DIFFERENTIAL FORMS, SPINORS AND BOUNDED CURVATURE COLLAPSE

John Lott

University of Michigan

November, 2000

From preprints

"Collapsing and the Differential Form Laplacian"

"On the Spectrum of a Finite-Volume Negatively-Curved Manifold"

"Collapsing and Dirac-Type Operators"

http://www.math.lsa.umich.edu/~lott

WHAT I WILL (NOT) TALK ABOUT

Fukaya (1987) : Studied the function Laplacian, under a bounded curvature collapse.

Cheeger-Colding (preprint) : Studied the function Laplacian, under a collapse with Ricci curvature bounded below.

Today : The differential form Laplacian and geometric Dirac-type operators, under a bounded curvature collapse.

MOTIVATION : CHEEGER'S INEQUALITY

Let M be a connected compact Riemannian manifold.

Spectrum of $\triangle^M$:

$$0 = \lambda_1(M) < \lambda_2(M) \leq \lambda_3(M) \leq \dots$$

Theorem 1. *(Cheeger 1969)*

$$\lambda_2(M) \ \geq \ \frac{h^2}{4}.$$

Definition 1.

$$h \ = \ \inf_A \frac{Area(A)}{\min(vol(M_1), vol(M_2))},$$

where A ranges over separating hypersurfaces.

Question (Cheeger) : Is there a similar inequality for the p-form Laplacian? (Problem # 79 on Yau's list.)

THE p-FORM LAPLACIAN

The p-form Laplacian is

$$\triangle_p^M = dd^* + d^*d : \Omega^p(M) \to \Omega^p(M).$$

Spectrum of $\triangle_p^M$:

$$0 \leq \lambda_{p,1}(M) \leq \lambda_{p,2}(M) \leq \lambda_{p,3}(M) \leq ...$$

By Hodge theory,

$$\mathrm{Ker}(\triangle_p^M) \cong \mathrm{H}^p(M;\mathbb{R}),$$

so

$$0 = \lambda_{p,1}(M) = \ldots = \lambda_{p,b_p(M)}(M) < \lambda_{p,b_p(M)+1}(M) \leq \ldots .$$

COLLAPSING (Cheeger, Gromov)

Notation :

$$R^M = \text{ Riemann sectional curvatures.}$$
$$\operatorname{diam}(M) = \sup_{p,q \in M} d_M(p,q).$$

Fix a number $K \geq 0$. Consider

$\{$ connected Riem. manifolds $M^n : \| R^M \|_\infty \leq K$ and $\operatorname{diam}(M) \leq 1\}$.

There is a number $v_0(n, K) > 0$ such that one has the following dichotomy :

I. Noncollapsing case : $\operatorname{vol}(M) \geq v_0$.

Finite number of topological types, $C^{1,\alpha}$-metric rigidity. In particular, uniform bounds on eigenvalues of $\triangle_p$.

or

II. Collapsing case : $\operatorname{vol}(M) < v_0$.

Special structure. Need to analyze $\triangle_p$ in this case.

BERGER EXAMPLE OF COLLAPSING

Hopf fibration $\pi : S^3 \to S^2$

Shrink the circles to radius ϵ. Look at the 1-form Laplacian $\triangle_1$. Since $\mathrm{H}^1(S^3;\mathbb{R}) = 0$, the first eigenvalue $\lambda_{1,1}$ of $\triangle_1$ is positive.

Fact (Colbois-Courtois 1990)

$$\lim_{\epsilon \to 0} \lambda_{1,1} = 0.$$

New phenomenon : (uncontrollably) small eigenvalues.

When does this happen?

FUKAYA'S WORK ON THE FUNCTION LAPLACIAN

Suppose that $\{M_i\}_{i=1}^{\infty}$ are connected n-dimensional Riemannian manifolds with $\| R^{M_i} \|_{\infty} \leq K$ and $\mathrm{diam}(M_i) \leq 1$.

Suppose that $M_i \stackrel{GH}{\longrightarrow} X$.

Consider the function Laplacian on M_i, with eigenvalues $\{\lambda_j(M_i)\}_{j=1}^{\infty}$.

Question : Suppose that X is a smooth Riemannian manifold. Is it true that $\lim_{i\to\infty} \lambda_j(M_i) = \lambda_j(X)$?

Answer : In general, no. Need to add a probability measure μ to X.

Laplacian on weighted L^2-space :

$$\frac{< f, \triangle^{X,\mu} f >}{< f, f >} = \frac{\int_X |\nabla f|^2 \, d\mu}{\int_X f^2 \, d\mu}.$$

Theorem 2. *(Fukaya 1987) If*

$$\lim_{i\to\infty} \left(M_i, \frac{dvol_{M_i}}{vol(M_i)} \right) = (X, \mu)$$

in the measured Gromov-Hausdorff topology then

$$\lim_{i\to\infty} \lambda_j(M_i) = \lambda_j(X, \mu).$$

GOAL

Want a "p-form Laplacian" on the limit space X so that after taking a subsequence,

$$\lambda_{p,j}(M_i) \longrightarrow \lambda_{p,j}(X).$$

Question : What kind of structure do we need on X?

SUPERCONNECTIONS (Quillen 1985, Bismut-L. 1995)

Input :

B a smooth manifold,

$E = \bigoplus_{j=0}^{m} E^j$ a $\mathbb{Z}$-graded real vector bundle on B.

The (degree-1) superconnections A' that we need will be formal sums of the form

$$A' \;=\; A'_{[0]} \;+\; A'_{[1]} \;+\; A'_{[2]}$$

where
- $A'_{[0]} \in C^\infty\left(B; \mathrm{Hom}(E^*, E^{*+1})\right)$,
- $A'_{[1]}$ is a grading-preserving connection ∇^E on E and
- $A'_{[2]} \in \Omega^2\left(B; \mathrm{Hom}(E^*, E^{*-1})\right)$.

Then $A' : C^\infty(B; E) \to \Omega(B; E)$ extends by Leibniz' rule to an operator $A' \,:\, \Omega(B; E) \to \Omega(B; E)$.

Flatness condition : $(A')^2 = 0$,

i.e.
- $\left(A'_{[0]}\right)^2 = \left(A'_{[2]}\right)^2 = 0$,
- $\nabla^E A'_{[0]} = \nabla^E A'_{[2]} = 0$ and
- $\left(\nabla^E\right)^2 + A'_{[0]} A'_{[2]} + A'_{[2]} A'_{[0]} = 0$.

Note : $A'_{[0]}$ gives a differential complex on each fiber of E.

THE NEEDED STRUCTURE ON THE LIMIT SPACE X

A triple (E, A', h^E), where

1. E is a $\mathbb{Z}$-graded real vector bundle on X,
2. A' is a flat degree-1 superconnection on E and
3. h^E is a Euclidean inner product on E.

We have

$$A' : \Omega(X; E) \to \Omega(X; E).$$

Using g^{TX} and h^E, we get

$$(A')^* : \Omega(X; E) \to \Omega(X; E).$$

Put

$$\triangle^E = A'(A')^* + (A')^* A',$$

the superconnection Laplacian.

Example : If E is the trivial $\mathbb{R}$-bundle on X, A' is the trivial connection and h^E is the standard inner product on E then $\triangle^E$ is the Hodge Laplacian.

Theorem 3. *If $M_i \xrightarrow{GH} X$ with bounded sectional curvature then after taking a subsequence, there is a certain triple (E, A', h^E) on X such that*

$$\lim_{i\to\infty} \sigma\left(\triangle_p^{M_i}\right) \; = \; \sigma\left(\triangle_p^{E}\right).$$

Remark 1 : This is a pointwise convergence statement, i.e. for each j, the j-th eigenvalue converges.

Remark 2 : Here the limit space X is assumed to be a Riemannian manifold. There is an extension to singular limit spaces (see later).

Remark 3 : The relation to Fukaya's work on functions:

For functions, only $\Omega^0(X; E^0)$ is relevant. Here E^0 is a trivial $\mathbb{R}$-bundle on X with a trivial connection. But its metric h^{E^0} may be nontrivial and corresponds to Fukaya's measure μ.

IDEA OF PROOF

1. The individual eigenvalues $\lambda_{p,j}$ are continuous with respect to the metric on M, in the C^0-topology (Cheeger-Dodziuk).

2. By Cheeger-Fukaya-Gromov, if M is Gromov-Hausdorff close to X then we can slightly perturb the metric to get a Riemannian affine fiber bundle. That is,

affine fiber bundle : M is the total space of a fiber bundle $M \to X$ with infranil fiber Z, whose holonomy can be reduced from $\mathrm{Diff}(Z)$ to $\mathrm{Aff}(Z)$.

Riemannian affine fiber bundle : In addition, one has
a. A horizontal distribution $T^H M$ on M with holonomy in $\mathrm{Aff}(Z)$, and
b. Fiber metrics g^{TZ} which are fiberwise affine-parallel.

So it's enough to just consider Riemannian affine fiber bundles.

3. If M is a Riemannian affine fiber bundle then $\sigma(\triangle_p^M)$ **equals** $\sigma(\triangle_p^E)$ up to a high level, which is on the order of $d_{GH}(M,X)^{-2}$. Here E is the vector bundle on X whose fiber over $x \in X$ is

$$E_x = \{\text{affine-parallel forms on } Z_x\}.$$

4. Show that the ensuing triples $\{(E_i, A_i', h^{E_i})\}_{i=1}^{\infty}$ have a convergent subsequence (modulo gauge transformation).

APPLICATION TO SMALL EIGENVALUES

Fix M and $K \geq 0$. Consider

$$\{g : \| R^M(g) \|_\infty \leq K \text{ and } \operatorname{diam}(M, g) \leq 1\}.$$

Question : Among these metrics, are there more than $\mathrm{b}_p(M)$ small eigenvalues of $\triangle_p^M$?

Suppose so, i.e. that for some $j > \mathrm{b}_p(M)$, there are metrics $\{g_i\}_{i=1}^\infty$ so that

$$\lambda_{p,j}(M, g_i) \stackrel{i \to \infty}{\longrightarrow} 0.$$

Step 1. Using Gromov precompactness, take a convergent subsequence of spaces

$$(M, g_i) \stackrel{i \to \infty}{\longrightarrow} X.$$

Since there are small positive eigenvalues, we must be in the collapsing situation.

Step 2. Using the analytic compactness theorem, take a further subsequence to get a triple (E, A', h^E) on X. Then

$$\lambda_{p,j}(\triangle^E) = 0.$$

In the limit, we've turned the small eigenvalues into **extra zero eigenvalues**.

Recall that $\triangle^E$ has the Hodge form $A'(A')^* + (A')^* A'$. Then from Hodge theory,

$$\dim(\mathrm{H}^p(A')) \geq j.$$

Analysis $\longrightarrow$ Topology

Fact : There is a spectral sequence to compute $\mathrm{H}^p(A')$. Analyze the spectral sequence.

RESULTS ABOUT SMALL EIGENVALUES

Theorem 4. *Given M, there are no more than* $\mathrm{b}_1(M)+\dim(M)$ *small eigenvalues of the* 1*-form Laplacian.*

More precisely, if there are j small eigenvalues and $j > \mathrm{b}_1(M)$ then in terms of the limit space X,

$$j \leq \mathrm{b}_1(X) + \dim(M) - \dim(X).$$

(Sharp in the case of the Berger sphere.)

More generally, where do small eigenvalues come from?

Theorem 5. *Let M be the total space of an affine fiber bundle $M \to X$, which collapses to X. Suppose that there are small positive eigenvalues of $\triangle_p$ in the collapse. Then there are exactly three possibilities :*

1. *The infranil fiber Z has small eigenvalues of its q-form Laplacian for some $0 \leq q \leq p$. That is,* $\mathrm{b}_q(Z) < \dim\{$*affine-parallel q-forms on Z*$\}$.

OR

2. *The "direct image" cohomology bundle* H^q *on X has a holonomy representation* $\pi_1(X) \to \mathrm{Aut}(\mathrm{H}^q(Z;\mathbb{R}))$ *which fails to be semisimple, for some $0 \leq q \leq p$.*

OR

3. *The Leray spectral sequence to compute* $\mathrm{H}^p(M;\mathbb{R})$ *does not degenerate at the E_2 term.*

Each of these cases occurs in examples.

UPPER EIGENVALUE BOUNDS

Theorem 6. *Fix M. If there is* not *a uniform upper bound on $\lambda_{p,j}$ (among metrics with $\| R^M \|_\infty \leq K$ and* $\mathrm{diam}(M) = 1$*) then M collapses to a limit space X with* $1 \leq \dim(X) \leq p-1$.

In addition, the generic fiber Z of the fiber bundle $M \to X$ is an infranilmanifold which does not admit nonzero affine-parallel k-forms for $p - \dim(X) \leq k \leq p$.

Example : Given M, if there is *not* a uniform upper bound on the j-th eigenvalue of the 2-form Laplacian then M collapses with bounded curvature to a 1-dimensional limit space. We know what such M look like.

SINGULAR LIMIT SPACES

Technical problem : in general, a limit space of a bounded-curvature collapse is not a manifold.

Theorem 7. *(Fukaya 1986) : A limit space X is of the form $\check{X}/G$, where $\check{X}$ is a Riemannian manifold and $G \subset \mathrm{Isom}(\check{X})$.*

What should the "forms on X" be? Answer : the basic forms on $\check{X}$.
$\Omega^*_{basic}(\check{X}) = \{\omega \in \Omega^*(\check{X}) : \omega$ is G-invariant and for all $\mathfrak{x} \in \mathfrak{g}$, $i_{\mathfrak{x}}\omega = 0\}$.

Fact : One can do analysis on the singular space X by working G-equivariantly on $\check{X}$, i.e. construct superconnection Laplacians, etc. The preceding results extend to this setting.

FINITE-VOLUME NEGATIVELY-CURVED MANIFOLDS

Theorem 8. *Let M^n be a complete connected Riemannian manifold with* $\mathrm{vol}(M) < \infty$ *and* $-b^2 \leq R^M \leq -a^2$*, with* $0 < a \leq b$*. Then the space of square-integrable harmonic p-forms on M is finite-dimensional.*

Previously known to be true if $p \neq \frac{n-1}{2}$ and $\frac{b}{a}$ is close enough to one (Donnelly-Xavier).

The result is also true if M just has bounded curvature and asymptotically-cylindrical ends, as long as the cross-sections of the ends are not too big.

Theorem 9. *There is a number $\delta(n) > 0$ such that if*
1. M^n is a complete connected Riemannian manifold,
2. $\| R^M \|_\infty \leq b^2$ and
3. The ends of M are $\delta(n)\, b^{-1}$-Gromov-Hausdorff close to rays
then the space of square-integrable harmonic p-forms on M is finite-dimensional.

Theorem 10. *If M is a finite-volume negatively-curved manifold as above then one can write down an explicit ordinary differential operator whose essential spectrum coincides with that of the p-form Laplacian on M.*

GEOMETRIC DIRAC-TYPE OPERATORS

Spinor modules V :

Say G is $SO(n)$ or $\mathrm{Spin}(n)$, and V is a Hermitian G-module. Suppose that there is a G-equivariant map $\gamma : \mathbb{R}^n \to \mathrm{End}(V)$ such that

$$\gamma(v)^2 \;=\; |v|^2 \,\mathrm{Id}.$$

Geometric Dirac-type operators :

Let M^n be a closed Riemannian manifold which is oriented or spin. Let V be a spinor module and let D^M be the corresponding Dirac-type operator. (Special cases: signature operator, pure Dirac operator.)

Theorem 11. *Suppose that $M_i \xrightarrow{GH} X$ with bounded curvature, with X smooth. Then after taking a subsequence, there are a Clifford-module E on X and a certain first-order elliptic operator D^E on $C^\infty(X;E)$ such that*

$$\lim_{i\to\infty} \sigma\left(D^{M_i}\right) \;=\; \sigma\left(D^E\right).$$

DIRAC OPERATORS ON SINGULAR SPACES

Suppose now that $M_i \xrightarrow{GH} X$ with bounded curvature, but with X singular. To describe the limit of $\sigma(D^{M_i})$, we need a Dirac-type operator on the singular space X. How to do this?

Let P_i be the principal G-bundle on M_i. Following Fukaya, we can assume that $P_i \xrightarrow{GH} \check{X}$, with $\check{X}$ a G-manifold. We want to define a Dirac-type operator on $X = \check{X}/G$.

Fundamental Problem : There is no notion of a "G-basic spinor".

Resolution : Observe that a spinor field on M_i is a G-invariant element of $C^\infty(P_i) \otimes V$. Take $P_i \longrightarrow \check{X}$.

Definition 2. *A "spinor field on X" is a G-invariant element of $C^\infty(\check{X}) \otimes V$.*

Fact : There's a certain first-order transversally elliptic operator $\check{D}$ on $C^\infty(\check{X}) \otimes V$.

Definition 3. *The Dirac-type operator D on X is the restriction of $\check{D}$ to the G-invariant subspace of $C^\infty(\check{X}) \otimes V$.*

APPLICATIONS TO SPECTRAL ANALYSIS OF DIRAC-TYPE OPERATORS

With this notion of the Dirac operator on X, one can prove a general convergence theorem for $\sigma(D^{M_i})$.

An application to upper eigenvalue bounds :

Theorem 12. *Fix M and the spinor module V. If there is* not *a uniform upper bound on the j-th eigenvalue of $|D^M|$ (among metrics with $\| R^M \|_\infty \leq K$ and* $\mathrm{diam}(M) = 1$*) then M collapses to a limit space X. Furthermore, the generic fiber Z of the map $M \to X$ is an infranil-manifold which does not admit any affine-parallel spinor fields.*

Finally, one can characterize the essential spectrum of a geometric Dirac-type operator on a finite-volume negatively-curved manifold. That is, one can show that it equals the essential spectrum of a certain first-order ordinary differential operator associated to the ends.