarding its geometry, we show that (1.3) holds on the limit.

THEOREM 1.4
Let $\{(M_i, p_i, g_i)\}_{i=1}^\infty$ be a sequence of pointed n-dimensional complete Kähler manifolds with $BK \geq K$. Suppose that there is some $v_0 > 0$ so that for all i, we have $\mathrm{vol}(B(p_i, 1)) \geq v_0$. Then after passing to a subsequence, there is a pointed Gromov-Hausdorff limit $(X_\infty, p_\infty, d_\infty)$ with the following properties.
(1) *X_∞ is a complex manifold.*
(2) *Embedded holomorphic disks Σ in X_∞ satisfy (1.3), where dA is now the 2-dimensional Hausdorff measure coming from d_∞.*

Some simple examples of such limit spaces come from 2-dimensional length spaces with Alexandrov curvature bounded below. The proof of Theorem 1.4 uses local Ricci flow techniques as developed by Bamler, Cabezas-Rivas, and Wilking [1], Hochard [16], Lee and Tam [21], and Simon and Topping [38].

The content of the article is as follows. In Section 2 we briefly recall some facts about Riemannian manifolds with nonnegative sectional curvature, and their Gromov–Hausdorff limits. In Section 3 we show that:

- a complete Kähler manifold has $BK \geq K$ if and only if $\sqrt{-1}\partial\overline{\partial} d_{K,p}^2 / 2 \leq \omega$ as currents;
- Theorem 1.2 holds;
- if a Hermitian manifold satisfies (1.3), then it must be Kähler;
- a domain M in a model space (of constant holomorphic sectional curvature) satisfies (1.3) if and only if the length metric on M is the same as the restricted metric from the model space.

Section 4 is about noncollapsed pointed Gromov–Hausdorff limits. We prove Theorem 1.4 and construct local Kähler potentials $\{\phi_\alpha\}$ on the limit space.

In Section 5 we give a notion of "$BK \geq K$" (enclosed in quotation marks to distinguish it from the condition $BK \geq K$ for smooth Kähler manifolds) for possibly singular complex spaces. We use the notion of Kähler spaces from [31], which is formulated in terms of local potential functions $\{\phi_\alpha\}$. We define metric Kähler spaces and an associated complex Gromov–Hausdorff convergence, which may be of independent interest. We say that a metric Kähler space has "$BK \geq K$" if $\phi_\alpha - d^2_{K,p}/2$ is plurisubharmonic for all α and p. For normal complex spaces, this is equivalent to (1.3) being satisfied. The following properties hold.

- Given a sequence of metric Kähler spaces with "$BK \geq K$," if it converges in the pointed complex Gromov–Hausdorff sense, then the limit space has "$BK \geq K$."
- Under the assumptions of Theorem 1.4, a subsequence converges in the pointed complex Gromov–Hausdorff sense.
- If a Kähler orbifold has "$BK \geq K$" in the sense of curvature tensors, then its underlying length space has "$BK \geq K$."

Section 6 is about tangent cones of the limit spaces from Theorem 1.4. We show that:

- a tangent cone is a Kähler cone that is biholomorphic to $\mathbb{C}^n$;
- when the distance function from the vertex is radially homogeneous on $\mathbb{C}^n$, the tangent cone is an affine cone over a copy of $\mathbb{C}P^{n-1}$ with "$BK \geq 2$," in the sense of the previous section.

2. Some facts from Riemannian comparison geometry

Let (M, g) be a complete Riemannian manifold. We consider lower sectional curvature bounds; for simplicity, we assume that (M, g) has nonnegative sectional curvature. Given $p \in M$, let $d_p \in C(M)$ denote the Riemannian distance from p. Then

$$\mathrm{Hess}(d_p^2/2) \leq g \tag{2.1}$$

away from the cut locus C_p of p.

Let $\{\gamma(t)\}_{t\in[0,L]}$ be a unit-speed geodesic in $M - C_p$. For brevity, we write $d_p(t)$ for $d_p(\gamma(t))$. It follows from (2.1) that $\frac{d^2}{dt^2}(d_p^2(t)/2) \leq 1$, that is, $\frac{d^2}{dt^2}(d_p^2(t)/2 - t^2/2) \leq 0$. In other words, $d_p^2(t) - t^2$ is concave on $[0, L]$. Then

$$d_p^2(t) - t^2 \geq \frac{t}{L}\big(d_p^2(L) - L^2\big) + \Big(1 - \frac{t}{L}\Big) d_p^2(0) \tag{2.2}$$

or

$$d_p^2(t) \geq \frac{t}{L} d_p^2(L) + \Big(1 - \frac{t}{L}\Big) d_p^2(0) - t(L - t). \tag{2.3}$$

Toponogov's theorem says that (2.3) remains true without the restriction that γ lies in $M - C_p$.

Remark 2.4
We state some facts without proof.

(1) Equation (2.3), when applied to minimizing geodesics, passes to pointed Gromov–Hausdorff limits. That is, such a limit is a complete length space with nonnegative Alexandrov curvature.
(2) A noncollapsed limit is a topological manifold (see [34]).
(3) A tangent cone of a noncollapsed limit is a metric cone. Its link has Alexandrov curvature bounded below by 1 (see [2, Corollary 7.10]) and is homeomorphic to a sphere (see [17, Theorem 1.3]).
(4) A Finsler manifold with nonnegative Alexandrov curvature is a Riemannian manifold.
(5) A polytope in Euclidean space, that is, a connected finite union of top-dimensional simplices, has nonnegative Alexandrov curvature, with respect to the length metric, if and only if it is convex.

3. Comparison geometry for Kähler manifolds with lower bounds on holomorphic bisectional curvature

3.1. Holomorphic bisectional curvature

Let M be an n-dimensional Kähler manifold. We let ω denote its Kähler form. In terms of holomorphic normal coordinates at a point p, we have $\omega(p) = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz^i \wedge d\overline{z}^i$.

Suppose that $n \geq 2$. Given $p \in M$, if σ and σ' are J-invariant 2-planes (i.e., complex lines) in T_pM, write $\sigma = \operatorname{span}(X, JX)$ and $\sigma' = \operatorname{span}(Y, JY)$ for unit vectors X and Y. The holomorphic bisectional curvature of σ and σ' is $H(\sigma, \sigma') = R(X, JX, Y, JY)$. If $\sigma = \sigma'$, then the holomorphic sectional curvature of σ is $H(\sigma, \sigma)$. From the Bianchi identity,

$$R(X, JX, Y, JY) = R(X, Y, X, Y) + R(X, JY, X, JY). \tag{3.1}$$

In particular,

$$\begin{aligned}(\text{sect. curv. } \geq \text{const.}) &\implies (\text{holo. bisec. curv. } \geq \text{const.}) \\ &\implies (\text{Ricci curv. } \geq \text{const.})\end{aligned} \tag{3.2}$$

where the constants are related by n-dependent factors. Given $K \in \mathbb{R}$, we say that $BK \geq K$ if all of the holomorphic bisectional curvatures are bounded below by K.

We use the curvature notation of [20, Chapter 9]. In particular, if $\{e_i, e_j\}$ are elements of a unitary frame, then the corresponding holomorphic bisectional curvature is $-R_{i\bar{i}j\bar{j}}$. (Note the minus sign.) Hence $BK \geq K$ if and only if we have

$$- R(X, \overline{X}, Y, \overline{Y}) \geq K\big(\langle X, \overline{X}\rangle\langle Y, \overline{Y}\rangle + \langle X, \overline{Y}\rangle\langle Y, \overline{X}\rangle\big) \tag{3.3}$$

for all $X, Y \in T^{(1,0)}M$. (If $n = 1$, then to be consistent with (3.3), we say that $BK \geq K$ if the holomorphic sectional curvatures are bounded below by $2K$.)

The metric on $\mathbb{C}P^n$ with constant holomorphic sectional curvature c is

$$g_{i\bar{j}} = \frac{4}{c}\partial_i\overline{\partial}_j \log\Big(1 + \frac{c}{4}|z|^2\Big) \tag{3.4}$$

with curvature tensor

$$R_{i\bar{j}k\bar{l}} = -\frac{c}{2}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}). \tag{3.5}$$

The Riemannian sectional curvatures lie in $[\frac{c}{4}, c]$. The holomorphic bisectional curvatures lie in $[\frac{c}{2}, c]$. The diameter is $\pi c^{-\frac{1}{2}}$. (If $n = 1$, then the Riemannian sectional curvature and the holomorphic bisectional curvature are c, and the diameter is $\pi c^{-\frac{1}{2}}$.)

If $BK \geq K > 0$, then $\operatorname{diam}(M) \leq \frac{\pi}{\sqrt{2K}}$ (see [23]). It seems to be open whether equality implies that (M, g) is the Fubini–Study metric on $\mathbb{C}P^n$, up to a constant (see [28], [40]).

A compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to a complex projective space (see [33], [39]). The nonnegative case was described in [32]. Alternative proofs of these results, along with extensions to transverse Sasakian geometry, are in [14] and [15].

3.2. *Differential inequality for smooth Kähler manifolds*

We now give a Kähler analogue of (2.1), for $BK \geq K$.

For $p \in M$, let d_p denote the distance function from p, and define $d_{K,p}$ using (1.1), with d replaced by d_p.

PROPOSITION 3.6

Let M be a complete Kähler manifold. If $BK \geq K$, then for all $p \in M$,

$$\sqrt{-1}\partial\overline{\partial}d_{K,p}^2/2 \leq \omega \tag{3.7}$$

as currents on M.

Proof

Suppose that $BK \geq K$. (If $K > 0$, then we initially restrict to the case when $\operatorname{diam}(M) < \frac{\pi}{\sqrt{2K}}$.) It follows from [40, Theorem 2.1], along with some calculation, that (3.7) is satisfied smoothly away from the cut locus of p. Given $q \in M - \{p\}$, let ϕ be a local Kähler potential in a neighborhood U of q, that is, $\omega = \sqrt{-1}\partial\overline{\partial}\phi$. We can assume that $p \notin U$. To prove (3.7), we wish to show that $\phi - d_{K,p}^2/2$ is plurisubharmonic. For this, it suffices to show that it is subharmonic on any embedded holomorphic disk Σ in U, that is, that $\triangle_\Sigma d_{K,p}^2 \leq 4$ as measures on Σ.

Given $m \in \Sigma$, we will construct a barrier function at m. Let $\gamma : [0, d(p,m)] \to M$ be a minimizing unit-speed geodesic from p to m. Let F_K be the function appearing on the right-hand side of (1.1), so $d_K^2 = F_K \circ d$. Then $F_K' \geq 0$ and $F_K'' \geq 0$. For small $\epsilon > 0$, consider $F_K \circ (d_{\gamma(\epsilon)} + \epsilon)$. Its value at m is $d_{K,p}^2(m)$. As $d_{\gamma(\epsilon)} + \epsilon \geq d_p$, it follows that $F_K \circ (d_{\gamma(\epsilon)} + \epsilon) \geq d_{K,p}^2$.

Since m is not in the cut locus of $\gamma(\epsilon)$, we now know that

$$\triangle_\Sigma (F_K \circ d_{\gamma(\epsilon)}) \leq 4 \tag{3.8}$$

in a neighborhood of m in Σ. As

$$\triangle_\Sigma (F_K \circ d_{\gamma(\epsilon)}) = (F_K'' \circ d_{\gamma(\epsilon)}) |\nabla_\Sigma d_{\gamma(\epsilon)}|^2 + (F_K' \circ d_{\gamma(\epsilon)}) \triangle_\Sigma d_{\gamma(\epsilon)}, \tag{3.9}$$

it follows that

$$(F_K' \circ d_{\gamma(\epsilon)}) \triangle_\Sigma d_{\gamma(\epsilon)} \leq 4 - (F_K'' \circ d_{\gamma(\epsilon)}) |\nabla_\Sigma d_{\gamma(\epsilon)}|^2 \leq 4, \tag{3.10}$$

so

$$\triangle_\Sigma d_{\gamma(\epsilon)} \leq \frac{4}{F_K' \circ d_{\gamma(\epsilon)}}, \tag{3.11}$$

where the denominator is strictly positive in a neighborhood of m.

Similarly,

$$\begin{aligned} &\triangle_\Sigma \big(F_K \circ (d_{\gamma(\epsilon)} + \epsilon)\big) \\ &\qquad = \big(F_K'' \circ (d_{\gamma(\epsilon)} + \epsilon)\big) |\nabla_\Sigma d_{\gamma(\epsilon)}|^2 + \big(F_K' \circ (d_{\gamma(\epsilon)} + \epsilon)\big) \triangle_\Sigma d_{\gamma(\epsilon)}. \end{aligned} \tag{3.12}$$

Combining with (3.10) and (3.11) gives

$$\begin{aligned} &\triangle_\Sigma \big(F_K \circ (d_{\gamma(\epsilon)} + \epsilon)\big) \\ &\quad \leq \Big(\big(F_K'' \circ (d_{\gamma(\epsilon)} + \epsilon)\big) - (F_K'' \circ d_{\gamma(\epsilon)})\Big) |\nabla_\Sigma d_{\gamma(\epsilon)}|^2 \\ &\qquad + \Big(\big(F_K' \circ (d_{\gamma(\epsilon)} + \epsilon)\big) - (F_K' \circ d_{\gamma(\epsilon)})\Big) \triangle_\Sigma d_{\gamma(\epsilon)} + 4 \\ &\quad = \Big(\big(F_K'' \circ (d_{\gamma(\epsilon)} + \epsilon)\big) - (F_K'' \circ d_{\gamma(\epsilon)})\Big) \\ &\qquad + \Big(\big(F_K' \circ (d_{\gamma(\epsilon)} + \epsilon)\big) - (F_K' \circ d_{\gamma(\epsilon)})\Big) \triangle_\Sigma d_{\gamma(\epsilon)} + 4 \\ &\quad \leq \Big(\big(F_K'' \circ (d_{\gamma(\epsilon)} + \epsilon)\big) - (F_K'' \circ d_{\gamma(\epsilon)})\Big) \end{aligned}$$

$$+\big((F_K' \circ (d_{\gamma(\epsilon)}+\epsilon)) - (F_K' \circ d_{\gamma(\epsilon)})\big)\frac{4}{F_K' \circ d_{\gamma(\epsilon)}} + 4. \tag{3.13}$$

Given $\epsilon' > 0$, using the continuity of F_K' and F_K'', by choosing ϵ small enough we can ensure that $\triangle_\Sigma(F_K \circ (d_{\gamma(\epsilon)}+\epsilon)) \leq 4+\epsilon'$ in a neighborhood of m in Σ. Thus $\triangle_\Sigma d_{K,p}^2 \leq 4$ in the barrier sense, hence in the viscosity sense and in the distributional sense. This means that $\phi - d_{K,p}^2/2$ is subharmonic on Σ. Thus (3.7) holds.

Now suppose that $K > 0$ and $\operatorname{diam}(M) = \frac{\pi}{\sqrt{2K}}$. Given $\lambda \in (0,1)$, the metric g also has $BK \geq \lambda^2 K$, while $\operatorname{diam}(M) < \frac{\pi}{\sqrt{2\lambda^2 K}}$. Hence $\phi + \frac{2}{\lambda^2 K}\log\cos(\lambda d_p\sqrt{\frac{K}{2}})$ is plurisubharmonic, that is, $\lambda^2\phi + \frac{2}{K}\log\cos(\lambda d_p\sqrt{\frac{K}{2}})$ is plurisubharmonic. Using the fact that $\frac{2}{K}\log\cos(\lambda d_p\sqrt{\frac{K}{2}})$ is monotonically nonincreasing in λ as $\lambda \to 1$, we can pass to the limit to conclude that $\phi + \frac{2}{K}\log\cos(d_p\sqrt{\frac{K}{2}})$ is plurisubharmonic (cf. [8, Proofs of Theorems I.4.15 and I.5.4]). This proves the proposition. □

Remark 3.14
If $K = 0$, then Proposition 3.6 was proved in [3] by very different means.

3.3. *Integral comparison inequality*

We now wish to give an analogue of (2.3). Comparing (3.7) with (2.1), it is clear that instead of integrating over geodesics—that is, real curves—we should now integrate over 2-dimensional objects, that is, complex curves.

PROPOSITION 3.15
Let M be a complete Kähler manifold. Given $K \in \mathbb{R}$, the manifold M has $BK \geq K$ if and only if it satisfies the following property. Let $i : \overline{D^2} \to M$ be an embedding of a disk into M that is holomorphic on D^2. Let Σ be the image of i. Let dA denote the area form on Σ. Let z be the local coordinate on D^2, and let $\theta \in [0,2\pi)$ be the local coordinate on $\partial\overline{D^2}$. Then

$$d_{K,p}^2(0) \geq \frac{2}{\pi}\iint_\Sigma \log|z|\,dA + \frac{1}{2\pi}\int_{\partial\Sigma} d_{K,p}^2(\theta)\,d\theta, \tag{3.16}$$

where the "0" on the left-hand side denotes $i(0)$, the center of Σ.

Proof
Suppose that $BK \geq K$. From Proposition 3.6, or more precisely its proof, we know that $\sqrt{-1}\partial\bar\partial d_{K,p}^2/2 \leq \omega_\Sigma$ as currents on Σ. The solution to $\sqrt{-1}\partial\bar\partial f/2 = \omega_\Sigma$ on Σ, with $f|_{\partial\Sigma} = d_{K,p}^2|_{\partial\Sigma}$ has

$$f(0) = \frac{2}{\pi} \iint_{\Sigma} \log |z| dA + \frac{1}{2\pi} \int_{\partial\Sigma} d^2_{K,p}(\theta)\, d\theta. \tag{3.17}$$

As $f - d^2_{K,p}$ is subharmonic on Σ, and vanishes on $\partial\Sigma$, inequality (3.16) follows.

Now suppose that the inequality $BK \geq K$ is violated at some point p. In complex normal coordinates around p, the metric is

$$g_{i\overline{j}} = \delta_{i\overline{j}} + \frac{1}{2} R_{i\overline{j}k\overline{l}} z^k \overline{z}^l + o\big(|z|^2\big), \tag{3.18}$$

where $R_{i\overline{j}k\overline{l}}$ is evaluated at p. Correspondingly,

$$\omega = \frac{1}{2}\sqrt{-1}\, dz^i \wedge d\overline{z}^i + \frac{1}{4}\sqrt{-1} R_{i\overline{j}k\overline{l}} z^k \overline{z}^l\, dz^i \wedge d\overline{z}^j + o\big(|z|^2\big). \tag{3.19}$$

In general, $d^2(p_0, p_1)$ is the minimum over γ of the energy

$$E(\gamma) = \int_0^1 g_{ij} \frac{d\gamma^i}{dt} \frac{d\gamma^i}{dt}\, dt, \tag{3.20}$$

where $\gamma : [0,1] \to M$ has $\gamma(0) = p_0$ and $\gamma(1) = p_1$. If γ is a unique minimizer and we perturb the metric by δg, then to leading order, the squared distance changes by

$$\delta d^2(p_0, p_1) = \int_0^1 \delta g_{ij} \frac{d\gamma^i}{dt} \frac{d\gamma^i}{dt}\, dt. \tag{3.21}$$

In our case, for the flat metric the minimizer between $0 \in \mathbb{C}^n$ and $z \in \mathbb{C}^n$ is $\gamma(t) = tz$. Treating the second term in (3.18) as the perturbation, the change in squared distance is

$$\frac{1}{2} \int_0^1 R_{i\overline{j}k\overline{l}} z^i z^{\overline{j}} (tz^k)(t\overline{z}^l)\, dt = \frac{1}{6} R_{i\overline{j}k\overline{l}} z^i z^{\overline{j}} z^k \overline{z}^l. \tag{3.22}$$

Hence since $p = 0$ in the local coordinates,

$$d^2_p(z) = |z|^2 + \frac{1}{6} R_{i\overline{j}k\overline{l}} z^i z^{\overline{j}} z^k \overline{z}^l + o\big(|z|^4\big). \tag{3.23}$$

From (1.1),

$$d^2_{K,p} = d^2_p + \frac{1}{12} K d^4_p + o(d^4_p), \tag{3.24}$$

so

$$d^2_{K,p}(z) = |z|^2 + \frac{1}{6} R_{i\overline{j}k\overline{l}} z^i z^{\overline{j}} z^k \overline{z}^l + \frac{1}{12} K |z|^4 + o\big(|z|^4\big). \tag{3.25}$$

This gives

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}d_{K,p}^2/2 &= \frac{1}{2}\sqrt{-1}\,dz^i \wedge d\overline{z}^i + \frac{1}{3}\sqrt{-1}R_{i\overline{j}k\overline{l}}z^k z^{\overline{l}}\,dz^i \wedge d\overline{z}^j \\ &\quad + \frac{1}{12}\sqrt{-1}K\overline{z}^i z^j\,dz^i \wedge d\overline{z}^j \\ &\quad + \frac{1}{12}\sqrt{-1}K|z|^2\,dz^i \wedge d\overline{z}^i + o\big(|z|^2\big). \end{aligned} \tag{3.26}$$

Equations (3.19) and (3.26) give

$$\sqrt{-1}\partial\overline{\partial}d_{K,p}^2/2 - \omega = \frac{1}{12}\sqrt{-1}R'_{i\overline{j}k\overline{l}}z^k z^{\overline{l}}\,dz^i \wedge d\overline{z}^j + o\big(|z|^2\big), \tag{3.27}$$

where

$$R'_{i\overline{j}k\overline{l}} = R_{i\overline{j}k\overline{l}} + K(\delta_{i\overline{j}}\delta_{k\overline{l}} + \delta_{i\overline{l}}\delta_{\overline{j}k}). \tag{3.28}$$

If Σ is an embedded holomorphic disk in M, then

$$\begin{aligned} &d_{K,p}^2(0) - \frac{2}{\pi}\iint_\Sigma \log|z|\,dA - \frac{1}{2\pi}\int_{\partial\Sigma} d_{K,p}^2(\theta)\,d\theta \\ &\quad = \frac{2}{\pi}\iint_\Sigma \log|z|(\sqrt{-1}\partial\overline{\partial}d_{K,p}^2/2 - \omega). \end{aligned} \tag{3.29}$$

Since M does not have $BK \geq K$ at p, there are unit vectors $X, Y \in T_p^{(1,0)}M$ so that $R'(X, \overline{X}, Y, \overline{Y}) > 0$. (Recall the minus sign in (3.3).)

Given $0 < \epsilon_1 \ll \epsilon_2 \ll 1$, consider a holomorphic disk $i : \overline{D^2} \to M$ given in complex normal coordinates by $i(w) = \epsilon_1 wX + \epsilon_2 Y$. Let Σ be the image of i. Using (3.27), the right-hand side of (3.29) is approximately

$$\begin{aligned} &\frac{1}{6\pi}\sqrt{-1}\epsilon_1^2\epsilon_2^2(\log\epsilon_2)R'(X, \overline{X}, Y, \overline{Y})\iint_{D^2} dw \wedge d\overline{w} \\ &\quad = \frac{1}{3\pi}\epsilon_1^2\epsilon_2^2(\log\epsilon_2)R'(X, \overline{X}, Y, \overline{Y})\iint_{D^2} dA_{D^2}. \end{aligned} \tag{3.30}$$

Since $\log\epsilon_2 < 0$, we conclude that

$$d_{K,p}^2(0) - \frac{2}{\pi}\iint_\Sigma \log|z|\,dA - \frac{1}{2\pi}\int_{\partial\Sigma} d_{K,p}^2(\theta)\,d\theta < 0, \tag{3.31}$$

contradicting (3.16). $\square$

Remark 3.32
There is an analogy between (2.3), with $t = \frac{L}{2}$, and (3.16), where $\frac{1}{2}(d_p^2(L) + d_p^2(0))$ is replaced by $\frac{1}{2\pi}\int_{\partial\Sigma} d_{K,p}^2(\theta)\,d\theta$ and $-\frac{L^2}{4}$ is replaced by $\frac{2}{\pi}\iint_\Sigma \log|z|\,dA$.

For any point q in the disk, there is an inequality similar to (3.16) with 0 replaced by q, obtained by performing a holomorphic automorphism of the disk.

Note that the area form dA in (3.16) can also be described as the 2-dimensional Hausdorff measure on Σ. Hence the statement of (3.16) only depends on the complex structure and the metric d.

3.4. *Hermitian manifolds*

One can ask when (3.16) holds more generally in the setting of Hermitian manifolds, rather than Kähler manifolds. It turns out that if (3.16) holds for a Hermitian manifold, then it is forced to be Kähler. We now give an analogue of Remark 2.4(4), in which Finsler manifolds are replaced by Hermitian manifolds, and Riemannian manifolds are replaced by Kähler manifolds.

PROPOSITION 3.33

If a Hermitian manifold M satisfies (3.16), for all $p \in M$ and all holomorphic disks Σ, then it is Kähler.

Proof

Choose complex coordinates around p. After a change of coordinates, we can write the metric locally as

$$g = dz^i \, d\overline{z}^i + T_{\overline{i}jk} z^j \, dz^k \, d\overline{z}^i + \overline{T_{\overline{i}jk}} \overline{z}^j \, d\overline{z}^k \, dz^i + O\big(|z|^2\big). \tag{3.34}$$

Here $T_{\overline{i}jk}$ is a constant times the torsion tensor at p, and is antisymmetric in j and k.

We first compute the leading-order terms in d_p^2, using (3.21). For the flat metric the minimizer between $0 \in \mathbb{C}^n$ and $z \in \mathbb{C}^n$ is $\gamma(t) = tz$. Treating the second and third terms in (3.34) as the perturbation, the change in squared distance is

$$\int_0^1 (T_{\overline{i}jk})(tz^j) z^k \overline{z}^i \, dt + \text{complex conjugate}. \tag{3.35}$$

This would be the $O(|z|^3)$-term in d_p^2, but it vanishes because of the (jk)-antisymmetry of $T_{\overline{i}jk}$. Hence $d_p^2(z) = |z|^2 + O(|z|^4)$. From (3.24), it follows that $d_{K,p}^2(z) = |z|^2 + O(|z|^4)$.

Then

$$\sqrt{-1}\partial\overline{\partial} d_{K,p}^2/2 = \frac{1}{2}\sqrt{-1}\, dz^i \wedge d\overline{z}^i + O\big(|z|^2\big). \tag{3.36}$$

On the other hand,

$$\begin{aligned} \omega = {} & \frac{1}{2}\sqrt{-1}\, dz^i \wedge d\overline{z}^i + \frac{1}{2}\sqrt{-1} T_{\overline{i}jk} z^j \, dz^k \wedge d\overline{z}^i \\ & + \frac{1}{2}\sqrt{-1}\overline{T_{\overline{i}jk}} \overline{z}^j \, d\overline{z}^k \wedge dz^i + O\big(|z|^2\big), \end{aligned} \tag{3.37}$$

so

$$\sqrt{-1}\partial\overline{\partial} d_{K,p}^2/2 - \omega = -\frac{1}{2}\sqrt{-1} T_{\overline{i}jk} z^j \, dz^k \wedge d\overline{z}^i - \frac{1}{2}\sqrt{-1}\overline{T_{\overline{i}jk}}\overline{z}^j \, d\overline{z}^k \wedge dz^i + O\big(|z|^2\big). \tag{3.38}$$

Suppose that M is non-Kähler, so it has a nonzero torsion tensor at some point p. Let $\vec{b} \in \mathbb{C}^n$ be such that $\sum_j b^j T_{\overline{i}jk}$ is a nonzero matrix in $(\overline{i}, k)$. Let $\vec{a} \in \mathbb{C}^n$ be such that $\sum_{i,j,k} \overline{a^i} b^j T_{\overline{i}jk} a^k \neq 0$. Multiplying $\vec{b}$ by a constant, we can assume that $\sum_{i,j,k} \overline{a^i} b^j T_{\overline{i}jk} a^k$ is a negative real number. Given $0 < \epsilon_1 \ll \epsilon_2 \ll 1$, consider a small disk $i : \overline{D^2} \to M$ given by $i(w) = \epsilon_1 w \vec{a} + \epsilon_2 \vec{b}$. Let Σ be the image of i. As in the proof of Proposition 3.15, it follows from (3.38) that the right-hand side of (3.29) is approximately

$$-4\epsilon_1^2 \epsilon_2 \log\big(\epsilon_2 |\vec{b}|\big) \sum_{i,j,k} \overline{a^i} b^j T_{\overline{i}jk} a^k < 0. \tag{3.39}$$

Thus (3.16) is violated for Σ, which is a contradiction. □

3.5. *Domains in model spaces*

We now give an analogue of Remark 2.4(5). That is, we look at regions in $\mathbb{C}^n$ or, more generally, in model spaces of constant holomorphic sectional curvature. Since we want to characterize when (3.16) holds, we need a complex structure everywhere. For that reason, we do not allow boundary, but simply consider when a domain in the model space satisfies (3.16). One might initially expect that it has something with pseudoconvexity of the domain. However, the latter notion is invariant under biholomorphisms, whereas we have a metric d in addition. It turns out that the answer is essentially given by convexity in the usual sense.

Given $K \in \mathbb{R}$, let M_K be the complete simply connected Kähler manifold with constant holomorphic sectional curvature $2K$. Its metric is given by (3.4), with $c = 2K$. One can check that equality is achieved in (3.7), away from the cut locus of p if $K > 0$.

PROPOSITION 3.40

Let M be a connected open subset of M_K. Let d be the length metric on M. Then M satisfies (3.16) if and only if d coincides with the restriction $\mathcal{D}$ of the metric from M_K.

Proof

If $d = \mathcal{D}$, then (3.16) follows immediately from the corresponding inequality for M_K.

Suppose that (3.16) is satisfied for M, but $d \neq \mathcal{D}$. Let $m_1, m_2 \in M$ be points such that $d(m_1, m_2) > \mathcal{D}(m_1, m_2)$. If $K > 0$, let D denote the cut locus of m_1, a copy of $\mathbb{C}P^{n-1}$. By continuity of the distance functions, we can assume that $m_2 \notin D$.

Let $\gamma : [0,1] \to M$ be a smooth embedding with $\gamma(0) = m_1$ and $\gamma(1) = m_2$. If $K > 0$, then we can assume that γ is disjoint from D. By approximation, we can assume that γ is real analytic. We can then extend γ to a real analytic embedding $\gamma : [-\epsilon, 1+\epsilon] \to M$ for some $\epsilon > 0$.

We claim that after possibly reducing ϵ, there is some $\epsilon' > 0$, and a continuous embedding $\Gamma : [-\epsilon, 1+\epsilon] \times [-\epsilon', \epsilon'] \to M$ that is holomorphic on the interior, so that $\Gamma(t, 0) = \gamma(t)$ for all $t \in [-\epsilon, 1+\epsilon]$. To see this, suppose first that $K = 0$, so $M_K = \mathbb{C}^n$. Let $\{\gamma^i(t)\}_{i=1}^n$ be the components of γ. As γ^i is real analytic, it extends to a holomorphic function $\Gamma^i : (-\epsilon, 1+\epsilon) \times (-\epsilon'_i, \epsilon'_i) \to \mathbb{C}$ for some $\epsilon'_i > 0$. Taking $\epsilon' = \min_i \epsilon'_i$, the functions $\{\Gamma^i\}_{i=1}^n$ combine to give a holomorphic map $\Gamma : (-\epsilon, 1+\epsilon) \times (-\epsilon', \epsilon') \to \mathbb{C}^n$. The image of $d\Gamma_{(t,0)}$ is the span of $\gamma'(t)$ and $J\gamma'(t)$, a 2-dimensional space. Hence by reducing ϵ and ϵ', we can ensure that Γ is a continuous embedding from $[-\epsilon, 1+\epsilon] \times [-\epsilon', \epsilon']$ to M, which is holomorphic on the interior.

If $K < 0$, then the underlying complex structure of M_K is the unit ball in $\mathbb{C}^n$, so the same argument can be applied. If $K > 0$, then $M_K - D$ is biholomorphic to $\mathbb{C}^n$, so again the same argument can be applied.

As Γ reparameterizes to a holomorphic disk $i : \overline{D^2} \to M$ with image Σ, by a holomorphic automorphism of the disk we can assume that $i(0) = m_1$. The equality case of (3.7) with $p = m_1$ implies that

$$0 = \frac{2}{\pi} \iint_\Sigma \log|z| \, dA + \frac{1}{2\pi} \int_{\partial\Sigma} \mathcal{D}^2_{K,m_1}(\theta)\, d\theta. \tag{3.41}$$

Note that the 2-dimensional Hausdorff measure dA is the same for d and $\mathcal{D}$. Since $d(m_1, m_2) > \mathcal{D}(m_1, m_2)$, if ϵ and ϵ' are small enough, then $d^2_{K,m_1}(\theta) > \mathcal{D}^2_{K,m_1}(\theta)$ for some θ. By continuity of the distance functions, this will also be true for all θ in some open interval. Thus

$$0 < \frac{2}{\pi} \iint_\Sigma \log|z| \, dA + \frac{1}{2\pi} \int_{\partial\Sigma} d^2_{K,m_1}(\theta)\, d\theta, \tag{3.42}$$

which contradicts (3.16). □

4. Noncollapsed Gromov–Hausdorff limits

We consider a noncollapsed pointed Gromov–Hausdorff limit of a sequence of complete Kähler manifolds with $BK \geq K$. Lee and Tam [22] proved that the limit has the structure of a complex manifold. This extends earlier results of Liu in [25] and [26], and is an analogue of Remark 2.4(2). We wish to study the geometry of the limit.

Although the metric d on the limit is generally not smooth, we show that it satisfies the comparison inequality (3.16). This is an analogue of Remark 2.4(1).

The method of proof is by running the Ricci flow on the approximants and passing to a limiting Ricci flow that exists for positive time (locally). Then one is reduced to understanding the $t \to 0$ limit of a single Ricci flow, as opposed to a sequence of Riemannian manifolds. This approach has been applied in many other contexts. Since we are not assuming an upper curvature bound, we apply recent results on local Ricci flow.

The proof also relies on local Kähler potentials. We actually prove the existence of local Kähler potentials, of a certain regularity, on the limit space.

PROPOSITION 4.1

Let $\{(M_i, p_i, g_i)\}_{i=1}^{\infty}$ be a sequence of pointed n-dimensional complete Kähler manifolds with $BK \geq K$. Suppose that there is some $v_0 > 0$ so that for all i, we have $\mathrm{vol}(B(p_i, 1)) \geq v_0$. *Then after passing to a subsequence, there is a pointed Gromov–Hausdorff limit $(X_\infty, p_\infty, d_\infty)$ with the following properties.*

(1) *X_∞ is a complex manifold and d_∞ is locally bi-Hölder-equivalent to the distance metric of a smooth Riemannian metric on X_∞.*

(2) *There is an open covering $\{U_\alpha\}_{\alpha \in A}$ of X_∞ and plurisubharmonic potentials $\phi_\alpha \in C(U_\alpha)$, locally Lipschitz with respect to d_∞, so that $\phi_\alpha - \phi_\beta$ is pluriharmonic on $U_\alpha \cap U_\beta$, and the following holds. Let Σ be a holomorphic disk in X_∞. Let $\phi_\alpha|_{\Sigma \cap U_\alpha}$ be the restriction of ϕ_α to $\Sigma \cap U_\alpha$, and put $\omega_\infty|_\Sigma = \sqrt{-1}\partial\bar{\partial}\phi_\alpha|_{\Sigma \cap U_\alpha}$, a globally defined measurable $(1,1)$-form on Σ. Then $\omega_\infty|_\Sigma$ equals the 2-dimensional Hausdorff measure μ_∞ coming from $d_\infty|_\Sigma$.*

(3) *We have*

$$d^2_{K,p}(0) \geq \frac{2}{\pi} \int_\Sigma \log|z|\, d\mu_\infty + \frac{1}{2\pi} \int_{\partial\Sigma} d^2_{K,p}(\theta)\, d\theta. \tag{4.2}$$

Proof

(1) We claim first that there are nondecreasing sequences $\alpha_k, \beta_k \geq 1$ and a nonincreasing sequence $S_k > 0$ such that for any i, there is a Kähler–Ricci flow $g_i(t)$ defined on $\bigcup_{k=1}^{\infty}(B_{g_i}(p_i, 2k) \times [0, S_k])$ with $g_i(0) = g_i$, such that

$$\left|\mathrm{Rm}\big(g_i(t)\big)\right| \leq \frac{\alpha_k}{t}, \tag{4.3}$$

$$\mathrm{Ric}\big(g_i(t)\big) \geq -\beta_k, \tag{4.4}$$

and

$$\mathrm{inj}_{g_i(t)} \geq \alpha_k^{-1}\sqrt{t} \tag{4.5}$$

on $B_{g_i}(p_i, 2k) \times [0, S_k]$. This follows from the pyramid Ricci flow constructed in [22, Theorem 1.2] (see also the proofs of [21, Theorem 5.1] and [30, Theorem 1.3]).

From distance distortion estimates as in [18, Section 27], there is then a constant $C_k < \infty$ so that for $t_1 \leq t_2$, we have

$$d_{g_i(t_1)} - C_k(\sqrt{t_2} - \sqrt{t_1}) \leq d_{g_i(t_2)} \leq e^{\beta_k(t_2 - t_1)} d_{g_i(t_1)} \tag{4.6}$$

on $B_{g_i}(p_i, 2k) \times [0, S_k]$.

Using a local version of Hamilton compactness [18, Appendix E], after passing to a subsequence of the i's, there is a pointed smooth manifold (X_∞, p_∞) and an exhaustion of X_∞ by precompact open sets $\{V_k\}_{k=1}^\infty$ containing p_∞, along with a limiting pointed Ricci flow $g_\infty(\cdot)$ defined on $\bigcup_{k=1}^\infty (V_k \times (0, S_k))$ (cf. [30, Theorem 1.5]). More precisely, for each $k \in \mathbb{Z}^+$, for large i there is a pointed embedding $\phi_{i,k} : V_k \to M_i$ so that

$$g_\infty(\cdot) = \lim_{i \to \infty} \phi_{i,k}^* g_i(\cdot) \tag{4.7}$$

on compact subsets of $V_k \times (0, S_k)$, in the smooth topology.

The distance distortion estimate (4.6) passes to the limiting Ricci flow. It follows that there is a pointed Gromov–Hausdorff limit $\lim_{t \to 0}(X_\infty, p_\infty, g_\infty(t)) = (X_\infty, p_\infty, d_\infty)$ for some complete metric d_∞. It then follows that $\lim_{i \to \infty}(M_i, p_i, g_i) = (X_\infty, p_\infty, d_\infty)$ in the pointed Gromov–Hausdorff topology. We can take V_k to be the metric ball $B(p_\infty, k)$ with respect to d_∞, so

$$d_{g_\infty(t_1)} - C_k(\sqrt{t_2} - \sqrt{t_1}) \leq d_{g_\infty(t_2)} \leq e^{\beta_k(t_2 - t_1)} d_{g_\infty(t_1)} \tag{4.8}$$

on $B(p_\infty, k) \times (0, S_k)$. Also,

$$\left|\mathrm{Rm}\big(g_\infty(t)\big)\right| \leq \frac{\alpha_k}{t} \tag{4.9}$$

on $B(p_\infty, k) \times (0, S_k)$.

From [38, Lemma 3.1], for any $t \in (0, S_k)$, the metric ball $B(p_\infty, k) \subset X_\infty$ with the metric d_∞ is bi-Hölder homeomorphic to the same ball with the metric $g_\infty(t)$.

Given $k \in \mathbb{Z}^+$ and considering the time interval $(0, S_k)$, since the complex structures J_i on $B_{g_i}(p_i, 2k) \subset M_i$ satisfy $\nabla_{g_i(t)} J_i = 0$, after passing to a subsequence of i's we can assume that they converge to a complex structure $J_{\infty,k}$ on $B(p_\infty, k)$ that satisfies $\nabla_{g_\infty(t)} J_{\infty,k} = 0$. After passing to a further subsequence of i's, we obtain a complex structure J_∞ on X_∞ that, on $B(p_\infty, k)$, satisfies $\nabla_{g_\infty(t)} J_\infty = 0$ for $t \in (0, S_k)$. Let $\omega(t)$ denote the corresponding Kähler form.

(2) Fix $k \in \mathbb{Z}^+$, and fix $t' \in (0, S_k)$. For $t \in (0, t']$, put

$$u(t) = -\int_t^{t'} \log \frac{\omega^n(s)}{\omega^n(t')}\, ds. \tag{4.10}$$

Then

$$\omega(t) = \omega(t') - (t - t')\,\mathrm{Ric}\big(\omega(t')\big) + \sqrt{-1}\partial\overline{\partial}u(t), \tag{4.11}$$

as can be seen by differentiating in t.

Since

$$\frac{\partial \omega}{\partial t} = -\,\mathrm{Ric}\big(\omega(t)\big), \tag{4.12}$$

the estimate (4.9) implies that

$$\left|\log \frac{\omega^n(s)}{\omega^n(t')}\right| \leq \mathrm{const.} \log \frac{t'}{s} \tag{4.13}$$

for $s \in (0, t']$, where "const." is an n-dependent factor times α_k. Then

$$\begin{aligned} \big|u(t_1) - u(t_2)\big| &\leq \mathrm{const.} \int_{t_1}^{t_2} \log \frac{t'}{s}\, ds \\ &= \mathrm{const.}\big((t_2 - t_1)\log(t') - t_2 \log(t_2) + t_1 \log(t_1)\big). \end{aligned} \tag{4.14}$$

Hence $\{u(1/j)\}$ is a uniformly Cauchy sequence and has a limit $u(0) \in C(B(p_\infty, k))$.

Given $x \in B(p_\infty, k)$, let U be a neighborhood of x that is biholomorphic to the unit ball in $\mathbb{C}^n$. There are $v_U, w_U \in C^\infty(U)$ so that we can write $\omega(t')$ on U as $\sqrt{-1}\partial\overline{\partial}v_U$, and we can write $\mathrm{Ric}(\omega(t'))$ on U as $\sqrt{-1}\partial\overline{\partial}w_U$. Doing the same for another point $p' \in B(p_\infty, k)$, we have $\sqrt{-1}\partial\overline{\partial}(v_U - v_{U'}) = 0$ and $\sqrt{-1}\partial\overline{\partial}(w_U - w_{U'}) = 0$ on $U \cap U'$. For $t \in [0, S_k)$, put

$$\phi_U(t) = v_U - (t - t')w_U + u(t)|_U. \tag{4.15}$$

If $t > 0$, then (4.11) gives $\sqrt{-1}\partial\overline{\partial}\phi_U(t) = \omega(t)$, so $\sqrt{-1}\partial\overline{\partial}(\phi_U(t) - \phi_{U'}(t)) = 0$ on $U \cap U'$. Let $\eta \in \Omega^{n-1,n-1}(U \cap U')$ be a smooth compactly supported form. Then

$$\begin{aligned} \int_{X_\infty} \big(\phi_U(t) - \phi_{U'}(t)\big) \wedge \sqrt{-1}\partial\overline{\partial}\eta &= \int_{X_\infty} \sqrt{-1}\partial\overline{\partial}\big(\phi_U(t) - \phi_{U'}(t)\big) \wedge \eta \\ &= 0. \end{aligned} \tag{4.16}$$

Using the uniform convergence $\lim_{t \to 0} u(t) = u(0)$, it follows that

$$\int_{X_\infty} \big(\phi_U(0) - \phi_{U'}(0)\big) \wedge \sqrt{-1}\partial\overline{\partial}\eta = 0, \tag{4.17}$$

so $\sqrt{-1}\partial\overline{\partial}(\phi_U(0) - \phi_{U'}(0)) = 0$ as a current. That is, $\phi_U(0) - \phi_{U'}(0)$ is pluriharmonic. Similarly, if η has compact support in U and is strongly positive in the sense of [8, Chapter 3], then for $t > 0$, we have

$$\int_{X_\infty} \phi_U(t) \wedge \sqrt{-1}\partial\overline{\partial}\eta = \int_{X_\infty} \sqrt{-1}\partial\overline{\partial}\phi_U(t) \wedge \eta = \int_{X_\infty} \omega(t) \wedge \eta \geq 0. \tag{4.18}$$

Passing to the limit as $t \to 0$ gives

$$\int_{X_\infty} \phi_U(0) \wedge \sqrt{-1}\partial\overline{\partial}\eta \geq 0. \tag{4.19}$$

Hence $\sqrt{-1}\partial\overline{\partial}\phi_U(0) \geq 0$ in the sense of currents, that is, $\phi_U(0)$ is plurisubharmonic.

From [7, Theorem 6], there is a bound on $|\nabla\phi_U(t)|$ in terms of K and the oscillation of $\phi_U(t)$, the latter of which is uniformly bounded in t. Hence $\phi_U(t)$ is uniformly Lipschitz in t, with respect to $d_{g_\infty(t)}$. This passes to the limit, to show that $\phi_U(0)$ is Lipschitz with respect to d_∞.

Taking an open cover $\{U_\alpha\}$ of X_∞ by such neighborhoods, we obtain such plurisubharmonic functions $\phi_\alpha = \phi_{U_\alpha}(0) \in C(U_\alpha)$ so that $\phi_\alpha - \phi_\beta$ is pluriharmonic on $U_\alpha \cap U_\beta$.

Fixing k, for $t \in (0, S_k)$ put $\widehat{d}_t = e^{-\beta_k t} d_{g_\infty(t)}$. From (4.8), we know that $\widehat{d}_t$ is nonincreasing in t. In addition, it follows from (4.8) that

$$\widehat{d}_t \leq d_\infty \leq e^{\beta_k t}\widehat{d}_t + C_k\sqrt{t}. \tag{4.20}$$

Let Σ be a holomorphic disk in $B(p_\infty, k)$. Then for $t \in (0, S_k)$, the 2-dimensional Hausdorff measure $\widehat{\mu}_t$ on Σ coming from $\widehat{d}_t|_\Sigma$ is $e^{-2\beta_k t}$ times $\omega(t)|_\Sigma = \sqrt{-1}\partial\overline{\partial}\phi_U(t)|_\Sigma$. It follows that $\lim_{t\to 0}\widehat{\mu}_t$ equals $\sqrt{-1}\partial\overline{\partial}\phi_U(0)|_\Sigma = \omega_\infty|_\Sigma$.

We claim that $\lim_{t\to 0}\widehat{\mu}_t$ also equals μ_∞, the 2-dimensional Hausdorff measure coming from $d_\infty|_\Sigma$. To see this, let $K \subset \Sigma$ be a compact set lying in some $B(p_\infty, k)$. Then $\mu_\infty(K) = \lim_{\delta\to 0} H^2_{d_\infty,\delta}(K)$, where

$$H^2_{d_\infty,\delta}(K) = \frac{\pi}{4}\inf\sum_l (\operatorname{diam}_{d_\infty} W_l)^2, \tag{4.21}$$

and $\{W_l\}$ ranges over finite covers of K by open sets $W_l \subset \Sigma$ with $\operatorname{diam}_{d_\infty}(W_l) < \delta$. The definition of $\widehat{\mu}_t$ is similar, using $\widehat{d}_t$. Note that $H^2_{d_\infty,\delta}(K)$ is nonincreasing in δ. Since $\widehat{d}_t$ is monotonically nondecreasing as $t \to 0$, with limit d_∞, it follows from (4.21) that $\widehat{\mu}_t(K)$ is monotonically nondecreasing as $t \to 0$, and $\lim_{t\to 0}\widehat{\mu}_t(K) \leq \mu_\infty(K)$. To show equality, suppose first that $\mu_\infty(K) < \infty$. Given t, δ, and ϵ, let $\{W_l\}$ be a finite open cover of K with

$$\frac{\pi}{4}\sum_l (\operatorname{diam}_{\widehat{d}_t} W_l)^2 \leq H^2_{\widehat{d}_t,\delta}(K) + \epsilon \tag{4.22}$$

and $\operatorname{diam}_{\widehat{d}_t} W_l < \delta$ for each l. Now

$$\frac{\pi}{4}\sum_l (\operatorname{diam}_{d_\infty} W_l)^2 \le \frac{\pi}{4}\sum_l (e^{\beta_k t}\operatorname{diam}_{\widehat{d}_t} W_l + C_k\sqrt{t})^2 \tag{4.23}$$

and $\operatorname{diam}_{d_\infty} W_l < e^{\beta_k t}\delta + C_k\sqrt{t}$ for each l. Since $\{W_l\}$ is finite, if t is small enough, then

$$\frac{\pi}{4}\sum_l (e^{\beta_k t}\operatorname{diam}_{\widehat{d}_t} W_l + C_k\sqrt{t})^2 \le \frac{\pi}{4}\sum_l (\operatorname{diam}_{\widehat{d}_t} W_l)^2 + \epsilon. \tag{4.24}$$

Put $\delta' = e^{\beta_k t}\delta + C_k\sqrt{t}$. Then

$$H^2_{d_\infty,\delta'}(K) \le H^2_{\widehat{d}_t,\delta}(K) + 2\epsilon \le \widehat{\mu}_t(K) + 2\epsilon \le \lim_{t'\to 0}\widehat{\mu}_{t'}(K) + 2\epsilon. \tag{4.25}$$

As ϵ is arbitrary, this shows that $H^2_{d_\infty,\delta'}(K) \le \lim_{t'\to 0}\widehat{\mu}_{t'}(K)$. A similar argument shows that if $\mu_\infty(K) = \infty$, then $\lim_{t'\to 0}\widehat{\mu}_{t'}(K) = \infty$. Hence $\mu_\infty \le \lim_{t'\to 0}\widehat{\mu}_{t'}$.

(3) Given $p \in X_\infty$, let $d_p \in C(X_\infty)$ be the distance function from p. Given $x \in X_\infty$, choose $k \in \mathbb{Z}^+$ so that $x \in B(p_\infty, k/2)$. Let $U \subset B(p_\infty, k/2)$ be a ball neighborhood of x on which the potential function $\phi_U(0) \in C(U)$ is defined.

Using the comparison maps in (4.7), we can assume that each Ricci flow $g_i(\cdot)$ is defined on $B(p_\infty, k) \times (0, S_k)$. As $\lim_{i\to\infty} J_i = J_\infty$ smoothly (say, relative to $g_\infty(t')$ for a given $t' \in (0, S_k)$), there is a sequence of holomorphic maps $\mu_i : (U, J_\infty) \to (B(p_\infty, k), J_i)$, for large i, with $\{\mu_i\}_{i=1}^\infty$ smoothly approaching the identity map (see [13]). The pullback Ricci flows $\{\mu_i^* g_i(\cdot)\}_{i=1}^\infty$ live on U and are all Kähler relative to the fixed complex structure J_∞.

Let $\{p_i\}_{i=1}^\infty$ be a sequence of points, with $p_i \in M_i$, that converges to p in the Gromov–Hausdorff sense. We first show that $\lim_{i\to\infty}\mu_i^* d_{p_i} = d_p$ uniformly on U. To see this, we apply (4.6) with $t_1 = 0$ and $t_2 = t$ to get that for all $q \in U$, we have

$$e^{-\beta_k t} d_{g_i(t)}\big(q, \mu_i(q)\big) \le d_i\big(q, \mu_i(q)\big) \le d_{g_i(t)}\big(q, \mu_i(q)\big) + C_k\sqrt{t}. \tag{4.26}$$

For fixed t, we have $\lim_{i\to\infty} d_{g_i(t)}(q, \mu_i(q)) = 0$ uniformly in q. Taking t to zero, we conclude from (4.26) that $\lim_{i\to\infty} d_i(q, \mu_i(q)) = 0$ uniformly in q. Now

$$\begin{aligned}\big|(\mu_i^* d_{p_i})(q) - d_p(q)\big| &= \big|d_i\big(p_i, \mu_i(q)\big) - d_\infty(p, q)\big| \\ &\le |d_i(p_i, q)) - d_\infty(p, q)| + \big|d_i\big(q, \mu_i(q)\big)\big|. \end{aligned}\tag{4.27}$$

Using the Gromov–Hausdorff convergence of d_i to d_∞, relative to the identity comparison map, equation (4.27) gives that $\lim_{i\to\infty}\mu_i^* d_{p_i} = d_p$ uniformly on U.

We will show that there are local Kähler potentials $\{\eta_i\}$ on M_i so that $\lim_{i\to\infty}\mu_i^*\eta_i = \phi_U(0)$ uniformly on U. Pulling back by μ_i, it suffices to construct such Kähler potentials for the pullback metrics on U, which we again denote by g_i, that are compatible with J_∞.

Construct $u_i(\cdot)$ as in the proof of part (2) of the proposition, except for the flow $g_i(\cdot)$ instead of $g_\infty(\cdot)$. From (4.10), we have

$$u_i(0) - u(0) = -\int_0^{t'} \log \frac{\omega^n(s)}{\omega_i^n(s)}\, ds. \tag{4.28}$$

Then

$$\left\| u_i(0) - u(0) \right\|_{C(U)} \leq \int_0^{t'} \left\| \log \frac{\omega^n(s)}{\omega_i^n(s)} \right\|_{C(U)} ds. \tag{4.29}$$

Using (4.13) and dominated convergence, it follows that $\lim_{i\to\infty} u_i(0) = u(0)$ uniformly on U.

Recall the functions v_U and w_U constructed in part (2), using the $\partial\overline{\partial}$-lemma. Construct functions v_i and w_i analogously for the metric g_i. From the smooth convergence of $\{g_i(t')\}_{i=1}^\infty$ to $g_\infty(t')$, and the explicit proof of the $\partial\overline{\partial}$-lemma (see [8, Lemma I.(3.29) and Proposition III.(1.19)]), we can assume that $\{v_i\}_{i=1}^\infty$ converges smoothly to v_∞, and $\{w_i\}_{i=1}^\infty$ converges smoothly to w_∞. Put

$$\phi_i(0) = v_i + t' w_i + u_i(0). \tag{4.30}$$

By construction, $\phi_i(0)$ is a Kähler potential for ω_i on U or, more precisely, for $\mu_i^* \omega_i$. We have shown that $\lim_{i\to\infty} \phi_i(0) = \phi_U(0)$ uniformly on U. Finally, for large i, put $\eta_i = (\mu_i^{-1})^* \phi_i(0)$. Then η_i is a smooth local Kähler potential for g_i on $\mu_i(U)$.

We momentarily exclude the case when $K > 0$ and $\operatorname{diam}(X_\infty, d_\infty) = \frac{\pi}{\sqrt{2K}}$. We know that $\eta_i - d^2_{K,p_i}/2$ is plurisubharmonic. As

$$\lim_{i\to\infty} \mu_i^*(\eta_i - d^2_{K,p_i}/2) = \phi_U(0) - d^2_{K,p}/2 \tag{4.31}$$

uniformly on U, it follows that $\phi_U(0) - d^2_{K,p}/2$ is plurisubharmonic on U.

If $K > 0$ and $\operatorname{diam}(X_\infty, d_\infty) = \frac{\pi}{\sqrt{2K}}$, then we use the fact that $BK \geq \lambda^2 K$ for $\lambda \in (0,1)$, and $\operatorname{diam}(X_\infty, d_\infty) < \frac{\pi}{\lambda\sqrt{2K}}$, so $\phi_U(0) - d^2_{\lambda^2 K,p}/2$ is plurisubharmonic on U. We take the limit as $\lambda \to 1$, as in the proof of Proposition 3.6, to again conclude that $\phi_U(0) - d^2_{K,p}/2$ is plurisubharmonic on U.

Given the holomorphic disk $\Sigma \in X_\infty$. we know that the restriction of $\phi_U(0) - d^2_{K,p}/2$ to $\Sigma \cap U$ is subharmonic. Hence

$$\sqrt{-1}\partial\overline{\partial} d^2_{K,p}|_{\Sigma\cap U}/2 \leq \sqrt{-1}\partial\overline{\partial}\phi_U(0)|_{\Sigma\cap U} = \mu_\infty|_{\Sigma\cap U}. \tag{4.32}$$

Then

$$\sqrt{-1}\partial\overline{\partial} d^2_{K,p}|_\Sigma/2 \leq \mu_\infty \tag{4.33}$$

globally, as measures on Σ.

Given $\epsilon \in (0, \frac{1}{10})$, define $f_\epsilon : D^2 \to \mathbb{R}$ by

$$f_\epsilon(re^{i\theta}) = \begin{cases} \log(\epsilon) + \epsilon & \text{if } 0 \le r \le \epsilon, \\ \log(r) + \epsilon & \text{if } \epsilon \le r \le e^{-\epsilon}, \\ 0 & \text{if } e^{-\epsilon} \le r < 1. \end{cases} \tag{4.34}$$

Then $\log(|z|) \le f_\epsilon(z) \le 0$, and $\sqrt{-1}\partial\overline{\partial} f_\epsilon$ exists as a measure. We have

$$\begin{aligned} \int_\Sigma (\sqrt{-1}\partial\overline{\partial} f_\epsilon) d^2_{K,p} =& \frac{1}{2}\int_0^{2\pi} \int_0^1 \big(\partial_r(r\partial_r f_\epsilon)\big) d^2_{K,p}(r,\theta)\, dr\, d\theta \\ =& \frac{1}{2}\int_0^{2\pi} \int_0^1 \big(\delta_\epsilon(r) - \delta_{e^{-\epsilon}}(r)\big) d^2_{K,p}(r,\theta)\, dr\, d\theta \\ =& \frac{1}{2}\int_0^{2\pi} \big(d^2_{K,p}(\epsilon,\theta) - d^2_{K,p}(e^{-\epsilon},\theta)\big)\, d\theta. \end{aligned} \tag{4.35}$$

Let $\widehat{f_\epsilon} \in C^\infty_c(D^2)$ be a smooth nonpositive approximation to f_ϵ, obtained by rounding out the corners at $r = \epsilon$ and $r = e^{-\epsilon}$. Since $\widehat{f_\epsilon}$ is nonpositive, equation (4.33) gives

$$\frac{1}{2}\int_\Sigma \widehat{f_\epsilon} \cdot \sqrt{-1}\partial\overline{\partial} d^2_{K,p} \ge \int_\Sigma \widehat{f_\epsilon}\, d\mu_\infty. \tag{4.36}$$

Passing to a limit as $\widehat{f_\epsilon}$ approaches f_ϵ, it follows from (4.35) that

$$\frac{1}{4}\int_0^{2\pi} \big(d^2_{K,p}(\epsilon,\theta) - d^2_{K,p}(e^{-\epsilon},\theta)\big)\, d\theta \ge \int_\Sigma f_\epsilon\, d\mu_\infty \ge \int_\Sigma \log|z| d\mu_\infty. \tag{4.37}$$

Taking the limit as $\epsilon \to 0$ gives

$$\frac{\pi}{2} d^2_{K,p}(0) - \frac{1}{4}\int_0^{2\pi} d^2_{K,p}(e^{i\theta})\, d\theta \ge \int_\Sigma \log|z| d\mu_\infty \tag{4.38}$$

or

$$d^2_{K,p}(0) \ge \frac{1}{2\pi}\int_0^{2\pi} d^2_{K,p}(e^{i\theta})\, d\theta + \frac{2}{\pi}\int_\Sigma \log|z|\, d\mu_\infty. \tag{4.39}$$

This proves the proposition. □

Remark 4.40

In the collapsing case, that is, if $\lim_{i\to\infty} \mathrm{vol}(B(p_i, 1)) = 0$, there is no direct analogue of Proposition 4.1 since the limit space need not be Kähler, even if it is smooth. For example, a sequence of flat 2-tori can converge in the Gromov–Hausdorff sense to a circle.

If there are uniform two-sided sectional curvature bounds, then one can take a limit in the sense of étale groupoids (see [29, Section 5]), even in the collapsing case. The conclusion is that there is a $W^{2,p}$-regular Kähler metric on the unit space of the groupoid, with $BK \geq K$.

Natural examples in which there is collapsing with a Kähler limit space arise in the long-time behavior of the Kähler–Ricci flow.

As a consequence of Proposition 4.1, we see that if a noncollapsed pointed Gromov–Hausdorff limit of a sequence of Kähler manifolds happens to be a smooth Riemannian manifold, and if the Kähler manifolds in the sequence have $BK \geq K$, then the limit is a Kähler manifold with $BK \geq K$.

COROLLARY 4.41

Let $\{(M_i, p_i, g_i)\}_{i=1}^{\infty}$ be a sequence of pointed n-dimensional complete Kähler manifolds with $BK \geq K$, that converges in the pointed Gromov–Hausdorff topology to a smooth pointed n-dimensional Riemannian manifold $(M_\infty, p_\infty, g_\infty)$. Then (M_∞, g_∞) is a Kähler manifold with $BK \geq K$.

Proof

This follows from Propositions 3.15 and 4.1. □

As an example of what the limits in Proposition 4.1 look like, consider the case of two real dimensions. A smooth oriented surface with a Riemannian metric is also a Kähler manifold. A lower bound on the sectional curvature is equivalent to a lower bound on the holomorphic bisectional curvature. Hence one would expect that oriented surfaces with lower curvature bounds, in the Alexandrov sense, could also be limits in the sense of Proposition 4.1.

PROPOSITION 4.42

Let (X, d) be a compact boundaryless 2-dimensional length space with Alexandrov curvature bounded below by $2K$. It follows that X is a topological manifold; assume that it is oriented. Then X satisfies the conclusions of Proposition 4.1.

Proof

One knows that X acquires a conformal structure (see [36, Theorem 7.1.2]). From [37], there is a smooth Ricci flow $g(\cdot)$ on $X \times (0, T]$, preserving the conformal structure, so that the sectional curvature of $g(t)$ is bounded below by $2K$, and $\lim_{t \to 0}(X, g(t)) = (X, d)$ in the Gromov–Hausdorff topology. Hence the proof of Proposition 4.1 applies. □

Remark 4.43
The examples in Proposition 4.42 show the sharpness of the regularity estimates in Proposition 4.1. Consider a conical metric on $\mathbb{R}^2$ given by $ds^2 = r^{-2\alpha}(dr^2 + r^2\, d\theta^2)$, with $\alpha \in (0,1)$. A Kähler potential is $\phi = \text{const.}\, r^{2-2\alpha}$, which is only Hölder continuous with respect to the standard metric on $\mathbb{R}^2$. On the other hand, the distance function from the origin is $d_0 = \text{const.}\, r^{1-\alpha}$, so ϕ is Lipschitz regular with respect to d.

5. Singular spaces with lower bounds on holomorphic bisectional curvature

In Section 4 the underlying topological spaces were manifolds, both in the noncollapsing sequences and in the limit spaces. In analogy with Alexandrov geometry, it is natural to ask if there is a notion for singular spaces of a lower bound on the holomorphic bisectional curvature.

5.1. Metric Kähler spaces

In the proof of Proposition 4.1, an important role was played by local Kähler potentials. This fits well with the notion of Kähler spaces, which are defined using local potentials on possibly singular complex spaces.

Let X be a reduced complex space of pure dimension n (see [8, Chapter 2.5]). For each $x \in X$, there is a neighborhood U_x of x and an embedding $e_x : U_x \to \mathbb{C}^{N_x}$ so that $e(U_x)$ is the zero set of a finite number of analytic functions defined on an open set $V_x \subset \mathbb{C}^{N_x}$.

If X_1 and X_2 are complex spaces, then a map $F : X_1 \to X_2$ is holomorphic if for each $x \in X_1$, there are such U_x and $U_{F(x)}$, with $F(U_x) \subset U_{F(x)}$, so that the composite map $e_{F(x)} \circ F|_{U_x} : U_x \to \mathbb{C}^{N_{F(x)}}$ equals $\widehat{F} \circ e_x$, where $\widehat{F} : V_x \to \mathbb{C}^{N_{F(x)}}$ is holomorphic (see [12, Section 1.3]).

A function ϕ on U_x is plurisubharmonic if it is the pullback under e_x of a plurisubharmonic function on $V_x \subset \mathbb{C}^{N_x}$. A pluriharmonic function on U_x is defined similarly. If X is normal and $\phi \in C(U_x)$ is plurisubharmonic on $U_x \cap X_{\text{reg}}$, then it is plurisubharmonic on U_x (see [10]).

As in [9] and [31], a (semi)-*Kähler space* consists of a complex space with a covering $\{U_j\}_{j=1}^{\infty}$ by such open sets, along with continuous plurisubharmonic functions ϕ_j on U_j, so $\phi_j - \phi_{j'}$ is pluriharmonic on each $U_j \cap U_{j'} \neq \emptyset$. Two such collections $\{(U_j, \phi_j)\}$ and $\{(\widehat{U}_k, \widehat{\phi}_k)\}$ are equivalent if $\phi_j - \widehat{\phi}_k$ is pluriharmonic on each $U_j \cap \widehat{U}_k \neq \emptyset$. (In the papers [9] and [31], the functions ϕ_j are taken to be smooth and strictly plurisubharmonic, but there is clearly some flexibility in the definitions.)

We wish to define a metric Kähler space, meaning a Kähler space with a metric d. Naturally, we want some compatibility between the Kähler space structure and the metric structure. If the Kähler potentials are smooth, then there is a corresponding

Riemannian metric and one can require that d be the corresponding length metric. If the Kähler potentials are only continuous, then it is not clear how to construct a length metric (see, however, [24, Theorem 1.3]).

An indication of a reasonable compatibility condition for us comes from the use of dA in (3.16). In the smooth setting dA is both the restriction of the Kähler form to a holomorphic disk, and its 2-dimensional Hausdorff measure. Again in the smooth setting, the complex structure and the 2-dimensional Hausdorff measure determine the Kähler form and the Riemannian metric. Based on this, we make the following definition.

Definition 5.1
A *metric Kähler space* is a Kähler space X equipped with a metric d that induces the topology of the complex space X, so that if Σ is an embedded holomorphic disk, then for all j, $\sqrt{-1}\partial\overline{\partial}\phi_j|_\Sigma$ equals the 2-dimensional Hausdorff measure on each $\Sigma \cap U_j \neq \emptyset$.

We now define a notion of "$BK \geq K$" for metric Kähler spaces, which we put in quotes in order to distinguish it from the condition $BK \geq K$ for smooth Kähler manifolds.

Definition 5.2
A metric Kähler space X has "$BK \geq K$" if for every $p \in X$ and every j, $\phi_j - d_{K,p}^2/2$ is plurisubharmonic on U_j.

If S is a subset of X and d_S denotes the distance to S, then we define $d_{K,S}$ in terms of d_S as in (1.1). The next lemma will be used in Section 6.

LEMMA 5.3
If X has "$BK \geq K$," then for any $S \subset X$, the function $\phi_j - d_{K,S}^2/2$ is plurisubharmonic on U_j.

Proof
As $d_S = \inf_{p \in S} d_p$, it follows that $d_{K,S} = \inf_{p \in S} d_{K,p}$ and $\phi_j - d_{K,S}^2/2 = \sup_{p \in S}(\phi_j - d_{K,p}^2/2)$. Now the supremum of a family of plurisubharmonic functions, when upper semicontinuous, is also plurisubharmonic (see [8, Chapter 1, Theorem 5.7]). As $\phi_j - d_{K,S}^2/2$ is continuous, it is hence plurisubharmonic. □

We now show the essential equivalence between "$BK \geq K$" and (3.16).

PROPOSITION 5.4
If X has "$BK \geq K$," then for all embedded holomorphic disks ϕ in X, equation (3.16) holds. If X is normal, then the converse is true.

Proof
If X has "$BK \geq K$," then by [10, Theorem 5.3.1], $\phi_j - d^2_{K,p}/2$ is subharmonic on $U_j \cap \Sigma$. Hence $\sqrt{-1}\partial\overline{\partial} d_{K,p}|^2_\Sigma/2 \leq dA$ globally on Σ. As in the proof of Proposition 4.1(3), it follows that (3.16) holds.

Suppose that X is normal and (3.16) holds. Taking embedded holomorphic disks Σ in $U_j \cap X_{\mathrm{reg}}$, it follows that $\phi_j - d^2_{K,p}/2$ is plurisubharmonic on $U_j \cap X_{\mathrm{reg}}$. As $\phi_j - d^2_{K,p}/2$ is continuous on U_j, it is then also plurisubharmonic on U_j. $\square$

We show that if a Kähler orbifold has $BK \geq K$, in the sense of curvature tensors, then the underlying length space has "$BK \geq K$." For a summary of the relevant topology and geometry of orbifolds, we refer to [19, Section 2].

PROPOSITION 5.5
If $\mathcal{O}$ is a smooth effective Kähler orbifold with $BK \geq K$, in terms of the curvature tensor on local coverings, then the underlying topological space $|\mathcal{O}|$ with the length metric has "$BK \geq K$."

Proof
Given $x \in |\mathcal{O}|$, let G_x be its local group. There is a local model $(\widehat{U}, G_x)$ around x, where $\widehat{U}$ is an open subset of $\mathbb{C}^n$ containing 0, and G_x acts effectively by holomorphic isometries on $\widehat{U}$ while fixing 0. Put $U = \widehat{U}/G_x$, a neighborhood of x, with projection $\pi : \widehat{U} \to U$. By shrinking $\widehat{U}$ if necessary, we can assume that there is a Kähler potential $\widehat{\phi}$ on it. Averaging $\widehat{\phi}$ over G_x, we can assume that it is G_x-invariant. Then there is a unique $\phi \in C(U)$ with $\pi^*\phi = \widehat{\phi}$. This gives $|\mathcal{O}|$ the structure of a Kähler space. With the natural length space structure on $|\mathcal{O}|$, it becomes a metric Kähler space.

The regular subset $|\mathcal{O}|_{\mathrm{reg}}$ consists of the points with trivial local group. It is convex in the sense that if $x_1, x_2 \in |\mathcal{O}|_{\mathrm{reg}}$, then any minimizing geodesic in $|\mathcal{O}|$ from x_1 to x_2 lies in $|\mathcal{O}|_{\mathrm{reg}}$, as follows for example from [35, Corollary of Theorem 1.2(A)]. Given $p \in |\mathcal{O}|_{\mathrm{reg}}$ and a local potential ϕ defined on an open set U, the convexity and the fact that $BK \geq K$ on $|\mathcal{O}|_{\mathrm{reg}}$ implies that $\phi - d^2_{K,p}/2$ is plurisubharmonic on $U \cap |\mathcal{O}|_{\mathrm{reg}}$. Since $|\mathcal{O}|$ is a normal complex space (see [4]), it follows that $\phi - d^2_{K,p}/2$ is plurisubharmonic on U.

For any $p \in |\mathcal{O}|$, we can find a sequence $\{p_i\}$ in $|\mathcal{O}|_{\mathrm{reg}}$ converging to p. As each $\phi - d^2_{K,p_i}/2$ is plurisubharmonic on U, we can pass to the limit and deduce that $\phi - d^2_{K,p}/2$ is plurisubharmonic on U. Hence $|\mathcal{O}|$ has "$BK \geq K$." $\square$

Remark 5.6
Proposition 5.5 shows that quotient singularities can occur as singularities of metric Kähler spaces with a lower bound on the holomorphic bisectional curvature. We do not know what other singularities can occur.

5.2. *Complex Gromov–Hausdorff convergence*

We now give a notion of Gromov–Hausdorff convergence that is adapted to metric Kähler spaces. One's first inclination may be to require the Gromov–Hausdorff approximants to be holomorphic. However, requiring this globally would be too restrictive. Instead we consider Gromov–Hausdorff approximants in the usual sense, which in turn can be locally approximated by holomorphic maps.

Definition 5.7
A collection $\{(X_i, p_i, d_i)\}_{i=1}^{\infty}$ of pointed complete metric Kähler spaces converges to a pointed complete metric Kähler space $(X_\infty, p_\infty, d_\infty)$ in the pointed complex Gromov–Hausdorff topology if for every $k \in \mathbb{Z}^+$, there is a covering of $B(p_\infty, k)$ by bounded open sets $\{U_{\infty,j}\}$ and associated plurisubharmonic functions $\{\phi_{\infty,j}\}$ so that for every $\epsilon > 0$, if i is sufficiently large, then there are

- a pointed ϵ-Gromov–Hausdorff approximation $h_i : B(p_\infty, k) \to B(p_i, k)$, and
- holomorphic maps $r_{i,j} : U_{\infty,j} \to M_i$ that are ϵ-close to h_i on $U_{\infty,j} \cap B(p_\infty, k)$, so that $r_{i,j}(U_{\infty,j})$ is contained in a set $V_{i,j}$ with an associated plurisubharmonic function $\phi_{i,j}$, and
- $r_{i,j}^* \phi_{i,j}$ is uniformly ϵ-close to $\phi_{\infty,j}$.

Note that in Definition 5.7, the limit space can have lower dimension than the approximants. In using Definition 5.7, we allow ourselves to pass to equivalent choices of $\{(V_{i,j}, \phi_{i,j})\}$ on M_i.

We now show that the "$BK \geq K$" condition is preserved under complex Gromov–Hausdorff limits.

PROPOSITION 5.8
If $\lim_{i \to \infty}(X_i, p_i, d_i) = (X_\infty, p_\infty, d_\infty)$ *in the pointed complex Gromov–Hausdorff topology, and each* (X_i, d_i) *has "*$BK \geq K$*," then* (X_∞, p_∞) *has "*$BK \geq K$*."*

Proof
Fix k. Given $p \in X_\infty$, let $\{m_i\}$ be points that approach it relative to the Gromov–Hausdorff convergence. Given $U_{\infty,j}$ as in Definition 5.7, we have

$$\lim_{i \to \infty} r_{i,j}^* (\phi_{i,j} - d_{K,m_i}^2/2) = \phi_{\infty,j} - d_{K,p}^2/2 \tag{5.9}$$

in $L^\infty(U_{\infty,j})$. As $r_{i,j}$ is holomorphic, it follows that $\phi_{\infty,j} - d^2_{K,p}/2$ is plurisubharmonic. $\square$

Finally, in the setting of Proposition 4.1, a subsequence converges in the complex Gromov–Hausdorff sense.

PROPOSITION 5.10
Let $\{(M_i, p_i, g_i)\}_{i=1}^\infty$ be a sequence of pointed n-dimensional complete Kähler manifolds with $BK \geq K$. Suppose that there is some $v_0 > 0$ so that for all i, $\mathrm{vol}(B(p_i, 1)) \geq v_0$. Then a subsequence converges in the pointed complex Gromov–Hausdorff topology.

Proof
This follows from the proof of Proposition 4.1(3). $\square$

6. Tangent cones

In this section, we prove an analogue of Remark 2.4(3).

6.1. Tangent cones as Kähler cones

We first characterize tangent cones of noncollapsed limit spaces.

PROPOSITION 6.1
Let $(X_\infty, p_\infty, d_\infty)$ be a limit space from Proposition 4.1. Let $T_{p_\infty} X_\infty$ be a tangent cone of X_∞ at p_∞. Then $T_{p_\infty} X_\infty$ is a Kähler cone that is biholomorphic to $\mathbb{C}^n$, with $r^2/2$ as a Kähler potential. It has "$BK \geq 0$."

Proof
As X_∞ is a noncollapsed limit of Riemannian manifolds with a uniform lower Ricci bound, $T_{p_\infty} X_\infty$ is a metric cone of the same dimension whose link has diameter at most π (see [6, Theorem 5.2]). After passing to a subsequence, we can write $(T_{p_\infty} X_\infty, 0) = \lim_{i\to\infty}(M_i, p_i, \mu_i^2 g_i)$, a pointed Gromov–Hausdorff limit, where $\lim_{i\to\infty} \mu_i = \infty$. Hence $(T_{p_\infty} X_\infty, 0)$ is a noncollapsed pointed limit of manifolds with the lower bound on BK going to zero. Proposition 4.1 implies that it satisfies (3.16) with $K = 0$.

Since a neighborhood of $x_\infty \in X_\infty$ is biholomorphic to a ball in $\mathbb{C}^n$, and $T_{p_\infty} X_\infty$ is a blowup limit, it makes sense that it should be biholomorphic to $\mathbb{C}^n$. To show this, we first construct the complex structure on $T_{p_\infty} X_\infty$, using the Kähler–Ricci flow.

By definition, $(T_{p_\infty} X_\infty, 0) = \lim_{k\to\infty}(X_\infty, p_\infty, \lambda_k d_\infty)$ as a pointed Gromov–Hausdorff limit, where $\lim_{k\to\infty} \lambda_k = \infty$. Let $g_\infty(\cdot)$ be the Kähler–Ricci flow constructed as in the proof of Proposition 4.1, with $t \to 0$ limit given by (X_∞, d_∞). The estimates (4.3)–(4.5) are valid for $g_\infty(\cdot)$. Define the parabolically rescaled Ricci flows $g_{\infty,k}(u) = \lambda_k^2 g_\infty(\lambda_k^{-2} u)$. After passing to a subsequence of the k's, we can assume that there is a pointed Cheeger–Hamilton limit

$$\left(T_{p_\infty} X_\infty, 0, g_{\infty,\infty}(\cdot)\right) = \lim_{k\to\infty} \left(X_\infty, p_\infty, g_{\infty,k}(\cdot)\right) \tag{6.2}$$

on the time interval $(0,\infty)$. Letting $B(0,l)$ denote the l-ball around the vertex 0 in $T_{p_\infty} X_\infty$, in taking the limit there are implicit embeddings $\sigma_{k,l} : B(0,l) \to X_\infty$ for large k so that $g_{\infty,\infty}(\cdot) = \lim_{k\to\infty} \sigma_{k,l}^* g_{\infty,k}(\cdot)$ on $[l^{-1}, l] \times B(0,l)$. In particular, $\sigma_{k,l}$ decreases distances by approximately λ_k, when going from $T_{p_\infty} X_\infty$ to (X_∞, d_∞).

As in the proof of Proposition 4.1, after passing to a subsequence, the pullbacks $\sigma_{k,l}^* J_\infty$ converge, as $k \to \infty$, to a complex structure on $B(0,l)$ (say, relative to the metric $g_{\infty,\infty}(1)$). Applying a diagonal argument, we obtain the complex structure $J_{\infty,\infty}$ on $T_{p_\infty} X_\infty$.

Let $\{z^a\}_{a=1}^n$ be local complex coordinates around p_∞ for X_∞. Note that $\sum_{a=1}^n |z^a|^2$ is strictly plurisubharmonic near p_∞. Put $z_{k,l}^a = \sigma_{k,l}^* z^a$, which for large k is a function on $B(0,l)$ that is holomorphic relative to $\sigma_{k,l}^* J_\infty$ and harmonic relative to $\sigma_{k,l}^* g_{\infty,k}(1)$. After a linear transformation, we can assume that $\int_{B(0,1)} z_{k,l}^a \overline{z_{k,l}^b}\, d\mu = \delta_{ab}$, where $d\mu$ is the n-dimensional Hausdorff measure on $T_{p_\infty} X_\infty$.

After passing to a subsequence of k's, there is a limit $z_{\infty,l}^a = \lim_{k\to\infty} z_{k,l}^a$, where $\{z_{\infty,l}^a\}_{a=1}^n$ are holomorphic functions on $B(0,l)$ with $\int_{B(0,1)} z_{\infty,l}^a \overline{z_{\infty,l}^b}\, d\mu = \delta_{ab}$. By a diagonal argument, we obtain independent holomorphic functions $\{z_\infty^a\}_{a=1}^n$ on $T_{p_\infty} X_\infty$. Let $F : T_{p_\infty} X_\infty \to \mathbb{C}^n$ be given by $F(q) = \{z_\infty^a(q)\}_{a=1}^n$. One sees by approximation that F is a proper holomorphic map of degree 1, and the level sets of $|F|^2$ are Stein domains. The preimage $F^{-1}(w)$ of a point $w \in \mathbb{C}^n$ is a compact subvariety in $T_{p_\infty} X_\infty$, so by the Stein property it is a finite set of points. It now follows from [11, Proposition 14.7, p. 87] that F is biholomorphic. Proposition 5.4 implies that $T_{p_\infty} X_\infty$ has "$BK \geq 0$."

To see that $r^2/2$ is a Kähler potential, we use an argument similar to [25, Section 4]. Let (M_i, p_i, g_i) be a sequence as in the beginning of the proof. Put $\widetilde{g}_i = \mu_i^2 g_i$ and $\widetilde{d}_{p_i} = \mu_i d_{p_i}$. Given $0 < a < b < \infty$ and $\epsilon > 0$, by [5, Proposition 4.38, Corollary 4.42, Corollary 4.83] there is a smooth approximate distance-squared function ρ_i for $(M_i, p_i, \widetilde{g}_i)$, defined on the metric annulus $\widetilde{d}_{p_i}^{-1}(a,b)$, so that

$$\begin{aligned} \|\rho_i - \widetilde{d}_{p_i}^2\|_{L^2}^2 &= o(i^0), \\ \|\widetilde{\nabla}\rho_i - \widetilde{\nabla}\widetilde{d}_{p_i}^2\|_{L^2}^2 &= o(i^0), \\ \left\|\widetilde{\mathrm{Hess}}\rho_i - \frac{1}{n}(\widetilde{\triangle}\rho_i)\widetilde{g}_i\right\|_{L^1} &= o(i^0). \end{aligned} \tag{6.3}$$

From [5, (4.25), Proposition 4.35], we also have

$$\|\widetilde{\triangle}\rho_i - n\|_{L^1} = o(i^0). \tag{6.4}$$

Hence

$$\|\widetilde{\mathrm{Hess}}\rho_i - \widetilde{g}_i\|_{L^1} = o(i^0). \tag{6.5}$$

In particular,

$$\|\sqrt{-1}\partial\overline{\partial}\rho_i - \widetilde{\omega}_i\|_{L^1} = o(i^0). \tag{6.6}$$

From Proposition 5.10, after passing to a subsequence, $\lim_{i\to\infty}(M_i, p_i, \widetilde{g}_i) = (T_{p_\infty}X_\infty, 0)$ in the pointed complex Gromov–Hausdorff topology. It follows from (6.6) that if ϕ_∞ is a local Kähler potential for $T_{p_\infty}X_\infty$, supported away from 0, then $\sqrt{-1}\partial\overline{\partial}(\frac{r^2}{2} - \phi_\infty) = 0$ as a current. Hence $\frac{r^2}{2}$ is a Kähler potential for $T_{p_\infty}X_\infty - 0$.

There is some continuous Kähler potential ϕ_0 defined in a neighborhood U_0 of 0. Then $\frac{r^2}{2} - \phi_0$ is continuous on U_0 and pluriharmonic on $U_0 - 0$. Thinking of it as a function in a neighborhood of $0 \in \mathbb{C}^n$, it follows that $\frac{r^2}{2} - \phi_0$ extends to a continuous pluriharmonic function on U_0 (which is then actually smooth). Hence $\frac{r^2}{2}$ is a Kähler potential on $T_{p_\infty}X_\infty$. □

6.2. *Curvature of the $\mathbb{C}P^{n-1}$ quotient*

We denote the generator of radial rescaling on $T_{p_\infty}X_\infty$ by $r\partial_r$. From [27, Proof of Proposition 15], $r\partial_r$ and $J_{\infty,\infty}(r\partial_r)$ generate 1-parameter groups that are holomorphic on an open dense subset of $\mathbb{C}^n \cong T_{p_\infty}X_\infty$. The 1-parameter group $\{\sigma_t\}$ generated by $J_{\infty,\infty}(r\partial_r)$ acts isometrically on $T_{p_\infty}X_\infty$ and preserves level sets of the distance function d_0 from the vertex p_∞. Following terminology about Sasaki manifolds, we say that the structure is regular if $\{\sigma_t\}$ comes from a free S^1-action. Then the quotient of $T_{p_\infty}X_\infty$ by the group action is a cone over a manifold.

In order to put ourselves in the setting of a regular structure, we assume that d_0 is a radially homogeneous function on $\mathbb{C}^n \cong T_{p_\infty}X_\infty$. That is, letting $\zeta : \mathbb{C}^n - 0 \to \mathbb{C}P^{n-1}$ denote the quotient map, we assume that there are a number $\delta > 0$ and a function $H \in C(\mathbb{C}P^{n-1})$ so that

$$d_0(z) = |z|^\delta H\big(\zeta(z)\big) \tag{6.7}$$

on $\mathbb{C}^n - 0$. (As an example, this is the case for a 2-dimensional cone.) Then

$$r\partial_r = \delta^{-1}\Big(\sum_{\alpha=1}^n z^\alpha \partial_{z^\alpha} + \sum_{\alpha=1}^n \overline{z}^\alpha \partial_{\overline{z}^\alpha}\Big) \tag{6.8}$$

and $\{\sigma_t\}$ is the Hopf action on the level sets of d_0. The quotient of the link $d_0^{-1}(1) = S^{2n-1}$ by the Hopf action is $\mathbb{C}P^{n-1}$, with a possibly nonstandard quotient metric $d_{\mathbb{C}P^{n-1}}$.

Let T be the tautological complex line bundle over $\mathbb{C}P^{n-1}$, whose fibers are lines through the origin in $\mathbb{C}^n$. The complement of the zero section in T is biholomorphic to $\mathbb{C}^n - 0$. We will also let $\zeta : T \to \mathbb{C}P^{n-1}$ denote the projection map from T to the base. Consider a local holomorphic trivialization of T, and let w be the fiber coordinate, with $w = 0$ corresponding to the vertex $0 \in T_{p_\infty} X_\infty$. Then $d_0^2 = h|w|^{2\delta}$ for some locally defined continuous function h on the base. We put a Kähler space structure on $\mathbb{C}P^{n-1}$ by saying that $\frac{1}{2}\log h$ is a local potential.

PROPOSITION 6.9
We have that $(\mathbb{C}P^{n-1}, d_{\mathbb{C}P^{n-1}})$ is a metric Kähler space with "$BK \geq 2$."

Proof
Let $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$ be the quotient map. Fix $z' \in \mathbb{C}P^{n-1}$, and let $S \subset \mathbb{C}^n$ be the corresponding complex line.

LEMMA 6.10
Let (r, s) denote a point in the metric cone $T_{p_\infty} X_\infty$ where $r \geq 0$ and $s \in S^{2n-1}$. Put $z = \pi(s)$. Then $d((r,s), S) = r \sin(d_{\mathbb{C}P^{n-1}}(z, z'))$.

Proof
By the definition of the metric cone,

$$d\big((r,s),(r',s')\big) = \sqrt{r^2 + (r')^2 - 2rr' \cos\big(d_{S^{2n-1}}(s,s')\big)}. \tag{6.11}$$

Minimizing over r' gives

$$d\big((r,s), S\big) = r \min_{s' \in S \cap S^{2n-1}} \sin\big(d_{S^{2n-1}}(s,s')\big). \tag{6.12}$$

As the S^1-action is isometric, the lemma follows from the definition of the quotient metric. □

From Lemma 5.3, we know that

$$\phi - d_S^2/2 = \frac{1}{2} r^2 \zeta^* \cos^2 d_{z'}^2 \tag{6.13}$$

is plurisubharmonic on $T_{p_\infty} X_\infty - 0 \cong \mathbb{C}^n - 0$.

Working locally on $\mathbb{C}P^{n-1}$ and putting

$$\begin{aligned} Dw &= \delta dw + wh^{-1}\partial h, \\ D\overline{w} &= \delta\, d\overline{w} + \overline{w}h^{-1}\overline{\partial}h, \\ \Omega &= \sqrt{-1}\partial\overline{\partial}\log h, \end{aligned} \tag{6.14}$$

one finds that

$$\begin{aligned} \partial r^2 &= |w|^{2\delta} h w^{-1} Dw, \\ \overline{\partial} r^2 &= |w|^{2\delta} h \overline{w}^{-1} D\overline{w}, \\ \sqrt{-1}\partial\overline{\partial} r^2 &= \sqrt{-1}|w|^{2(\delta-1)} h Dw \wedge D\overline{w} + |w|^{2\delta} h\Omega \end{aligned} \tag{6.15}$$

as currents.

To show that $(\mathbb{C}P^{n-1}, d_{\mathbb{C}P^{n-1}})$ is a metric Kähler space, it remains to show that if Σ is a holomorphic disk in the domain of h, then $\frac{1}{2}\sqrt{-1}\partial\overline{\partial}\log h|_{\mathrm{Dom}(h)\cap\Sigma}$ equals the 2-dimensional Hausdorff measure dA on $\mathrm{Dom}(h) \cap \Sigma$. Put $\Gamma = \zeta^{-1}(\Sigma)$, a 4-dimensional submanifold of $T_{p_\infty}X_\infty - 0$. Let $\mathscr{H}$ denote the 4-dimensional Hausdorff measure on Γ. As in the proof of Proposition 6.1, there is a Kähler–Ricci flow whose pointed Gromov–Hausdorff limit as $t \to 0$ is $T_{p_\infty}X_\infty$. Let $\mathscr{H}_t$ be the 4-dimensional Hausdorff measure on Γ coming from $d_t|_\Gamma$. It equals $\frac{1}{2}(\sqrt{-1}\partial\overline{\partial}\phi(t))^2$, where $\phi(t)$ is a local Kähler potential for the flow. Using [8, Chapter 3.3] and proceeding as in the proof of Proposition 4.1(2), it follows that $\lim_{t\to 0}\mathscr{H}_t = \frac{1}{2}(\sqrt{-1}\partial\overline{\partial}r^2/2)^2$. Also as in the proof of Proposition 4.1(2), we have $\lim_{t\to 0}\mathscr{H}_t = \mathscr{H}$. Hence

$$\mathscr{H} = \frac{1}{2}(\sqrt{-1}\partial\overline{\partial}r^2/2)^2 = \frac{1}{4}\sqrt{-1}|w|^{4\delta-2}h^2 Dw \wedge D\overline{w} \wedge \Omega \tag{6.16}$$

as a measure on Γ.

From (6.15), the area form on a preimage of ζ is

$$\frac{1}{2}\sqrt{-1}\delta^2|w|^{2(\delta-1)}h\, dw \wedge d\overline{w}. \tag{6.17}$$

Since the area of a level set of w is proportionate to $h|w|^{2\delta}$, doing a fiberwise integration on Γ gives

$$\int_{|w|\le 1}\mathscr{H} = \Big(\int_{B^2}\delta^2|z|^{4\delta-2}\cdot\frac{1}{2}\sqrt{-1}\,dz\wedge d\overline{z}\Big)h^2\,dA. \tag{6.18}$$

On the other hand, from (6.16),

$$\int_{|w|\le 1}\mathscr{H} = \Big(\int_{B^2}\delta^2|z|^{4\delta-2}\cdot\frac{1}{2}\sqrt{-1}\,dz\wedge d\overline{z}\Big)\cdot\frac{1}{2}h^2\Omega. \tag{6.19}$$

Thus $dA = \frac{1}{2}\Omega$ on $\mathrm{Dom}(h) \cap \Sigma$. Since Ω equals $\sqrt{-1}\partial\overline{\partial}\log h$, this shows that $(\mathbb{C}P^{n-1}, d_{\mathbb{C}P^{n-1}})$ is a metric Kähler space.

Finally, put $C = \cos d_{z'} \in C(\mathbb{C}P^{n-1})$, which we will identify with its pullback to T, and put

$$\begin{aligned} D_C w &= Dw + wC^{-2}\partial C^2, \\ D_C \overline{w} &= D\overline{w} + \overline{w}C^{-2}\overline{\partial}C^2. \end{aligned} \tag{6.20}$$

One finds that

$$\begin{aligned} &\sqrt{-1}C^{-2}\partial\overline{\partial}(r^2C^2) \\ &\quad = \sqrt{-1}|w|^{2(\delta-1)}hD_C w \wedge D_C\overline{w} \\ &\qquad + |w|^{2\delta}h(\Omega + \sqrt{-1}C^{-2}\partial\overline{\partial}C^2 - \sqrt{-1}C^{-4}\partial C^2 \wedge \overline{\partial}C^2), \end{aligned} \tag{6.21}$$

as equalities of currents. Hence from (6.13), it follows that

$$\Omega + \sqrt{-1}C^{-2}\partial\overline{\partial}C^2 - \sqrt{-1}C^{-4}\partial C^2 \wedge \overline{\partial}C^2 \geq 0 \tag{6.22}$$

or

$$-\sqrt{-1}\partial\overline{\partial}\log C^2 \leq \Omega. \tag{6.23}$$

Equivalently, $\frac{1}{2}\log h - d^2_{2,z'}/2$ is plurisubharmonic, where $d^2_{2,z'}$ is defined in (1.1), which means that $(\mathbb{C}P^{n-1}, d_{\mathbb{C}P^{n-1}})$ has "$BK \geq 2$." □

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