BOSONIZATION IN CURVED SPACETIME

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The bosonization of a pair of two-dimensional Weyl fermions, of opposite chiralities, is considered. When the fermions are in different metrics the bosonization procedure gives a unique expression for the gravitational anomaly.

The classical bosonization of a Dirac fermion in a two-dimensional background gauge field is closely related to the gauge anomaly [1]. Under a variation of the gauge field, one sees directly the anomalous variation of the effective action from the bosonized lagrangian. We wish to show that the gravitational anomaly [2] can also be seen in this way.

As is well known, in 2D Minkowski space a pair of right- and left-handed Weyl fermions feeling two different gauge fields can be interpreted as a Dirac fermion feeling one gauge field. If one requires vector invariance for such a theory, one obtains a unique effective action [3]. (This corresponds to requiring that the euclidean fermion determinant be invariant under conjugation.) We show that the same trick works for Weyl fermions feeling two different metrics; one obtains a Dirac fermion feeling one metric along with a C-valued gauge field, giving a unique effective action. The fermion determinant for a single Weyl fermion has been discussed in refs. [4,5] [‡1].

First, consider a Dirac fermion in background metric g and U(1) gauge field A. On euclidean S^2, upon integrating out the fermions, the effective action is $\Gamma = -(1/2)\ln\det \not{D}^2_{g,A}$. From the axial anomaly, this is

$$\Gamma = -\tfrac{1}{2}\ln\det \not{D}_g^2 + (1/4\pi)\langle F_{ab}, \Delta_g^{-1} F_{ab}\rangle\,, \qquad (1)$$

which is clearly diffeomorphism invariant. Under a conformal change of metric $\delta g = 2\sigma g$, we have

$$\delta(-\tfrac{1}{2}\ln\det \not{D}_g^2) = -\frac{1}{24\pi}\int \sigma R\sqrt{g}$$
$$= \delta(\tfrac{1}{2}\ln\det\Delta_g)\,. \qquad (2)$$

Thus Γ equals the effective action for the bosonic lagrangian

$$L_{\mathrm{B}} = \int \sqrt{g}[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{\mathrm{i}}{2\sqrt{\pi}}\phi\epsilon^{\mu\nu}F_{\mu\nu}]\,, \qquad (3)$$

namely

$$\Gamma = -\frac{1}{48\pi}\int\sqrt{g}R\Delta_g^{-1}R + \frac{1}{4\pi}\int\sqrt{g}F_{ab}\Delta_g^{-1}F^{ab}\,. \qquad (4)$$

When rotated to Minkowski space, this shows the equivalence between

$$L_{\mathrm{F}} = \int\sqrt{-g}\,\mathrm{i}\bar{\psi}\gamma^\mu \mathrm{D}_\mu\psi \qquad (5)$$

and

$$L_{\mathrm{B}} = \int\sqrt{-g}\,[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + (1/2\sqrt{\pi})\phi\epsilon^{\mu\nu}F_{\mu\nu}]\,, \qquad (6)$$

which is the content of the Sugawara relation [6,7]

$$T_{\mu\nu} = J_\mu J_\nu - \tfrac{1}{2}\eta_{\mu\nu}J^2\,.$$

Now let P be the principal SO(2) bundle over Minkowski R^2 and let d^i and e^i be soldering forms on P (local cross sections of which are zweibeins $d_\mu{}^i$ and $e_\mu{}^i$). The action for left- and right-handed fermions in the backgrounds of d^i and e^i respectively is

‡1 Our conventions are $\{\gamma^a,\gamma^b\} = 2\eta^{ab}$, $\sigma^{ab} = (1/4)[\gamma^a,\gamma^b]$, $\gamma^3 = [1/2(-\det\eta)^{1/2}][\gamma^0,\gamma^1]$, $P_+ = (1/2)(1+\gamma^3)$, $P_- = (1/2)(1-\gamma^3)$, $\psi_+ = P_+\psi$, $\psi_- = P_-\psi$, $X^\pm = (1/\sqrt{2})(X^0 \pm X^1)$ and $\Delta_g = -\nabla_g^2$.

$$L_F = \mathrm{i} \int [\bar{\psi}_+ d_j^{\mu} \gamma^j \nabla_\mu(d)\, \psi_+ d^1 \wedge d^2$$

$$+ \bar{\psi}_- e_j^{\mu} \gamma^j \nabla_\mu(e)\, \psi_- e^1 \wedge e^2] \,. \tag{7}$$

Define a new soldering form f by $f^+ = e^+$ and $f^- = d^-$, and a C-valued one-form A by

$$A = -\tfrac{1}{2}\, \mathrm{i} f_+^{\mu} [\omega_{-+\mu}(d) - \omega_{-+\mu}(f)] f^+$$

$$+ \tfrac{1}{2}\, \mathrm{i} f_-^{\mu} [\omega_{-+\mu} e) - \omega_{-+\mu}(f)] f^- \,. \tag{8}$$

Then one finds that

$$L_F = \mathrm{i} \int \bar{\psi} f_j^{\mu} \gamma^j [\nabla_\mu(f) + \mathrm{i} A_\mu]\, \psi f^1 \wedge f^2 \,. \tag{9}$$

If g is the metric coming from f, it follows that an equivalent bosonic action for (7) is (6).

The effective action for (7) is given by (4) when rotated to Minkowski space. To find the form of the gravitational anomalies, it is easiest to compute the anomalous variations of Γ in euclidean space, letting f again denote the background soldering form and A denote the background gauge field. The putative symmetry group is the group Aut(P) of automorphisms of P, which has the group Diff(S^2) as a quotient group. We will break up the formal Lie algebra aut(P) as $\{\rho + \nabla_X\}$ where ρ is an infinitesimal frame rotation and ∇_X is infinitesimal parallel transport with respect to $\omega(f)$ over the vector field X. This will act on the space of d^i's. [Because Diff(S^2) is not a subgroup of Aut(P), it generally does not make sense to talk of Lie derivatives of spinors or soldering forms.]

Under the frame rotation $\delta d^i = \rho_{01} \epsilon^i{}_j d^j$, one has

$$\delta \not{D} = \tfrac{1}{2} P_- \rho_{ab} \sigma^{ab} \not{D} - \not{D}\, \tfrac{1}{2} \rho_{ab} \sigma^{ab} P_+ \,, \tag{10}$$

and

$$\delta\Gamma = -\tfrac{1}{2}\, \delta \ln \det \not{D}^2 = -\lim_{s\to 0} \mathrm{Tr}(\not{D}^2)^{-s-1} \not{D} \,,$$

$$[\tfrac{1}{2} P_- \rho_{ab} \sigma^{ab} \not{D} - \not{D}\, \tfrac{1}{2}\, \rho_{ab} \sigma^{ab} P_+]$$

$$= \lim_{s\to 0} \tfrac{1}{2}\, \mathrm{i}\, \mathrm{Tr}\, \rho_{01} (\not{D}^2)^{-s} = \frac{\mathrm{i}}{24\pi} \int \sqrt{g} R(g) \rho_{01} \,. \tag{11}$$

Similarly, given a vector field X on S^2, under the variation $\delta d^i = \nabla(d) X^i + [\omega_{01\sigma}(f) - \omega_{01\sigma}(d)] X^\sigma \epsilon^i{}_j d^j$, one has

$$\delta \not{D} = P_- \nabla_X(f) \not{D} - \not{D} \nabla_X(f) P_+ \,, \tag{12}$$

and

$$\delta\Gamma = -\tfrac{1}{2}\, \delta \ln \det \not{D}^2 = -\lim_{s\to 0} \mathrm{Tr}(\not{D}^2)^{-s-1} \not{D} \,,$$

$$[P_- \nabla_X(f) \not{D} - \not{D} \nabla_X(f) P_+]$$

$$= \lim_{s\to 0} \mathrm{Tr}\, \gamma^3 \nabla_X(f) (\not{D}^2)^{-s}$$

$$= -\frac{1}{8\pi} \int \sqrt{g} X^\mu \partial_\mu (\epsilon^{ab} F_{ab}) \,. \tag{13}$$

By construction, these anomaly expressions satisfy the Wess-Zumino consistency condition, and presumably differ by the variation of local counterterms from the index theorem expressions.

As with the classical bosonization, one wishes to also let ϕ vary under the above variations in such a way that the effective action for (6), (including the ϕ field), changes exactly by the anomaly. To do this it is necessary to include the diffeomorphism change of g and the axial change of A. If δg is written as $\mathcal{L}_V g + \alpha g$ for some vector field V, and $\delta A_\mu = (\mathcal{L}_V A)_\mu + \partial_\mu \eta + \epsilon_{\mu\nu} \partial^\nu \chi$ then one needs $\delta\phi = V^\mu \partial_\mu \phi + (1/\sqrt{\pi})\chi$. Under these variations, the effective action for (6) changes exactly by the above expressions.

Finally, if the Weyl spinors are in background gauge fields B and C then this can be handled by adding a gauge field D to (9) which satisfies $D_+ = B_+$ and $D_- = C_-$. In this way one obtains the mixed gauge–gravitation anomalies.

References

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