IMRN International Mathematics Research Notices
2001, No. 4

The Dirac Operator and Conformal Compactification

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1 Introduction

A general problem in spectral geometry is to understand the spectral properties of Dirac-type operators on complete Riemannian manifolds. In this paper we show how to use the conformal covariance of the Dirac operator, along with some simple arguments, to derive more precise results than are known for general Dirac-type operators. In particular, we give results about the L^2-kernel and the spectrum of the Dirac operator on a complete Riemannian manifold which is conformally equivalent to the interior of a Riemannian manifold with nonempty boundary.

For background information about Dirac operators, we refer to [19]. Let M be a connected smooth spin manifold of dimension $n > 1$. Let S denote the spin bundle of M, equipped with its natural Euclidean inner product.

Let g be a complete Riemannian metric on M, with Dirac operator D_g. Let $\mathrm{Ker}(D_g)$ denote the kernel of D_g when acting on $L^2(S, \mathrm{dvol}_g)$. Given $\sigma \in C^\infty(M)$, let h be the Riemannian metric on M given by

$$g = e^{2\sigma} h. \tag{1.1}$$

Definition 1. (M, g) has a conformal boundary component if one can find σ as above and a manifold-with-boundary N such that

(1) M is diffeomorphic to $\mathrm{Int}(N)$,
(2) $\partial N \neq \emptyset$,
(3) h extends to a smooth Riemannian metric on N, and
(4) $e^{-\sigma}$ extends to a locally Lipschitz function on N.

Received 1 April 2000. Revision received 18 October 2000.

If (M, g) has a conformal boundary component, then (M, g) is conformally hyperbolic in the sense of [20]. A basic example of a manifold with a conformal boundary component is the real hyperbolic space $H^n(\mathbb{R})$, in which case we can take $N = \overline{B^n}$ and $e^\sigma(x) = 2/(1-|x|^2)$ for $x \in B^n$. In general, we do not require that N be compact.

Theorem 1. If (M, g) has a conformal boundary component, then $\mathrm{Ker}(D_g) = 0$. □

Corollary 1. If (M, g) has a conformal boundary component and $4 \mid \dim(M)$, then zero lies in the essential spectrum of D_g. □

Corollary 2. Let Z be a closed connected spin manifold with $\widehat{A}(Z) \neq 0$. Let Γ be a countably infinite discrete group, and let $\widehat{Z}$ be a connected normal Γ-cover of Z. If g is a Riemannian metric on Z, let $\widehat{g}$ be the pullback metric on $\widehat{Z}$. Then $(\widehat{Z}, \widehat{g})$ does not admit a conformal boundary component. □

Examples. (1) Let M be $H^n(\mathbb{R})$. Then M satisfies the conditions of Theorem 1 and Corollary 1. In this case, the conclusions of Theorem 1 and Corollary 1 were previously known by explicit calculation (see [5], [7]).

(2) Let M be the complex hyperbolic space $H^{2n}(\mathbb{C}) = G/K$, where $G = SU(2n, 1)$ and $K = S(U(2n) \times U(1))$. There is a Lie algebra representation of $s(u(2n) \oplus u(1))$ on $R = \Lambda^*(\mathbb{C}^{2n})$ given by $(m, a) \to \Lambda^*(m) + a/2\,\mathrm{Id}$. This integrates to a spinor representation of a double cover $\widehat{K}$ of K on R. Using the isomorphism $\pi_1(G) \cong \pi_1(K)$, let $\widehat{G}$ be the corresponding double cover of G. Then $H^{2n}(\mathbb{C}) = \widehat{G}/\widehat{K}$ has a spinor bundle given by $S = \widehat{G} \times_{\widehat{K}} R$ and a Dirac operator D_g. In this case, $\dim(\mathrm{Ker}(D_g)) = \infty$ (see [7]). This shows the necessity of the condition in Theorem 1 that h be nondegenerate on all of N. Note that the numerator in the Poincaré metric

$$g = \frac{(1-|z|^2)\sum_{i=1}^{2n} dz_i \otimes d\bar{z}_i + \sum_{i,j=1}^{2n} \bar{z}_i\, dz_i \otimes z_j\, d\bar{z}_j}{(1-|z|^2)^2} \tag{1.2}$$

degenerates on the boundary of the unit disk.

(3) As remarked in [4, proof of Theorem 6], if m is odd, then the manifold $M_{n,m}$ constructed in [10] is an aspherical spin manifold with positive signature, which is the total space of a surface bundle over a surface. Let $\widehat{M}_{n,m}$ be the universal cover; it is diffeomorphic to $\mathbb{R}^4$. As $\widehat{A}(M_{n,m}) \neq 0$, Corollary 2 implies that, for any Riemannian metric g on $M_{n,m}$, $(\widehat{M}_{n,m}, \widehat{g})$ does not admit a conformal compactification (in the sense of Definition 1) to a 4-disk.

(4) The signature operator $d + d^*$ on $H^{2n}(\mathbb{R})$ has an infinite-dimensional L^2-kernel (see [6]). This shows that the analog of Theorem 1 for general Dirac-type operators is false.

Now let (Z, h) be a connected closed Riemannian spin manifold. Let X be a closed subset of Z. We consider complete Riemannian manifolds (M, g) which are conformally equivalent to $(Z-X, h)$. For example, $S^{n-m-1} \times H^{m+1}$ is conformally equivalent to $S^n - S^m$, where the metric on $S^{n-m-1} \times H^{m+1}$ is a product of constant curvature metrics.

Theorem 2. Let (M, g) be a complete connected Riemannian spin manifold of dimension $n > 1$. Suppose that there is a $\sigma \in C^\infty(M)$ such that

(1) $(M, e^{-2\sigma} g)$ is isometrically spin-diffeomorphic to $(Z - X, h)$,
(2) σ is bounded below on M, and
(3) X has finite $(n-2)$-dimensional Hausdorff mass.

Then $\dim(\mathrm{Ker}(D_g)) < \infty$. □

Corollary 3. In addition to the hypotheses of Theorem 2, suppose that $\int_M e^\sigma \, \mathrm{dvol}_h < \infty$. Then $\mathrm{Ker}(D_g) \cong \mathrm{Ker}(D_h)$. □

Corollary 4. Under the hypotheses of Corollary 3, if n is even, then the L^2-index of D_g is $\int_M \widehat{A}(M, g)$. □

Corollary 5. Let (Z, h) be a connected closed even-dimensional Riemannian spin manifold. Let ρ be a nonnegative smooth function on Z whose zero set is a submanifold $X \subset Z$, along which the Hessian of ρ is nondegenerate on the normal bundle $T_X Z / T_X X$. Put $M = Z - X$, with the Riemannian metric $g = \rho^{-1} h$. Then the L^2-index of D_g is

$$\mathrm{Ind}_{L^2}(D_g) = \begin{cases} 0 & \text{if } \dim(X) = \dim(Z) - 1, \\ \int_M \widehat{A}(M, g) & \text{if } \dim(X) < \dim(Z) - 1. \end{cases} \tag{1.3}$$

□

In Section 2 we prove the results stated in the introduction. In Section 3 we make some remarks.

The conformal covariance of the Dirac operator was applied to a different but related problem in [16].

2 Proofs

If h is as in (1.1), let D_h denote the Dirac operator associated to h. As explained in [13, Section 2], the notion of a spinor field on M is independent of the choice of Riemannian metric, and so it makes sense to compare D_g and D_h. From [9, Proposition 1.3] and [12, Proposition 2],

$$D_g = e^{-(n+1)\sigma/2} D_h e^{(n-1)\sigma/2}. \tag{2.1}$$

Let $\mathrm{Ker}(D_h)$ denote the kernel of D_h on $L^2(S, \mathrm{dvol}_h)$. Let $\mathrm{Ker}_{(\sigma)}(D_h)$ denote the kernel of D_h on $L^2(S, e^\sigma \, \mathrm{dvol}_h)$. Let $\mathcal{M}_{e^{((n-1)\sigma)/2}}$ denote the operator of multiplication by $e^{((n-1)\sigma)/2}$ on sections of S.

Proposition 1. $\mathcal{M}_{e^{((n-1)\sigma)/2}}$ is an isometric isomorphism between the Hilbert spaces $L^2(S, \mathrm{dvol}_g)$ and $L^2(S, e^\sigma \, \mathrm{dvol}_h)$ which restricts to an isometric isomorphism between $\mathrm{Ker}(D_g)$ and $\mathrm{Ker}_{(\sigma)}(D_h)$. □

Proof. Given $\psi \in L^2(S, \mathrm{dvol}_g)$, put $\psi_\sigma = e^{((n-1)\sigma)/2}\psi$. Then

$$\int_M |\psi|^2 \, \mathrm{dvol}_g = \int_M \left| e^{-(n-1)\sigma/2} \psi_\sigma \right|^2 e^{n\sigma} \, \mathrm{dvol}_h = \int_M e^\sigma |\psi_\sigma|^2 \, \mathrm{dvol}_h \,. \tag{2.2}$$

From (2.1),

$$D_g \psi = \left(e^{-(n+1)\sigma/2} \, D_h e^{(n-1)\sigma/2} \right) \left(e^{-(n-1)\sigma/2} \psi_\sigma \right) = e^{-(n+1)\sigma/2} D_h \psi_\sigma. \tag{2.3}$$

The proposition follows. ■

Proof of Theorem 1. Put $N' = N \cup_{\partial N} ((-1, 0] \times \partial N)$, the addition of a collar neighborhood to N. By assumption, h extends to a smooth Riemannian metric h' on N'. Let $D_{h'}$ be the corresponding Dirac operator on N'.

Given $\psi \in \mathrm{Ker}(D_g)$, put $\psi_\sigma = e^{(n-1)\sigma/2}\psi$. From Proposition 1, $\psi_\sigma \in \mathrm{Ker}_{(\sigma)}(D_h)$. Let ψ'_σ be the extension of ψ_σ by zero to N'. We claim that ψ'_σ is a smooth solution to $D_{h'}\psi'_\sigma = 0$. Once we show this, it will follow from the unique continuation property of $D^2_{h'}$ (see [1, Theorem, p. 235, and Remark 3, p. 248]), along with the vanishing of ψ'_σ on $(-1, 0) \times \partial N$, that $\psi'_\sigma = 0$ and hence $\psi = 0$.

To show that ψ'_σ is a smooth solution to $D_{h'}\psi'_\sigma = 0$, by elliptic regularity theory it is enough to show that it is a weak solution to $D_{h'}\psi'_\sigma = 0$. Let η'_σ be a smooth compactly supported spinor field on N'. Let η_σ be the restriction of η'_σ to $\mathrm{Int}(N) \cong M$. Then

$$\int_{N'} \left\langle D_{h'}\eta'_\sigma, \psi'_\sigma \right\rangle \, \mathrm{dvol}_{h'} = \int_M \left\langle D_h \eta_\sigma, \psi_\sigma \right\rangle \, \mathrm{dvol}_h \,. \tag{2.4}$$

We want to show that this vanishes for each choice of η'_σ.

If $\mathrm{supp}(\eta'_\sigma) \subset \mathrm{int}(N)$, then we can integrate by parts to obtain

$$\int_M \langle D_h \eta_\sigma, \psi_\sigma \rangle \, \mathrm{dvol}_h = \int_M \left\langle \eta_\sigma, D_h \psi_\sigma \right\rangle \, \mathrm{dvol}_h = 0. \tag{2.5}$$

Thus we may assume that $\mathrm{supp}(\eta'_\sigma) \cap \partial N \neq \emptyset$.

Let K be a compact codimension-0 submanifold-with-boundary of ∂N. For ϵ a small enough positive number, let $f : [0, \epsilon] \times K \to N$ be an embedding given by Fermi coordinates near K. That is, for any $k \in K$, the curve $t \to f(t,k)$ is a unit-speed geodesic with $f(0,k) = k$ and $(\partial_t f)(0,k) \perp T_k \partial N$. In terms of these coordinates, we can write

$$h = dt^2 + m_t, \tag{2.6}$$

where m_t is a Riemannian metric on K which depends smoothly on $t \in [0,\epsilon]$. To prove that ψ'_σ is a weak solution to $D_{h'}\psi'_\sigma = 0$, we may assume without loss of generality that $\mathrm{supp}(\eta_\sigma) \subset (0,\epsilon] \times K$ for some such K and ϵ. (We are thinking of η_σ as being defined on $M \cong \mathrm{int}(N)$. So $\mathrm{supp}(\eta_\sigma)$ is a subset of M which is closed in the topology of M or, equivalently, in the relative topology induced from N. In this sense, $(0,\epsilon] \times K$ is also a closed subset of M.)

As M is complete, $e^{-\sigma}$ vanishes on ∂N. Then, as $e^{-\sigma}$ is locally Lipschitz, there is some $C > 0$ such that, when restricted to $f([0,\epsilon] \times K)$,

$$e^{-\sigma}(t,k) \le Ct. \tag{2.7}$$

For $t \in (0,\epsilon]$, put $K_t = f(\{t\} \times K)$. As $\psi_\sigma \in L^2(S, e^\sigma \ \mathrm{dvol}_h)$, we have

$$\begin{aligned} \int_0^\epsilon C^{-1} t^{-1} \int_{K_t} \left|\psi_\sigma\right|^2 \ \mathrm{dvol}_{m_t} \ dt &\le \int_0^\epsilon \int_{K_t} e^\sigma |\psi_\sigma|^2 \ \mathrm{dvol}_{m_t} \ dt \\ &\le \int_M e^\sigma |\psi_\sigma|^2 \ \mathrm{dvol}_h \\ &< \infty. \end{aligned} \tag{2.8}$$

Thus there is a sequence $t_i \in (0,\epsilon]$ such that $\lim_{i\to\infty} t_i = 0$ and

$$\lim_{i\to\infty} \int_{K_{t_i}} |\psi_\sigma|^2 \ \mathrm{dvol}_{m_{t_i}} = 0. \tag{2.9}$$

As $\psi_\sigma \in L^2(S, e^\sigma \ \mathrm{dvol}_h)$, it follows that the restriction of ψ_σ to $f((0,\epsilon] \times K)$ is square-integrable with respect to dvol_h. Then, as $D_h\eta_\sigma \in L^2(S, \mathrm{dvol}_h)$, it follows that $\langle D_h\eta_\sigma, \psi_\sigma\rangle \in L^1(M, \mathrm{dvol}_h)$. Given $t \in (0,\epsilon]$, integration by parts gives

$$\int_{f([t,\epsilon]\times K)} \left\langle D_h\eta_\sigma, \psi_\sigma \right\rangle \ \mathrm{dvol}_h = \int_{K_t} \left\langle c(\partial_t)\eta_\sigma, \psi_\sigma \right\rangle \ \mathrm{dvol}_{m_t}, \tag{2.10}$$

where $c(\partial_t)$ denotes Clifford multiplication by the unit vector ∂_t. Then

$$\begin{aligned} \int_M \left\langle D_h\eta_\sigma, \psi_\sigma \right\rangle \ \mathrm{dvol}_h &= \lim_{i\to\infty} \int_{f([t_i,\epsilon]\times K)} \left\langle D_h\eta_\sigma, \psi_\sigma \right\rangle \ \mathrm{dvol}_h \\ &= \lim_{i\to\infty} \int_{K_{t_i}} \left\langle c(\partial_t)\eta_\sigma, \psi_\sigma \right\rangle \ \mathrm{dvol}_{m_{t_i}}. \end{aligned} \tag{2.11}$$

Thus

$$\begin{aligned}
&\left|\int_M \left\langle D_h \eta_\sigma, \psi_\sigma \right\rangle \, dvol_h\right| \\
&\quad \leq \lim_{i\to\infty} \left|\int_{K_{t_i}} \left\langle c(\partial_t)\eta_\sigma, \psi_\sigma \right\rangle \, dvol_{m_{t_i}}\right| \\
&\quad \leq \lim_{i\to\infty} \left(\int_{K_{t_i}} |\eta_\sigma|^2 \, dvol_{m_{t_i}}\right)^{1/2} \left(\int_{K_{t_i}} |\psi_\sigma|^2 \, dvol_{m_{t_i}}\right)^{1/2} \\
&\quad \leq \|\eta_\sigma\|_\infty \, \sup_{t\in[0,\epsilon]} \, vol^{1/2}(K_t) \lim_{i\to\infty} \left(\int_{K_{t_i}} |\psi_\sigma|^2 \, dvol_{m_{t_i}}\right)^{1/2} \\
&\quad = 0.
\end{aligned} \tag{2.12}$$

This proves the theorem. ■

Proof of Corollary 1. Suppose that zero does not lie in the essential spectrum of D_g. Then D_g is a Fredholm operator. Let X be a connected closed n-dimensional spin manifold with $\widehat{A}(X) \neq 0$. Let M' be the connected sum of M with X. Let $K \subset M$ and $K' \subset M'$ be sufficiently large compact sets, and let g' be a Riemannian metric on M' for which $M'-K'$ is isometric to $M-K$. Then $D_{g'}$ is also Fredholm. As Theorem 1 applies to M', we deduce that both D_g and $D_{g'}$ have vanishing L^2-index. However, by the relative index theorem (see [8], [18]), the difference of the L^2-indices is $\widehat{A}(X)$, which is a contradiction. ■

Proof of Corollary 2. Suppose that $(\widehat{Z}, \widehat{g})$ admits a conformal boundary component. By Theorem 1, $\mathrm{Ker}(D_{\widehat{g}}) = 0$. However, by Atiyah's L^2-index theorem (see [2]), $\dim(\mathrm{Ker}(D_{\widehat{g}})) = \infty$, which is a contradiction. ■

Proof of Theorem 2. If $\psi \in \mathrm{Ker}(D_g)$, put $\psi_\sigma = e^{((n-1)\sigma)/2}\psi$. Then ψ_σ lies in the kernel of the Dirac operator on $Z-X$. As $\psi_\sigma \in L^2(S, e^\sigma \, dvol_h)$ and σ is bounded below, it follows that $\psi_\sigma \in L^2(S, dvol_h)$. From [17, p. 267, 2.3.4], ψ_σ extends to an element of $\mathrm{Ker}(D_h)$ on Z. The unique continuation property of D_h^2 implies that we have constructed a well-defined map from $\mathrm{Ker}(D_g)$ to $\mathrm{Ker}(D_h)$, which is clearly injective. As Z is closed, $\dim(\mathrm{Ker}(D_h)) < \infty$. ■

Proof of Corollary 3. In the proof of Theorem 2, we constructed maps $\mathrm{Ker}(D_g) \to \mathrm{Ker}_{(\sigma)}(D_h) \to \mathrm{Ker}(D_h)$. By Proposition 1, the first map is an isomorphism. The second map is injective. As $\int_M e^\sigma \, dvol_h < \infty$, the second map is surjective. The corollary follows. ■

Proof of Corollary 4. From Corollary 3, the L^2-index of D_g equals the index of D_h on Z. From the Atiyah-Singer index theorem, this equals $\int_Z \widehat{A}(Z, h)$. From the conformal invariance of the $\widehat{A}$-form, this in turn equals $\int_M \widehat{A}(M, g)$. ■

Proof of Corollary 5. Put $e^{2\sigma} = \rho^{-1}$.

If $\dim(X) = \dim(Z) - 1$, let N be the metric completion of $Z - X$ with respect to h. Then we can apply Theorem 1 to conclude that $\mathrm{Ker}(D_g) = 0$.

If $\dim(X) < \dim(Z) - 1$, then the claim follows from Corollary 4. We remark that, in this case, it follows from [14] that D_g has closed range, as in the model case of $S^n - S^m$. (When $\dim(X) = \dim(Z) - 2$, we are using the fact that the spin structure on $Z - X$ is inherited from the spin structure on Z.) Hence, from Corollary 3 , D_g is actually Fredholm. ■

3 Remarks

Any connected complete noncompact Riemannian manifold (M, g) can be conformally compactified to a compact metric space. For example, fix a basepoint $m_0 \in M$. There are $\phi \in C^\infty(M)$ and $c > 0$ such that $\phi(m) \leq d_g(m_0, m) \leq \phi(m) + c$. Taking a function $f : \mathbb{R} \to \mathbb{R}$ which grows sufficiently fast, the metric completion of $(M, e^{-f\circ\phi} g)$ will be the Freudenthal compactification of M, in which one adds a point for each end of M.

In general, even if M has finite topological type, (M, g) will not have a conformal compactification that is a manifold (with or without boundary). The results of this paper suggest that the Dirac operator D_g on M can be studied in terms of the possibly singular spaces that arise as conformal compactifications of M. For example, let (X, g_X) be a closed Riemannian spin manifold. Then, for any $c > 0$, an end of M which is isometric to $([0, \infty) \times X, dr^2 + g_X)$ can be conformally compactified to the conical space $([0, 1] \times X, ds^2 + c^2 s^2 g_X)$. In this case, the L^2-index theorem for manifolds with cylindrical ends (see [3], [15]) says that there will be a contribution to the L^2-index formula for D_g of $-1/2$ times the eta-invariant of the link of the vertex point of the cone, that is, of X. This suggests a relationship between Dirac operators on certain complete manifolds and Dirac operators on singular spaces (see [11] and references therein for the latter), although, of course, the relevant L^2-spaces are different.

Acknowledgments

I thank Victor Nistor for comments on this paper. I thank the Max-Planck-Institut-Bonn for its hospitality while this research was performed.

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