

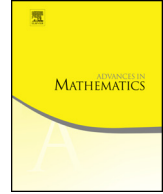


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# On scalar curvature lower bounds and scalar curvature measure <sup>☆</sup>



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## ABSTRACT

We relate the (non)existence of lower scalar curvature bounds to the existence of certain distance-decreasing maps. We also give a sufficient condition for the existence of a limiting scalar curvature measure in the backward limit of a Ricci flow solution.

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## 1. Introduction

In [11] we proved the following result, in which a lower bound on scalar curvature gives a restriction on the existence of distance-nonincreasing maps of nonzero degree. Let  $R$  denote scalar curvature and let  $H$  denote mean curvature.

**Theorem 1.1.** *Let  $N$  and  $M$  be compact connected Riemannian manifolds-with-boundary of the same even dimension. Let  $f : (N, \partial N) \rightarrow (M, \partial M)$  be a smooth spin map and let  $\partial f : \partial N \rightarrow \partial M$  denote the restriction to the boundary. Suppose that*

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- $f$  is  $\Lambda^2$ -nonincreasing and  $\partial f$  is distance-nonincreasing,
- $M$  has nonnegative curvature operator and  $\partial M$  has nonnegative second fundamental form,
- $R_N \geq f^*R_M$  and  $H_{\partial N} \geq (\partial f)^*H_{\partial M}$ ,
- $M$  has nonzero Euler characteristic and
- $f$  has nonzero degree.

Then  $R_N = f^*R_M$  and  $H_{\partial N} = (\partial f)^*H_{\partial M}$ .

Furthermore,

- If  $0 < \text{Ric}_M < \frac{1}{2}R_M g_M$  then  $f$  is a Riemannian covering map.
- If  $\text{Ric}_M > 0$  and  $f$  is distance-nonincreasing then  $f$  is a Riemannian covering map.
- If  $M$  is flat then  $N$  is Ricci-flat.

In particular, the lower scalar curvature bound  $R_N \geq f^*R_M$  means that it is impossible for  $f$  and  $\partial f$  to be distance-decreasing (i.e. have Lipschitz constant less than one), with  $f$  having nonzero degree, and to also have  $H_{\partial N} > (\partial f)^*H_{\partial M}$ . Theorem 1.1 follows earlier work by Llarull [8] and Goette-Semmelmann [4]; we refer to [11] for background and generalizations. The first main result of the present paper is a converse and shows that the lack of a lower bound on the scalar curvature implies that such distance-decreasing maps do exist.

**Theorem 1.2.** *Given  $n > 1$  and  $K \in \mathbb{R}$ , let  $Z$  be an  $n$ -dimensional Riemannian manifold and let  $z \in Z$  be a point where the scalar curvature is  $R_z < n(n - 1)K$ . Then for any neighborhood  $U$  of  $z$ , there are*

- (1) A codimension-zero compact submanifold-with-boundary  $N \subset U$  containing  $z$  that is diffeomorphic to a ball,
- (2) A codimension-zero compact submanifold-with-boundary  $M$  in the  $n$ -dimensional model space of constant curvature  $K$ , diffeomorphic to a ball,
- (3) A smooth map  $f : (N, \partial N) \rightarrow (M, \partial M)$  of nonzero degree so that  $f$  and  $\partial f$  are distance-decreasing, and the mean curvatures satisfy  $H_{\partial N} > (\partial f)^*H_{\partial M}$ , and
- (4) Numbers  $\delta, l > 0$  so that for all  $n \in \partial N$  and  $t \in [0, l)$ , one has  $f(\exp_n(t\nu_{\partial N})) = \exp_{f(n)}((1 - \delta)t\nu_{\partial M})$ , where  $\nu_{\partial N}$  and  $\nu_{\partial M}$  are the inward unit normals to  $\partial N$  and  $\partial M$ , respectively.

If  $K \leq 0$  then we can take  $M$  to be strictly convex.

Together, Theorems 1.1 and 1.2 essentially give a metric characterization of lower scalar curvature bounds. While the geometric meaning of scalar curvature may be hard to understand, the metric characterization is in terms of mean curvature, which is more tractable.

The proof of Theorem 1.2 is by induction on  $n$ , as in the proof of a related result by Gromov in [6]. Item (4) in the conclusion of the theorem is just for technical convenience, in order to simplify the induction argument. In the induction step, it is fairly easily to obtain cylindrical regions that satisfy the conclusions of the theorem, along the lines of [6], but have codimension-two singularities. The main technical issue is to smooth the singularities while simultaneously maintaining the distance-decreasing property and the inequality on mean curvatures. Given Theorem 1.2, one can somewhat simplify Gromov’s proof of the preservation of lower scalar curvature bounds under  $C^0$ -limits of smooth Riemannian metrics [6].

The second main result of the paper is about the existence of a limiting scalar curvature measure, as  $t \rightarrow 0$ , for a Ricci flow coming out of a metric space. If there is going to be a finite limiting measure then by looking at the total scalar curvature along the flow, one sees that the finiteness of  $\int_0^T \int_M (R^2 - 2|\text{Ric}|^2) \text{dvol}_{g(t)} dt$  is a necessary condition (equation (3.25)). The next theorem essentially says that it is also sufficient.

**Theorem 1.3.** *Let  $(M, g(t))$ ,  $t \in (0, T]$ , be a Ricci flow solution on a compact  $n$ -dimensional manifold  $M$  satisfying*

- (1)  $|\text{Rm}_{g(t)}| < \frac{A}{t}$  for some  $A < \infty$  and all  $t$ ,
- (2)  $\text{Ric}_{g(t)} \geq Eg(t)$  for some  $E > -\infty$  and all  $t$ , and
- (3)  $R^2 - 2|\text{Ric}|^2 \in L^1((0, T] \times M; dt \text{dvol}_{g(t)})$ .

*Then there is a limit  $\lim_{t \rightarrow 0} R_{g(t)} \text{dvol}_{g(t)} = \mu_0$  in the weak- $\star$  topology.*

One’s first approach to proving Theorem 1.3 might be to fix a test function  $f$  and consider the time evolution of  $\int_M f R_{g(t)} \text{dvol}_{g(t)}$ . This turns out to not be useful. Instead we let  $f$  evolve by a backward heat equation and use heat kernel estimates from [2].

Using Theorem 1.3, we show the existence of a subsequential limiting scalar curvature measure on a class of Ricci limit spaces. Recall that a Riemannian manifold has 2-nonnegative curvature operator if at each point, the sum of the two lowest eigenvalues of the curvature operator is nonnegative.

**Theorem 1.4.** *Given  $D, \hat{A} < \infty$  and  $v_0 > 0$ , let  $\{(M_i, g_i)\}_{i=1}^\infty$  be a sequence of compact  $n$ -dimensional Riemannian manifolds,  $n \geq 4$ , such that*

- (1)  $\text{diam}(M_i, g_i) \leq D$ ,
- (2)  $\text{vol}(M_i, g_i) \geq v_0$ ,
- (3)  $(M_i, g_i)$  has 2-nonnegative curvature operator, and
- (4)  $\int_{M_i} R_{g_i} \text{dvol}_{g_i} \leq \hat{A}$ .

Then after passing to a subsequence, there is a Gromov-Hausdorff limit  $(X_\infty, d_\infty)$  with a measure  $\mu_0$ , along with continuous Gromov-Hausdorff approximations  $\eta_i : M_i \rightarrow X_\infty$ , such that

- (1)  $\lim_{i \rightarrow \infty} (\eta_i)_* (R_{g_i} \text{dvol}_{g_i}) \stackrel{\text{weak-}\star}{=} \mu_0$ , and
- (2) There is a smooth 1-parameter family of Riemannian metrics  $\{g(t)\}_{t \in (0, T]}$  on  $X_\infty$ , with 2-nonnegative curvature operator, so that  $\lim_{t \rightarrow 0} (X_\infty, g(t)) \stackrel{GH}{=} (X_\infty, d_\infty)$  and  $\lim_{t \rightarrow 0} R_{g(t)} \text{dvol}_{g(t)} \stackrel{\text{weak-}\star}{=} \mu_0$ .

Condition (4) in the Theorem (for some  $\widehat{A}$ ) may in fact follow from the other three conditions. From the hypotheses of Theorem 1.4, the subsequential existence of a limiting metric space (in the Gromov-Hausdorff topology) and a limiting scalar curvature measure (in the weak- $\star$  topology) is automatic. The content of the theorem is that the metric space and the scalar curvature measure also arise as a continuous limit, coming from a Ricci flow.

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**2. Proof of Theorem 1.2**

It would be unwieldy to write out equations for all the steps in the proof of Theorem 1.2, so we give the main ingredients. The proof is by induction on  $n$ , as in [6, Section 4.9].

**Lemma 2.1.** *The theorem is true in dimension two.*

**Proof.** If  $n = 2$ , choose normal coordinates  $(r, \theta)$  around  $z$  and  $(r', \theta')$  around a point  $m \in M$ . Given  $\alpha \in (0, 1]$ , let  $N$  be given by  $r \leq r_0$  and let  $M$  be given by  $r' \leq r'_0 = \alpha r_0$ . In the normal coordinates, to leading order,

$$g_N \sim dr^2 + r^2(1 - \frac{1}{6}R_z r^2)d\theta^2, \tag{2.2}$$

$$g_M \sim (dr')^2 + (r')^2(1 - \frac{1}{3}K(r')^2)(d\theta')^2,$$

$$H_{\partial N} \sim \frac{1}{r_0} - \frac{1}{6}R_z r_0,$$

$$H_{\partial M} \sim \frac{1}{r'_0} - \frac{1}{3}K r'_0.$$

Define a Lipschitz function  $F$  by  $F(r, \theta) = (\alpha r, \theta)$ . Then

$$F^* g_M \sim \alpha^2 dr^2 + \alpha^2 r^2(1 - \frac{1}{3}K\alpha^2 r^2)d\theta^2, \tag{2.3}$$

$$(\partial F)^* H_{\partial M} \sim \frac{1}{\alpha r_0} - \frac{1}{3}K\alpha r_0.$$

For small  $r_0$ , if  $\alpha = 1$  then  $F$  is distance-nonincreasing,  $\partial F$  is distance-decreasing and  $H_{\partial N} > (\partial F)^*H_{\partial M}$ . By continuity, if  $\alpha$  is slightly less than one then  $F$  and  $\partial F$  will be distance-decreasing and we will still have  $H_{\partial N} > (\partial F)^*H_{\partial M}$ . If  $f$  is the result of slightly smoothing  $F$  near  $z$  then it will satisfy the conclusion of the theorem.  $\square$

Now suppose that  $n > 2$  and the theorem is true in dimension  $n - 1$ .

**Lemma 2.4.** *There is a (small) minimal hypersurface  $V \subset U$  containing  $z$  and a unit normal vector  $v \in T_z Z$  so that*

- *The second fundamental form  $A_z$  at  $z$  has  $|A_z| \ll 1$ ,*
- *$\text{Ric}(v, v) < (n - 1)K$ , and*
- *The scalar curvature of  $V$  at  $z$  satisfies  $R'_z < (n - 1)(n - 2)K$ .*

**Proof.** Multiplying  $g_Z$  by a large constant  $\lambda$ , the geometry of a unit ball around  $z$  becomes closer and closer to Euclidean. Given a unit vector  $w \in T_z Z$ , consider the foliation of the rescaled ball of radius two given by hyperplanes perpendicular to  $w$  with respect to the Euclidean metric in the normal coordinates. The leaves are minimal with respect to the Euclidean metric. Consider the part of the foliation whose height varies between  $-1.5$  and  $1.5$ . Using stability results as in [15,16], if  $\lambda$  is large enough then there is a small  $C^k$ -perturbation of the foliation by minimal hypersurfaces that preserves the intersections of the leaves with the sphere of radius two. This restricts to a minimal foliation of the unit ball with arbitrarily small second fundamental form, if  $\lambda$  is large enough. Let  $\alpha(w)$  be the choice of unit normal, to the leaf at  $z$ , that is close to  $w$ . For sufficiently large  $\lambda$ , the map  $\alpha : S^{n-1} \rightarrow S^{n-1}$  is a local diffeomorphism, hence is surjective. We will take  $w = \alpha^{-1}(v)$  for an appropriately chosen  $v$  that is specified below and let  $V$  be the corresponding minimal leaf through  $z$ .

From the Gauss-Codazzi equation, the scalar curvature  $R'_z$  of  $z$  in  $V$  is given by

$$R'_z = R_z - 2\text{Ric}(v, v) + (\text{Tr}(A))^2 - \text{Tr}(A^2). \tag{2.5}$$

Put

$$\widehat{\text{Ric}} = \text{Ric} - (n - 1)Kg, \tag{2.6}$$

with trace  $\widehat{R} = R - n(n - 1)K$ . Let  $\widehat{R}_{11} \leq \widehat{R}_{22} \leq \dots \leq \widehat{R}_{nn}$  be the eigenvalues of  $\widehat{\text{Ric}}_z$ .

If  $\widehat{R}_{nn} < 0$ , let  $v$  be a corresponding unit eigenvector. Then  $\text{Ric}(v, v) < (n - 1)K$  and

$$\begin{aligned} R_z - 2\text{Ric}(v, v) &= R_{11} + \dots + R_{n-2, n-2} + (R_{n-1, n-1} - R_{nn}) \\ &\leq R_{11} + \dots + R_{n-2, n-2} < (n - 1)(n - 2)K. \end{aligned} \tag{2.7}$$

If  $\widehat{R}_{nn} = 0$ , let  $v$  be a unit vector that is a slight perturbation from  $\text{Ker}(\widehat{\text{Ric}})$ . Then  $\text{Ric}(v, v) < (n - 1)K$  and we still have  $R_z - 2\text{Ric}(v, v) < (n - 1)(n - 2)K$ .

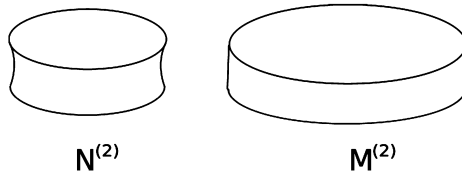


Fig. 1. Cylindrical regions.

If  $\widehat{R}_{nn} > 0$  then the quadratic form  $\widehat{\text{Ric}}$  is indefinite and for any  $\delta > 0$ , we can find a unit vector  $v$  so that  $-\delta < \widehat{\text{Ric}}(v, v) < 0$ . Then

$$(n - 1)K - \delta < \text{Ric}(v, v) < (n - 1)K, \tag{2.8}$$

so  $R_z - 2 \text{Ric}(v, v) < R_z - 2(n - 1)K + 2\delta$ . Taking  $\delta$  small enough, we can ensure that  $R_z - 2 \text{Ric}(v, v) < (n - 1)(n - 2)K$ .

In any case, we can achieve a negative upper bound on  $R_z - 2 \text{Ric}(v, v) - (n - 1)(n - 2)K$  that is independent of  $\lambda$ . Finally, taking  $\lambda$  large enough to ensure that  $|(\text{Tr}(A))^2 - \text{Tr}(A^2)|$  is small, we obtain from (2.5) that  $R'_z < (n - 1)(n - 2)K$ .  $\square$

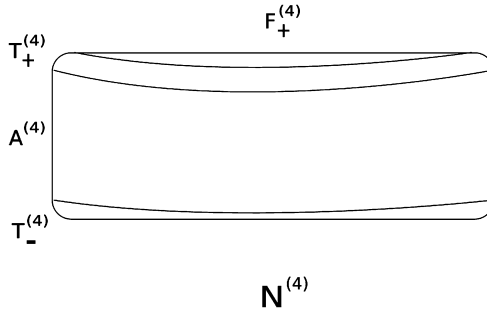
With reference to the  $V$  of Lemma 2.4, let  $N'$  be an  $(n - 1)$ -dimensional compact submanifold-with-boundary of  $V$  containing  $z$  obtained by applying the induction hypothesis, with corresponding submanifold  $M'$  of the  $(n - 1)$ -dimensional model space of constant curvature  $K$ , and with map  $f' : N' \rightarrow M'$  of nonzero degree so that  $f'$  and  $\partial f'$  are distance-decreasing, and  $H_{\partial N'} > (\partial f')^* H_{\partial M'}$ . Taking  $N'$  small enough, we can assume that the unit normal vector  $\nu_{N'}$  satisfies  $\text{Ric}(\nu_{N'}, \nu_{N'}) < (n - 1)K$  on  $N'$ .

For small  $\epsilon > 0$ , let  $N^{(2)}$  be the cylindrical region  $\{\exp(u\nu_{N'}) : |u| \leq \epsilon\}$  in  $Z$ . Similarly, put  $M^{(2)} = \{\exp(u\nu_{M'}) : |u| \leq (1 - \delta_{N'})\epsilon\}$  in the  $n$ -dimensional model space of constant curvature  $K$ , where  $\nu_{M'}$  is a unit normal field to  $M'$  and  $\delta_{N'}$  is the parameter appearing in the induction hypothesis. See Fig. 1, which illustrates the case  $K = 0$ . In what follows we can always reduce  $l_{N'}$  and  $\delta_{N'}$ . Define  $f^{(2)} : N^{(2)} \rightarrow M^{(2)}$  by  $f^{(2)}(\exp_{n'}(u\nu_{N'})) = \exp_{f'(n')}((1 - \delta_{N'})u\nu_{M'})$ . Note that  $\partial N^{(2)}$  has a top face and a bottom face, both diffeomorphic to  $N'$ , and an annular region diffeomorphic to  $[-\epsilon, \epsilon] \times \partial N'$ . The annular region meets the top face orthogonally in a codimension-two stratum diffeomorphic to  $\partial N'$ , with a similar statement for the bottom face. The maps  $f^{(2)}$  and  $\partial f^{(2)}$  are distance-decreasing.

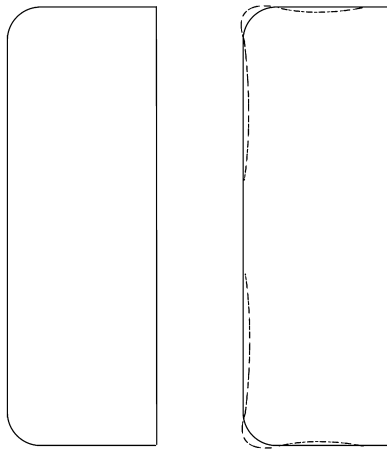
Along the geodesics in  $N^{(2)}$  normal to  $N'$ , we have

$$\frac{dH}{dt} + \text{Tr}(A^2) = -\text{Ric}(\gamma', \gamma'). \tag{2.9}$$

For small  $\epsilon$ , if  $N'$  is taken small enough then  $|A_z| \ll 1$  and on the top and bottom faces of  $N^{(2)}$ , we have  $H \sim -\epsilon \text{Ric}(\nu_{N'}, \nu_{N'})$ . Similarly, on the top and bottom faces of  $M^{(2)}$ , we have  $H \sim -(n - 1)\epsilon(1 - \delta_{N'})K$ . Including the annular region over  $\partial N'$ , if  $\epsilon$  and  $\delta_{N'}$  are small then  $H_{\partial N^{(2)}} > (\partial f^{(2)})^* H_{\partial M^{(2)}}$ .



**Fig. 2.** Boundary decomposition.



**Fig. 3.** Deformation.

We can assume that the parameter  $l_{N'}$  in the induction assumption is less than the focal radius of  $\partial N' \subset N'$ , and  $l_{N'} \ll \epsilon$ . Given  $\epsilon' \ll l_{N'}$ , let  $N^{(3)}$  be the points inside of  $N^{(2)}$  that have distance at least  $\epsilon'$  from  $\partial N^{(2)}$ . Let  $N^{(4)}$  be the  $\epsilon'$ -neighborhood of  $N^{(3)}$  in  $Z$ . Do a similar construction for  $M$ , to obtain  $M^{(4)}$ .

The boundary  $\partial N^{(4)}$  is  $C^{1,1}$ -regular and has a decomposition into a top face  $F_+^{(4)}$ , a bottom face  $F_-^{(4)}$ , an annular belt  $A^{(4)}$ , an upper tube  $T_+^{(4)}$  and a lower tube  $T_-^{(4)}$ . See Fig. 2. There is a similar decomposition of  $\partial M^{(4)}$ .

Consider the top tubular region  $T_+^{(4)}$ . It has two boundary components  $\partial T_{+,1}^{(4)}$  and  $\partial T_{+,2}^{(4)}$ , with  $\partial T_{+,1}^{(4)}$  also being a boundary component of  $F_+^{(4)}$  and  $\partial T_{+,2}^{(4)}$  also being a boundary component of  $A^{(4)}$ .

Let  $p : N^{(2)} \rightarrow N'$  be projection onto the second factor in the diffeomorphism  $N^{(2)} \cong [-\epsilon, \epsilon] \times N'$ . Let  $\hat{p} : M^{(2)} \rightarrow M'$  be the analogous map on  $M^{(2)}$ . Given  $n' \in \partial N'$ , put  $L_{n'} = \{\exp_{n'}(t\nu_{\partial N'}) : 0 \leq t \leq l_{N'}\} \subset N'$  and put  $G_{n'} = p^{-1}(L_{n'})$ . Put  $F_{\pm, n'}^{(4)} = F_{\pm}^{(4)} \cap G_{n'}$ ,  $T_{\pm, n'}^{(4)} = T_{\pm}^{(4)} \cap G_{n'}$  and  $A_{n'}^{(4)} = A^{(4)} \cap G_{n'}$ . See the left-hand picture in Fig. 3, which illustrates  $G_{n'} \cap N^{(4)}$ . Define  $\hat{G}_{m'}$ ,  $\hat{F}_{\pm, m'}^{(4)}$ ,  $\hat{T}_{\pm, m'}^{(4)}$  and  $\hat{A}_{m'}^{(4)}$  similarly for  $M^{(4)}$ .

We define a map  $\partial f^{(4)} : \partial N^{(4)} \rightarrow \partial M^{(4)}$  as follows. Given  $n' \in \partial N'$ , we first send the curve  $T_{+,n'}^{(4)} \cup A_{n'}^{(4)} \cup T_{-,n'}^{(4)}$  to  $\widehat{T}_{+,( \partial f')(n')}^{(4)} \cup \widehat{A}_{( \partial f')(n')}^{(4)} \cup \widehat{T}_{-,( \partial f')(n')}^{(4)}$  piecewise linearly with respect to arc length. Next, given the point  $x = \partial T_{+,1}^{(4)} \cap G_{n'}$ , write  $p(x)$  as  $\exp_{n'}(\tau_{n'} \nu_{\partial N'})$  for some  $\tau_{n'} \in (0, l_{N'})$ . The parameter  $\tau_{n'}$  is comparable to  $\epsilon'$ . Define  $\widehat{\tau}_{( \partial f')(n')}$  similarly for  $M'$ . Let  $\lambda_{n'}$  be the increasing linear bijection from  $[\tau_{n'}, l_{N'}]$  to  $[\widehat{\tau}_{( \partial f')(n')}, (1 - \delta_{N'})l_{N'}]$ . Given  $t \in [\tau_{n'}, l_{N'}]$ , let  $y_t$  be the point in  $F_{+,n'}^{(4)}$  with  $p(y_t) = \exp_{n'}(t \nu_{\partial N'})$  and let  $(\partial f^{(4)})(y_t)$  be the point in  $\widehat{F}_{+,( \partial f')(n')}^{(4)}$  whose image under  $\widehat{p}$  is  $\exp_{( \partial f')(n')}(\lambda_{n'}(t) \nu_{\partial M'})$ . Define  $\partial f^{(4)}$  on the remaining points of  $F_{+,n'}^{(4)}$  to be the same as  $\partial f^{(2)}$ . Finally, define  $\partial f^{(4)}$  on  $F_{-,n'}^{(4)}$  by a similar construction.

The lengths of  $T_{\pm,n'}^{(4)}$  and  $\widehat{T}_{\pm,m'}^{(4)}$  are  $\frac{\pi}{2\epsilon'} + O((\epsilon')^0)$  as  $\epsilon' \rightarrow \infty$ . The Lipschitz constant of  $\partial f^{(4)}$  is  $1 + O(\epsilon')$ . We can extend  $\partial f^{(4)}$  to a map  $f^{(4)} : N^{(4)} \rightarrow M^{(4)}$ , sending  $G_{n'} \cap N^{(4)}$  to  $\widehat{G}_{( \partial f')(n')} \cap M^{(4)}$ , whose Lipschitz constant is  $1 + O(\epsilon')$ .

By tube formulas, the mean curvature on  $T_{\pm}^{(4)}$  is  $\frac{1}{\epsilon'} + O(\epsilon')$ , and similarly for the tube regions of  $\partial M^{(4)}$  [5, Theorem 9.23]. We now perturb  $N^{(4)}$  to increase the mean curvature on  $T_{\pm}^{(4)}$ . To do so, we effectively borrow some of the mean curvature from  $F_{\pm}^{(4)}$  and  $A^{(4)}$ .

We do some preliminary calculations. Let  $\phi : [0, \infty) \rightarrow \mathbb{R}$  be a smooth nonnegative function such that  $\phi(0) = 0$ ,  $\phi'(0) = 1$  and  $\phi(x)$  vanishes when  $x \geq 2$ . Given constants  $c, c' > 0$  so that  $cl_{N'} \gg 1$  and  $c\epsilon' \ll 1$ , and  $L < \infty$ , define  $u_L \in C^{1,1}(\mathbb{R})$  by

$$u_L = \begin{cases} \frac{c'L}{2c} \phi(-cx) & x < 0 \\ \frac{c'}{2} x(x - L) & 0 \leq x \leq L \\ \frac{c'L}{2c} \phi(c(x - L)) & x > L. \end{cases} \tag{2.10}$$

Then

$$u_L'' = \begin{cases} \frac{c'cL}{2} \phi''(-cx) & x < 0 \\ c' & 0 < x < L \\ \frac{c'cL}{2} \phi''(c(x - L)) & x > L. \end{cases} \tag{2.11}$$

Let  $d_1$  be the intrinsic distance function on  $\partial N^{(4)}$  from  $\partial T_{+,1}^{(4)}$  and let  $d_2$  be the intrinsic distance function on  $\partial N^{(4)}$  from  $\partial T_{+,2}^{(4)}$ . Let  $\pi_1 : \partial N^{(4)} \rightarrow \partial T_{+,1}^{(4)}$  be nearest point projection with respect to the intrinsic distance on  $\partial N^{(4)}$ , and similarly for  $\pi_2$ . (In the application, the nearest point will be unique.) Define a function  $V_+ \in C^{1,1}(\partial N^{(4)})$  by



$$V_+ = \begin{cases} -\frac{c'}{2}d_1d_2 & \text{on } T_+^{(4)} \\ \frac{c'}{2c}(d_2 \circ \pi_1)\phi(cd_1) & \text{on } F_+^{(4)} \\ \frac{c'}{2c}(d_1 \circ \pi_2)\phi(cd_2) & \text{on } A_+^{(4)} \\ 0 & \text{on } T_-^{(4)} \cup F_-^{(4)}. \end{cases} \tag{2.12}$$

Define  $V_- \in C^{1,1}(\partial N^{(4)})$  similarly, replacing  $T_+^{(4)}$  by  $T_-^{(4)}$ . Put  $V = V_+ + V_-$ . Deform  $\partial N^{(4)}$  by distance  $V$  in the inward normal direction. See the right-hand picture in Fig. 3, where the deformation is indicated by dashed lines. Let  $N^{(5)}$  be the region bounded by the ensuing hypersurface, i.e.  $\partial N^{(5)}$  is the image of the  $C^{1,1}$ -diffeomorphism  $D : \partial N^{(4)} \rightarrow \partial N^{(5)}$  given by  $D(x) = \exp_x(V(x)\nu_{\partial N^{(4)}})$ . Note that the deformation is outward on  $T_{\pm}^{(4)}$ , of magnitude comparable to  $c'(\epsilon')^2$ , and inward on the rest of  $\partial N^{(4)}$ , of magnitude comparable to  $c'c^{-1}\epsilon'$ .

The first variation formula for mean curvature is

$$H' = \Delta V + (|A|^2 + \text{Ric}(\mu, \mu))V. \tag{2.13}$$

If  $x$  denotes the length variable on a minimal arc in  $T_+^{(4)}$  between  $q \in \partial T_{+,2}^{(4)}$  and  $\pi_1(q) \in \partial T_{+,1}^{(4)}$  then on  $T_+^{(4)}$ , to leading order  $\Delta V \sim \frac{d^2}{dx^2}V$  and  $|A|^2 \sim (\epsilon')^{-2}$ . From (2.11), we deduce that on  $T_{\pm}^{(4)}$  the change in  $H$  roughly ranges between  $c' \left(1 - \frac{\pi^2}{32}\right)$  and  $c'$ . On the rest of  $\partial N^{(4)}$ , the change in  $H$  is bounded in magnitude by  $\text{const.} \cdot c'\epsilon'(c + c^{-1})$ . Put  $M^{(5)} = M^{(4)}$ .

Define subsets of  $\partial N^{(5)}$  by  $\tilde{F}_{\pm}^{(5)} = D(F_{\pm}^{(4)})$ ,  $\tilde{T}_{\pm}^{(5)} = D(T_{\pm}^{(4)})$  and  $\tilde{A}^{(5)} = D(A^{(4)})$ . Define  $\partial f^{(5)} : \partial N^{(5)} \rightarrow \partial M^{(5)}$  by  $\partial f^{(5)} = (\partial f^{(4)}) \circ D^{-1}$ . On  $\tilde{T}_{\pm}^{(5)}$ , the map  $\partial f^{(5)}$  has a Lipschitz bound comparable to that of  $\partial f^{(4)}$ , namely  $1 + O(\epsilon')$ , using the fact that the perturbation on  $T_{\pm}^{(4)}$  is outward. (It may seem paradoxical that  $\tilde{T}_{\pm}^{(5)}$  lies outside of  $T_{\pm}^{(4)}$  but has a higher mean curvature. One way to understand this is by looking at Fig. 3 and comparing the total turning angle of  $T_{+,n'}^{(4)}$  with the total turning angle of the corresponding dotted segment on the right.) If the Lipschitz bound of  $\partial f^{(2)}$  is  $1 - \sigma$ , where  $\sigma > 0$ , then the Lipschitz bound of  $\partial f^{(5)}$  on the rest of  $\partial N^{(5)}$  is  $1 - \sigma + \text{const.} \cdot c'\epsilon'$ . In sum,  $\partial f^{(5)}$  has a Lipschitz bound that is  $1 + O(\epsilon')$ . We can extend  $\partial f^{(5)}$  to a map  $f^{(5)} : N^{(5)} \rightarrow M^{(5)}$  which also has a Lipschitz bound that is  $1 + O(\epsilon')$ . We can assume that  $f^{(5)}$  maps normal geodesics to normal geodesics, in a small neighborhood of  $\partial N^{(5)}$ .

On  $\tilde{T}_{\pm,n'}^{(5)}$ , the ratio  $\frac{H_{\partial N^{(5)}}}{(\partial f^{(5)})^* H_{\partial M^{(5)}}}$  is bounded below by  $(1 + \text{const.} \cdot c'\epsilon')$ . If  $c'\epsilon'(c + c^{-1}) \ll 1$  then on the rest of  $\partial N^{(5)}$ , there is a uniform lower bound for the ratio that is greater than one, coming from  $\partial N^{(2)}$ . Hence by taking  $c'$  sufficiently large and then  $\epsilon'$  sufficiently small, we can ensure that the Lipschitz constant of  $f^{(5)}$  is strictly less than the minimum of  $\frac{H_{\partial N^{(5)}}}{(\partial f^{(5)})^* H_{\partial M^{(5)}}}$ .

Put  $N^{(6)} = N^{(5)}$ . Taking normal coordinates around  $f^{(5)}(z) \in M^{(5)}$ , let  $M^{(6)}$  be the result of a slight radial shrinking of  $M^{(5)}$ . If  $f^{(6)} : N^{(6)} \rightarrow M^{(6)}$  is the composite

map then we can ensure that  $f^{(6)}$  and  $\partial f^{(6)}$  are distance-decreasing, while  $H_{\partial N^{(6)}} > (\partial f^{(6)})^* H_{\partial M^{(6)}}$ .

We run the mean curvature flow on  $\partial N^{(6)}$  and  $\partial M^{(6)}$  for a time  $\tau \ll (\epsilon')^2$  (cf. [3]) to obtain smooth hypersurfaces  $\partial N$  and  $\partial M$ , and hence  $N$  and  $M$ . As the mean curvature obeys a diffusion-type equation under mean curvature flow, the main effect on the mean curvature at a point will be to average the mean curvature with respect to a Gaussian centered at the point with scale on the order of  $\sqrt{\tau}$ . In particular, if  $\tau$  is small enough then for the map  $\partial f : \partial N \rightarrow \partial M$ , obtained from  $f^{(6)}$  by following the flows, the inequalities will be preserved. We can then extend it to a smooth distance-decreasing map  $f : N \rightarrow M$ . By construction, after choosing orientations on  $N$  and  $M$  (which are diffeomorphic to balls), the degree is nonzero. Finally, after a small perturbation of  $f$ , we can assume that there are numbers  $\delta, l > 0$  so that for all  $n \in \partial N$  and  $t \in [0, l)$ , one has  $f(\exp_n t\nu_{\partial N}) = \exp_{f(n)}((1 - \delta)t\nu_{\partial M})$ .

If  $K < 0$  then for sufficiently small  $\tau$ , the preceding steps preserve the strict convexity of  $M$ . If  $K = 0$  then they preserve the convexity of  $M$  and we can slightly perturb  $M$  at the end, for example by the mean curvature flow, to make it strictly convex.

**3. Proof of Theorem 1.3**

**Lemma 3.1.** *There is a constant  $C' = C'(n, A) < \infty$  so that if  $0 < s \leq t$  then*

$$d_s - C' (\sqrt{t} - \sqrt{s}) \leq d_t \leq e^{E(s-t)} d_s. \tag{3.2}$$

**Proof.** This follows from distance distortion estimates for Ricci flow, as in [7, Remark 27.5 and Corollary 27.16].  $\square$

**Corollary 3.3.** *The diameter of  $(M, g(t))$  is uniformly bounded above in  $t$ .*

**Lemma 3.4.** *There are some  $v_0, A' > 0$  so that for all  $(x, t) \in M \times (0, T]$ ,*

- (1)  $\text{vol}_{g(t)}(B_{g(t)}(x, 1)) \geq v_0$ , and
- (2)  $\text{vol}_{g(t)}(B_{g(t)}(x, \sqrt{t})) > A' t^{n/2}$

**Proof.** From the evolution of volume under Ricci flow,

$$\frac{d}{dt} \text{Vol}(M, g(t)) = - \int_M R_{g(t)} \, d\text{vol}_{g(t)} \leq -nE \text{Vol}(M, g(t)). \tag{3.5}$$

It follows that  $\text{Vol}(M, g(t)) \geq e^{nE(T-t)} \text{Vol}(M, g(T))$ . The lower Ricci curvature bound, the diameter bound from Corollary 3.3 and Bishop-Gromov comparison now give numbers  $v_0, A'$  as in the statement of the lemma.  $\square$

**Lemma 3.6.** *Let  $F$  be a solution of the backward heat equation*

$$\partial_t F = -\Delta F. \tag{3.7}$$

*Then  $\max |F|$  and  $e^{-2Et} \max |\nabla F|$  are nondecreasing in  $t$ .*

**Proof.** From the maximum principle,  $\max |F|$  is nondecreasing in  $t$ . Next, we have

$$\partial_t |\nabla F|^2 = 2 \operatorname{Ric}(\nabla F, \nabla F) - 2 \langle \nabla F, \nabla \Delta F \rangle \tag{3.8}$$

and

$$\Delta |\nabla F|^2 = 2 \langle \nabla F, \nabla \Delta F \rangle + 2 |\operatorname{Hess} F|^2 + 2 \operatorname{Ric}(\nabla F, \nabla F). \tag{3.9}$$

Hence

$$(\partial_t + \Delta) |\nabla F|^2 \geq 4 \operatorname{Ric}(\nabla F, \nabla F) \geq 4E |\nabla F|^2, \tag{3.10}$$

or

$$(\partial_t + \Delta) (e^{-4Et} |\nabla F|^2) \geq 0. \tag{3.11}$$

By the maximum principle,  $e^{-4Et} \max |\nabla F|^2$  is nondecreasing in  $t$ . This proves the lemma.  $\square$

**Lemma 3.12.** *Given  $f \in C^\infty(M)$ , there is a function  $\alpha : (0, T] \rightarrow \mathbb{R}^+$  with  $\lim_{t \rightarrow 0} \alpha(t) = 0$  having the following property. Given  $\hat{t} \in (0, T]$ , let  $F$  be the solution to (3.7) on  $(0, \hat{t}]$  with  $F(\hat{t}) = f$ . If  $s \in (0, \hat{t}/2]$  then  $\|F(s) - f\|_\infty \leq \alpha(\hat{t})$ .*

**Proof.** Let  $G(x, t; y, s)$ ,  $0 < s < t$ , be the Green's function for (3.7), meaning that for fixed  $(x, t)$ , the function  $G(x, t; \cdot, \cdot)$  satisfies

$$(\partial_s + \Delta_{y,s})G(x, t; \cdot, \cdot) = 0, \tag{3.13}$$

and  $\lim_{s \rightarrow t} G(x, t; y, s) = \delta_x(y)$ . Then  $G$  is positive and for given  $(y, s)$ , one has

$$\int_M G(x, t; y, s) \operatorname{dvol}_{g(t)}(x) = 1. \tag{3.14}$$

Also,

$$F(y, s) = \int_M G(x, \hat{t}; y, s) f(x) \operatorname{dvol}_{g(\hat{t})}(x). \tag{3.15}$$

By [2, Proposition 3.1], there is a constant  $C = C(n, A) < \infty$  so that

$$G(x, t; y, s) < Ct^{-\frac{n}{2}} e^{-\frac{d_s^2(x, y)}{Ct}} \tag{3.16}$$

whenever  $s \leq \frac{t}{2}$ .

Given  $L < \infty$ , we have

$$\begin{aligned} F(y, s) - f(y) &= \int_M G(x, \hat{t}; y, s) (f(x) - f(y)) \, \text{dvol}_{g(\hat{t})}(x) \\ &= \int_{B_{g(\hat{t})}(y, L\sqrt{\hat{t}})} G(x, \hat{t}; y, s) (f(x) - f(y)) \, \text{dvol}_{g(\hat{t})}(x) + \\ &\quad \int_{M - B_{g(\hat{t})}(y, L\sqrt{\hat{t}})} G(x, \hat{t}; y, s) (f(x) - f(y)) \, \text{dvol}_{g(\hat{t})}(x), \end{aligned} \tag{3.17}$$

so

$$\begin{aligned} |F(y, s) - f(y)| &\leq \max_{x \in B_{g(\hat{t})}(y, L\sqrt{\hat{t}})} |f(x) - f(y)| + \\ &\quad 2(\max |f|) \int_{M - B_{g(\hat{t})}(y, L\sqrt{\hat{t}})} Ct^{-\frac{n}{2}} e^{-\frac{d_s^2(x, y)}{Ct}} \, \text{dvol}_{g(\hat{t})}(x). \end{aligned} \tag{3.18}$$

From Lemma 3.6,

$$\max_{x \in B_{g(\hat{t})}(y, L\sqrt{\hat{t}})} |f(x) - f(y)| \leq L\sqrt{\hat{t}} \max |\nabla f|_{g(\hat{t})} \leq L\sqrt{\hat{t}} e^{2E(\hat{t}-T)} \max |\nabla f|_{g(T)}. \tag{3.19}$$

From (3.2), we have  $d_s^2(x, y) \geq e^{2E(\hat{t}-s)} d_{\hat{t}}^2(x, y) \geq e^{-2|E|\hat{t}} d_{\hat{t}}^2(x, y)$ . Then Bishop-Gromov comparison gives

$$\begin{aligned} &\int_{M - B_{g(\hat{t})}(y, L\sqrt{\hat{t}})} Ct^{-\frac{n}{2}} e^{-\frac{d_s^2(x, y)}{Ct}} \, \text{dvol}_{g(\hat{t})}(x) \leq \\ &C\hat{t}^{-\frac{n}{2}} \text{vol}(S^{n-1}) \int_{L\sqrt{\hat{t}}}^{\infty} e^{-\frac{e^{-2|E|\hat{t}}r^2}{Ct}} \left( \frac{1}{\sqrt{|E|}} \sinh(r\sqrt{|E|}) \right)^{n-1} dr = \\ &C \text{vol}(S^{n-1}) \int_L^{\infty} e^{-\frac{e^{-2|E|\hat{t}}u^2}{C}} \left( \frac{1}{\sqrt{|E|\hat{t}}} \sinh(u\sqrt{|E|\hat{t}}) \right)^{n-1} du \leq \\ &C \text{vol}(S^{n-1}) \int_L^{\infty} e^{-\frac{u^2}{2C}} (\sinh(u))^{n-1} du \end{aligned} \tag{3.20}$$

for  $\widehat{t}$  small. Taking  $L = \widehat{t}^{-\frac{1}{4}}$ , the lemma follows from (3.18), (3.19) and (3.20).  $\square$

**Lemma 3.21.** *If  $F$  is a solution of (3.7) then*

$$\frac{d}{dt} \int_M FR \, d\text{vol} = - \int_M F (R^2 - 2|\text{Ric}|^2) \, d\text{vol}. \tag{3.22}$$

**Proof.** We have

$$\begin{aligned} \frac{d}{dt} \int_M FR \, d\text{vol} &= \int_M \left( \frac{\partial F}{\partial t} R \, d\text{vol} + F \frac{\partial R}{\partial t} \, d\text{vol} + FR \frac{d \, d\text{vol}}{dt} \right) \\ &= \int_M \left( -(\Delta F)R \, d\text{vol} + F(\Delta R + 2|\text{Ric}|^2) \, d\text{vol} - FR^2 \, d\text{vol} \right) \\ &= - \int_M F (R^2 - 2|\text{Ric}|^2) \, d\text{vol}. \end{aligned} \tag{3.23}$$

This proves the lemma.  $\square$

**Lemma 3.24.** *There is a  $C'' < \infty$  such that  $\|R_{g(t)}\|_{L^1} \leq C''$  for all  $t \in (0, T]$ .*

**Proof.** Taking  $F = 1$  in (3.22) gives

$$\int_M R_{g(T)} \, d\text{vol}_{g(T)} - \int_M R_{g(\widehat{t})} \, d\text{vol}_{g(\widehat{t})} = - \int_{\widehat{t}}^T \int_M (R^2 - 2|\text{Ric}|^2)(x, t) \, d\text{vol}_{g(t)}(x) dt. \tag{3.25}$$

As  $\text{Ric}_{g(\widehat{t})} \geq Eg(\widehat{t})$ , it follows that  $R_{g(\widehat{t})} \geq nE$ . The lemma now follows from (3.25).  $\square$

With the hypotheses of Lemma 3.12, from (3.22) we obtain

$$\int_M fR_{g(\widehat{t})} \, d\text{vol}_{g(\widehat{t})} - \int_M F(s)R_{g(s)} \, d\text{vol}_{g(s)} = - \int_s^{\widehat{t}} F(t) (R^2 - 2|\text{Ric}|^2) \, d\text{vol}_{g(t)} \, dt. \tag{3.26}$$

Now

$$\begin{aligned} \left| \int_M fR_{g(\widehat{t})} \, d\text{vol}_{g(\widehat{t})} - \int_M fR_{g(s)} \, d\text{vol}_{g(s)} \right| &\leq \\ \left| \int_M fR_{g(\widehat{t})} \, d\text{vol}_{g(\widehat{t})} - \int_M F(s)R_{g(s)} \, d\text{vol}_{g(s)} \right| &+ \end{aligned} \tag{3.27}$$

$$\left| \int_M (F(s) - f) R_{g(s)} \, d\text{vol}_{g(s)} \right|.$$

From Lemma 3.6 and (3.26),

$$\left| \int_M f R_{g(\hat{t})} \, d\text{vol}_{g(\hat{t})} - \int_M F(s) R_{g(s)} \, d\text{vol}_{g(s)} \right| \leq (\max |f|) \int_0^{\hat{t}} |R^2 - 2|\text{Ric}|^2| \, d\text{vol}_{g(t)} \, dt. \tag{3.28}$$

From Lemmas 3.12 and 3.24,

$$\left| \int_M (F(s) - f) R_{g(s)} \, d\text{vol}_{g(s)} \right| \leq C'' \alpha(\hat{t}). \tag{3.29}$$

Hence

$$\begin{aligned} & \left| \int_M f R_{g(\hat{t})} \, d\text{vol}_{g(\hat{t})} - \int_M f R_{g(s)} \, d\text{vol}_{g(s)} \right| \leq \\ & (\max |f|) \int_0^{\hat{t}} |R^2 - 2|\text{Ric}|^2| \, d\text{vol}_{g(t)} \, dt + C'' \alpha(\hat{t}). \end{aligned} \tag{3.30}$$

From (3.30), the sequence  $\{\int_M f R_{g(2^{-j}T)} \, d\text{vol}_{g(2^{-j}T)}\}_{j=0}^\infty$  is a Cauchy sequence and so has a limit  $M_f \in \mathbb{R}$ . Then from (3.30),  $\lim_{t \rightarrow 0} \int_M f R_{g(t)} \, d\text{vol}_{g(t)} = M_f$ .

Given  $f, f' \in C^\infty(M)$ , we have

$$\left| \int_M f R_{g(t)} \, d\text{vol}_{g(t)} - \int_M f' R_{g(t)} \, d\text{vol}_{g(t)} \right| \leq C'' \|f - f'\|_\infty. \tag{3.31}$$

It follows that the map  $f \rightarrow M_f$  extends to a bounded linear function on  $C(M)$ , and so defines a Borel measure  $\mu_0$  on  $M$ .

This proves Theorem 1.3.

**Example 3.32.** In dimension two,  $R^2 - 2|\text{Ric}|^2 = 0$ . Let  $\Sigma$  be a compact boundaryless two dimensional Alexandrov space. It is known that there is a Ricci flow solution  $(M, g(t))$ , defined for an interval  $(0, T]$ , that satisfies the assumptions of Theorem 1.3 and for which  $\lim_{t \rightarrow 0} (M, g(t)) \stackrel{GH}{=} \Sigma$  [14]. Theorem 1.3 reproduces the canonical curvature measure on  $\Sigma$ , as defined in [13].

**Remark 3.33.** The scalar curvature measure  $\mu_0$  is defined using the Ricci flow. We do not know if it just depends on the Gromov-Hausdorff limit  $\lim_{t \rightarrow 0} (M, g(t))$ . There are

examples of distinct Ricci flows coming out of a cone [1]; however those Ricci flows do not have a lower bound on the Ricci curvature.

If the time slices  $(M, g(t))$  have nonnegative curvature operator then by the uniqueness result of [9], the scalar curvature measure  $\mu_0$  agrees with the measure constructed there, and hence only depends on the limit space.

**4. Proof of Theorem 1.4**

Note that if  $(M, g)$  has 2-nonnegative curvature operator then it has nonnegative Ricci curvature.

The next lemma is probably well known; we give the direct proof.

**Lemma 4.1.** *Let  $H$  be a finite dimensional real inner product space and let  $S$  be a symmetric operator on  $H$ . Let  $\lambda_1 \leq \lambda_2 \leq \dots$  be the eigenvalues of  $S$ , listed with multiplicity. If  $J$  is a  $j$ -dimensional subspace of  $H$ , let  $P_J$  be orthogonal projection onto  $J$ . Then*

$$\text{Tr}(P_J S P_J) \geq \sum_{i=1}^j \lambda_i. \tag{4.2}$$

**Proof.** By continuity and the compactness of the Grassmannian of  $j$ -planes in  $H$ , there is some  $J$  that minimizes the left-hand side of (4.2). Suppose that  $J$  is a minimizer. If  $U \in O(H)$  then  $P_{UJ} = U P_J U^{-1}$ , so  $\text{Tr}(S P_J) \leq \text{Tr}(S U P_J U^{-1})$ . Considering one-parameter subgroups of  $O(H)$ , it follows that

$$0 = \text{Tr}(S[\eta, P_J]) = \text{Tr}(\eta[P_J, S]) \tag{4.3}$$

for all skew-symmetric  $\eta$ . As  $[P_J, S]$  is skew-symmetric, it must vanish. If  $H = \bigoplus_k H_k$  is the spectral decomposition of  $S$  into eigenspaces of distinct eigenvalue then we must have  $J = \bigoplus_k W_k$ , where  $W_k \subset H_k$ . Considering  $J$ 's just of this form and minimizing  $\text{Tr}(P_J S P_J)$ , the lemma follows.  $\square$

**Lemma 4.4.** *Suppose that*

- (1)  $n = 3$  and  $(M, g)$  has nonnegative sectional curvature, or
- (2)  $n \geq 4$  and  $(M, g)$  has 2-nonnegative curvature operator.

Then  $R^2 - 2|\text{Ric}|^2 \geq 0$

**Proof.** We have

$$R^2 - 2|\text{Ric}|^2 = \text{Tr}(\text{Ric}(Rg - 2\text{Ric})). \tag{4.5}$$

Let  $\{e_i\}$  be an orthonormal basis of  $T_m M$  that diagonalizes  $\text{Ric}$ . Then

$$R^2 - 2|\text{Ric}|^2 = \sum_i R_{ii}(Rg - 2\text{Ric})_{ii}. \tag{4.6}$$

Now

$$(Rg - 2\text{Ric})_{ii} = R - 2\text{Ric}_{ii} = \sum_{j,k} R_{jkjk} - 2 \sum_j R_{ijij} = \sum_{j,k \neq i} R_{jkjk}. \tag{4.7}$$

If  $(M, g)$  has nonnegative sectional curvature then the last term in (4.7) is clearly non-negative. Suppose that  $(M, g)$  has 2-nonnegative curvature operator. The last term in (4.7) is

$$\sum_{j,k \neq i} R_{jkjk} = 2 \sum_{\substack{j,k \neq i \\ j < k}} \langle e_j \wedge e_k, \mathcal{R}(e_j \wedge e_k) \rangle. \tag{4.8}$$

We apply Lemma 4.1 with  $H = \Lambda^2(T_m M)$  and  $J = \text{span}\{e_j \wedge e_k\}_{\substack{j,k \neq i \\ j < k}}$ . We conclude that

$$\sum_{\substack{j,k \neq i \\ j < k}} \langle e_j \wedge e_k, \mathcal{R}(e_j \wedge e_k) \rangle \geq \sum_{l=1}^{\binom{n-1}{2}} \lambda_l. \tag{4.9}$$

If  $n \geq 4$  then  $\binom{n-1}{2} \geq 2$  and  $\sum_{l=1}^{\binom{n-1}{2}} \lambda_l \geq \lambda_1 + \lambda_2 \geq 0$ . This proves the lemma.  $\square$

**Corollary 4.10.** *Let  $(M, g(t))$ ,  $t \in (0, T]$ , be a Ricci flow solution on a compact  $n$ -dimensional manifold  $M$ . Suppose that*

- (1)  $|\text{Rm}_{g(t)}| < \frac{A}{t}$ ,
- (2)  $n = 3$  and each  $(M, g(t))$  has nonnegative sectional curvature, or  $n \geq 4$  and each  $(M, g(t))$  has 2-nonnegative curvature operator, and
- (3) There is a uniform upper bound on  $\int_M R_{g(t)} \, d\text{vol}_{g(t)}$ .

Then there is a limit  $\lim_{t \rightarrow 0} R_{g(t)} \, d\text{vol}_{g(t)} = \mu_0$  in the weak- $\star$  topology.

**Proof.** From Lemma 4.4 and (3.25), it follows that  $R^2 - 2|\text{Ric}|^2 \in L^1((0, T] \times M; dt \, d\text{vol}_{g(t)})$ . The corollary now follows from Theorem 1.3.  $\square$

We now prove Theorem 1.4. There is a uniform existence time  $[0, T]$  for the Ricci flow solutions  $(M_i, g_i(t))$  with initial condition  $g_i(0) = g_i$  [2]. The flows have 2-nonnegative curvature operator and satisfy  $|\text{Rm}_{g_i(t)}| < \frac{A}{t}$  for some  $A < \infty$ . By (3.25) and Lemma 4.4,  $\int_{M_i} R_{g_i(t)} \, d\text{vol}_{g_i(t)} \leq \hat{A}$ . By Cheeger-Hamilton compactness, after passing to a subsequence there are

- (1) A smooth manifold  $X_\infty$ ,



- (2) A Ricci flow solution  $(X_\infty, g_\infty(t))$  defined for  $t \in (0, T]$  and
- (3) Diffeomorphisms  $\phi_i : X_\infty \rightarrow M_i$

so that for any  $[a, b] \subset (0, T)$ ,  $\lim_{i \rightarrow \infty} \phi_i^* g_i = g_\infty$  smoothly on  $[a, b] \times X_\infty$ . Then  $g_\infty(t)$  has 2-nonnegative curvature operator, and satisfies  $|\text{Rm}_{g_\infty(t)}| < \frac{A}{t}$  and  $\int_{X_\infty} R_{g_\infty(t)} \text{dvol}_{g_\infty(t)} \leq \widehat{A}$ , for all  $t \in (0, T]$ . From Lemma 3.1, there is a Gromov-Hausdorff limit  $\lim_{t \rightarrow 0} (X_\infty, g_\infty(t)) = (X_\infty, d_\infty)$ . By Corollary 4.10, there is a limit  $\lim_{t \rightarrow 0} R_{g_\infty(t)} \text{dvol}_{g_\infty(t)} = \mu_0$  in the weak- $\star$  topology.

The proof of Theorem 1.3 can be made uniform in the underlying geometry. Hence given  $f \in C(X_\infty)$ , it follows that

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{X_\infty} f (\phi_i^{-1})_* (R_{g_i(t)} \text{dvol}_{g_i(t)}) &= \tag{4.11} \\ \lim_{i \rightarrow \infty} \int_{X_\infty} f R_{\phi_i^* g_i(t)} \text{dvol}_{\phi_i^* g_i(t)} &= \int_{X_\infty} f R_{g_\infty(t)} \text{dvol}_{g_\infty(t)}, \end{aligned}$$

uniformly in  $i$  and  $t \in (0, T]$ . Thus

$$\lim_{i \rightarrow \infty} \int_{X_\infty} f (\phi_i^{-1})_* (R_{g_i} \text{dvol}_{g_i}) = \int_{X_\infty} f \text{d}\mu_0, \tag{4.12}$$

which means that  $\lim_{i \rightarrow \infty} (\phi_i^{-1})_* (R_{g_i} \text{dvol}_{g_i}) \stackrel{\text{weak-}\star}{=} \mu_0$ .

**Remark 4.13.** There is a conjecture that for any  $n \in \mathbb{Z}^+$  and  $v > 0$ , there is some  $\widehat{A} = \widehat{A}(n, v) < \infty$  so that if  $(M, g)$  is a complete  $n$ -dimensional Riemannian manifold with  $\text{Ric} \geq -(n - 1)g$ , and  $B(m, 1)$  is a unit ball in  $M$  with  $\text{vol}(B(m, 1)) \geq v$ , then  $\int_{B(m, 1)} R \text{dvol}_g \leq \widehat{A}$ . This conjecture is known to be true if  $M$  is a polarized Kähler manifold [10, Proposition 1.7] or if  $M$  has sectional curvature bounded below by  $-1$  [12]. If the conjecture holds then condition (4) in Theorem 1.4 follows from the first three conditions.

**References**

- [1] S. Angenent, D. Knopf, Ricci solitons, conical singularities, and nonuniqueness, *Geom. Funct. Anal.* 32 (2022) 411–489.
- [2] R. Bamler, E. Cabezas-Rivas, B. Wilking, The Ricci flow under almost non-negative curvature conditions, *Invent. Math.* 217 (2019) 95–126.
- [3] K. Ecker, G. Huisken, Interior estimates for hypersurfaces moving by mean curvature, *Invent. Math.* 105 (1991) 547–569.
- [4] S. Goette, U. Semmelmann, Scalar curvature estimates for compact symmetric spaces, *Differ. Geom. Appl.* 16 (2002) 65–78.
- [5] A. Gray, *Tubes*, 2nd edition, Progress in Mathematics, vol. 221, Birkhäuser Verlag, Basel, 2004.
- [6] M. Gromov, Dirac and Plateau billiards in domains with corners, *Cent. Eur. J. Math.* 12 (2014) 1109–1156.

- [7] B. Kleiner, J. Lott, Notes on Perelman's papers, *Geom. Topol.* 12 (2008) 2587–2855.
- [8] M. Llarull, Sharp estimates and the Dirac operator, *Math. Ann.* 310 (1998) 55–71.
- [9] N. Lebedeva, A. Petrunin, Curvature tensor of smoothable Alexandrov spaces, preprint, <https://arxiv.org/abs/2202.13420>, 2022.
- [10] G. Liu, G. Székelyhidi, Gromov-Hausdorff limits of Kähler manifolds with Ricci curvature bounded below, *Geom. Funct. Anal.* 32 (2022) 236–279.
- [11] J. Lott, Index theory for scalar curvature on manifolds with boundary, *Proc. Am. Math. Soc.* 149 (2021) 4451–4459.
- [12] A. Petrunin, An upper bound for the curvature integral, *St. Petersburg Math. J.* 20 (2009) 255–265.
- [13] Y. Reshetnyak, Two-dimensional manifolds of bounded curvature, in: *Geometry IV*, in: *Encyclopaedia Math. Sci.*, vol. 70, Springer, Berlin, 1993, pp. 3–163.
- [14] T. Richard, Canonical smoothing of compact Aleksandrov surfaces via Ricci flow, *Ann. Sci. Éc. Norm. Supér.* 51 (2018) 263–279.
- [15] B. White, The space of  $m$ -dimensional surfaces that are stationary for a parametric elliptic functional, *Indiana Univ. Math. J.* 36 (1987) 567–602.
- [16] B. White, The space of minimal submanifolds for varying Riemannian metrics, *Indiana Univ. Math. J.* 40 (1991) 161–200.