# REMARK ABOUT SCALAR CURVATURE
# AND RIEMANNIAN SUBMERSIONS

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ABSTRACT. We consider modified scalar curvature functions for Riemannian manifolds equipped with smooth measures. Given a Riemannian submersion whose fiber transport is measure-preserving up to constants, we show that the modified scalar curvature of the base is bounded below in terms of the scalar curvatures of the total space and fibers. We give an application concerning scalar curvatures of smooth limit spaces arising in bounded curvature collapses.

## 1. INTRODUCTION

An important result in the study of manifolds of nonnegative sectional curvature is O'Neill's theorem, which says that sectional curvature is nondecreasing under a Riemannian submersion [1, Chapter 9]. In a previous paper we showed that there is a Ricci analog of O'Neill's theorem, provided that one uses a modified Ricci tensor and assumes that the fiber transport of the Riemannian submersion preserves measures up to multiplicative constants [6, Theorem 2]. We used this to show that, in the case of a bounded curvature collapse with a smooth limit space, a lower Ricci bound on the collapsing manifolds implies a lower bound on the modified Ricci curvature of the limit space [6, Theorem 3].

These results can be viewed as precursors to a synthetic treatment of Ricci curvature for metric-measure spaces [7, 12]. In particular, [6, Theorem 2] is given a synthetic proof in [7, Corollary 7.52], and [6, Theorem 3] follows from [7, Corollary 7.45]. (To be precise, with the notation of [7], if $N < \infty$, then one must restrict to the case of nonnegative Ricci curvature.)

In the present note we consider scalar curvature analogs of [6, Theorems 2 and 3]. A modified scalar curvature function for a smooth Riemannian manifold, equipped with a smooth measure, was defined by Perelman [8, Section 1.3]. Let $(M, g)$ be a Riemannian manifold. Let $\mathrm{dvol}_M$ denote the Riemannian density on $M$. Let $\phi$ be a smooth positive function on $M$. Then $(M, \phi \, \mathrm{dvol}_M)$ is a smooth metric-measure space. Perelman's modified scalar curvature is

$$(1.1) \qquad R_\infty \;=\; R \,-\, 2\,\frac{\nabla^2 \phi}{\phi} \,+\, \frac{|\nabla \phi|^2}{\phi^2}.$$

It is designed so that a weighted Lichnerowicz identity holds. In particular, if $M$ is a closed spin manifold and $R_\infty > 0$, then $\widehat{A}(M) = 0$. Perelman's entropy functional is the corresponding total scalar curvature, i.e. $\mathcal{F} = \int_M R_\infty \phi \, \mathrm{dvol}_M$. Note that $R_\infty$ is *not* the trace of the Bakry-Émery tensor $\mathrm{Ric}_\infty = \mathrm{Ric} - \frac{\mathrm{Hess}\,\phi}{\phi} + \frac{d\phi}{\phi} \otimes \frac{d\phi}{\phi}$.

More generally, for $q \in (0, \infty)$, we define

$$(1.2) \qquad R_q = R - 2\frac{\nabla^2 \phi}{\phi} + \left(1 - \frac{1}{q}\right)\frac{|\nabla\phi|^2}{\phi^2}.$$

We now state a scalar curvature analog of the Ricci O'Neill theorem of [6, Theorem 2]. Suppose that a Riemannian submersion $p : M \to B$ has compact fiber $F$. Put $F_b = p^{-1}(b)$. Given a smooth curve $\gamma : [0,1] \to B$ and a point $m \in F_{\gamma(0)}$, let $\rho(m)$ be the endpoint $\overline{\gamma}(1)$ of the horizontal lift $\overline{\gamma}$ of $\gamma$ that starts at $m$. Then $\rho$ is the fiber transport diffeomorphism from $F_{\gamma(0)}$ to $F_{\gamma(1)}$.

Given the positive function $\phi^M$ on $M$, define $\phi^B$, a smooth positive function on $B$, by

$$(1.3) \qquad p_*(\phi^M \, \mathrm{dvol}_M) = \phi^B \, \mathrm{dvol}_B.$$

Let $\phi^F$ denote the restriction of $\phi^M$ to a fiber $F$. Let $R_\infty^M$, $R_\infty^B$, $R_q^B$ and $R_\infty^F$ denote the modified scalar curvature functions. Let $\mathrm{dvol}_F$ denote the fiberwise Riemannian density. Let $\int_F$ denote fiberwise integration.

**Theorem 1.1.** *Suppose that fiber transport preserves the fiberwise measure $\phi^F \, \mathrm{dvol}_F$ up to multiplicative constants, i.e. for any smooth curve $\gamma : [0,1] \to B$, there is a constant $c_\gamma > 0$ such that $\rho^*\left(\phi^{F_{\gamma(1)}} \, \mathrm{dvol}_{F_{\gamma(1)}}\right) = c_\gamma \, \phi^{F_{\gamma(0)}} \, \mathrm{dvol}_{F_{\gamma(0)}}$.*

*1. We have an inequality of functions on $B$:*

$$(1.4) \qquad R_\infty^B \geq \frac{\int_F (R_\infty^M - R_\infty^F)\,\phi^M \, \mathrm{dvol}_F}{\int_F \phi^M \, \mathrm{dvol}_F}.$$

*2. Suppose in addition that $\phi^M = 1$. Put $q = \dim(F)$. Then*

$$(1.5) \qquad R_q^B \geq \frac{\int_F (R^M - R^F)\,\mathrm{dvol}_F}{\mathrm{vol}(F)}.$$

We now give an application of Theorem 1.1 to collapsing of Riemannian manifolds. A general phenomenon in collapsing is that when the limit space is smooth, a lower curvature bound on the collapsing manifolds tends to give a lower curvature bound on the limit space. This is clearly not true for scalar curvature, as can be seen by taking a product $B \times S^q$, with $q > 1$, and shrinking the $S^q$-factor. One can get around this particular problem by assuming an upper bound on the scalar curvatures of the collapsing manifolds. The next result is a scalar curvature analog of [6, Theorem 3].

**Theorem 1.2.** *Let $\{(M_i, g_i)\}_{i=1}^\infty$ be a sequence of $N$-dimensional connected closed Riemannian manifolds with sectional curvatures bounded above in absolute value by $\Lambda$ and diameters bounded above by $D$, for some $D, \Lambda \in \mathbb{R}^+$. Let $(X, \mu)$ be a limit point for $\{(M_i, g_i)\}_{i=1}^\infty$ in the measured Gromov-Hausdorff topology. Suppose that for some $r \in \mathbb{R}$ and all $i \in \mathbb{Z}^+$, $R(M_i, g_i) \geq r$. Suppose that $X$ is an $n$-dimensional closed manifold. Put $q = N - n$.*

*a. If $q = 0$, then $X$ has $R \geq r$.*

*b. If $q > 0$, then $X$ has $R_q \geq r$. Consequently, $\frac{\int_X R^X \, \mathrm{dvol}_X}{\mathrm{vol}(X)} \geq r$.*

In the statement of Theorem 1.2, the pair $(g_X, \phi^X)$ is $W^{2,p}$-regular for all $p \in [1, \infty)$. Hence $R$ and $R_q$ are well defined in $\bigcap_{p \in [1, \infty)} L^p(X)$.

The last inequality in Theorem 1.2 can be compared with the lower bound in [7, Corollary 7.49] on the mean scalar curvature of a smooth limit space when it arises from a sequence of collapsing manifolds with Ricci curvature bounded below by $K \in \mathbb{R}$. One could ask whether there is an analog of Theorem 1.2 if one assumes a two-sided bound on the Ricci curvature of the manifolds $M_i$, as opposed to the sectional curvature.

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## 2. Proof of Theorem 1.1

We use the notation of [6, Section 3], which in turn uses the notation of [1, Chapter 9]. If $X$ is a vector field on $B$, let $\overline{X}$ be its horizontal lift to $M$. Let $N$ be the mean curvature vector field to the fibers $F$. Let $A$ be the curvature of the horizontal distribution. Let $T$ be the second fundamental form tensor of the fibers $F$. Let $\nabla^M$ be the covariant derivative operator on $M$ and let $\nabla^B$ be the covariant derivative operator on $B$. Define $\check{\delta}N \in C^\infty(M)$ by saying that if $\{\overline{e}_\alpha\}$ is an orthonormal basis of $T_{hor}M$ at a point $m$, then $(\check{\delta}N)(m) = -\sum_\alpha \langle \nabla^M_{\overline{e}_\alpha} N, \overline{e}_\alpha \rangle$. From [1, Corollary 9.37], there is an identity of functions on $M$:

$$(2.1) \qquad R^M = R^B + R^F - |A|^2 - |T|^2 - |N|^2 - 2\check{\delta}N.$$

Given $f \in C^\infty(M)$, we write $\nabla^2_M f$ for the Laplacian of $f$. Define $\nabla^2_{hor} f \in C^\infty(M)$ by saying that $\left(\nabla^2_{hor} f\right)(m) = \sum_\alpha (\text{Hess } f)(\overline{e}_\alpha, \overline{e}_\alpha)$. Define $\nabla^2_F f \in C^\infty(M)$ by saying that if $m$ is in a fiber $F$, then $\left(\nabla^2_F f\right)(m)$ is the Laplacian of the function $f|_F$ on $F$, evaluated at $m$. One can check that

$$(2.2) \qquad \nabla^2_M f = \nabla^2_{hor} f + \nabla^2_F f - (\nabla_{hor} f, N).$$

Given $b \in B$, let $\{\theta_t\}_{t \in (-\epsilon, \epsilon)}$ be the flow of $X$ as defined in a neighborhood of $b$ and for $t$ in some interval $(-\epsilon, \epsilon)$. Let $\{\overline{\theta}_t\}_{t \in (-\epsilon, \epsilon)}$ be the flow of $\overline{X}$ that covers $\theta_t$. It sends fibers to fibers diffeomorphically. Hence it makes sense to define $\mathcal{L}_{\overline{X}} \text{dvol}_F$ by

$$(2.3) \qquad \left(\mathcal{L}_{\overline{X}} \text{dvol}_F\right)\Big|_{F_b} = \frac{d}{dt}\Big|_{t=0} \left(\overline{\theta}_t^* \text{dvol}_F\right)\Big|_{F_b}.$$

With our conventions,

$$(2.4) \qquad \mathcal{L}_{\overline{X}} \text{dvol}_F = -\left(\overline{X}, N\right) \text{dvol}_F.$$

We have

$$(2.5) \qquad \phi^B = \int_F \phi^M \text{ dvol}_F.$$

Then

$$(2.6) \qquad X\phi^B = \mathcal{L}_X \phi^B = \mathcal{L}_X \int_F \phi^M \text{ dvol}_F = \int_F \mathcal{L}_{\overline{X}} \left(\phi^M \text{ dvol}_F\right)$$

$$= \int_F \left(\overline{X}\phi^M - \left(\overline{X}, N\right)\phi^M\right) \text{ dvol}_F.$$

From [6, (3.7)],

$$(2.7) \quad \mathrm{Hess}(\phi^B)(X,X) = \int_F \left[ \frac{\mathrm{Hess}(\phi^M)(\overline{X},\overline{X})}{\phi^M} - \left(\frac{\overline{X}\phi^M}{\phi^M}\right)^2 - \left(\overline{X}, \nabla^M_{\overline{X}} N\right) \right.$$

$$\left. + \left(\frac{\overline{X}\phi^M}{\phi^M} - (\overline{X}, N)\right)^2 \right] \phi^M \, \mathrm{dvol}_F \,.$$

Then
(2.8)
$$\nabla^2_B \phi^B = \int_F \left[ \frac{\nabla^2_{hor}\phi^M}{\phi^M} - \left|\frac{\nabla_{hor}\phi^M}{\phi^M}\right|^2 + \check{\delta}N + \left|\frac{\nabla_{hor}\phi^M}{\phi^M} - N\right|^2 \right] \phi^M \, \mathrm{dvol}_F \,.$$

Given $b \in B$, put $F_b = p^{-1}(b)$. From (1.1), (2.1) and (2.8),

(2.9)

$$\phi^B(b) \, R^B_\infty(b) = \phi^B(b) \, R^B(b) - 2\,\nabla^2_B \phi^B + \frac{|\nabla_B \phi^B|^2}{\phi^B}$$

$$= \int_{F_b} R^B \, \phi^M \, \mathrm{dvol}_F - 2\,\nabla^2_B \phi^B + \frac{|\nabla_B \phi^B|^2}{\phi^B}$$

$$= \int_{F_b} \left[ \left(R^M - R^F + |A|^2 + |T|^2 + |N|^2\right) \phi^M \right.$$

$$\left. - 2\,\nabla^2_{hor}\phi^M + 2\frac{|\nabla_{hor}\phi^M|^2}{\phi^M} - 2\left|\frac{\nabla_{hor}\phi^M}{\phi^M} - N\right|^2 \phi^M \right] \mathrm{dvol}_F$$

$$+ \frac{|\nabla_B \phi^B|^2}{\phi^B}.$$

We have

$$(2.10) \qquad \mathcal{L}_{\overline{X}}(\phi^M \, \mathrm{dvol}_F) = \left(\frac{\overline{X}\phi^M}{\phi^M} - (\overline{X}, N)\right) \phi^M \, \mathrm{dvol}_F \,.$$

By assumption, $\frac{\overline{X}\phi^M}{\phi^M} - (\overline{X}, N)$ is constant on a fiber $F$. Let $\{e_\alpha\}$ be an orthonormal basis of $T_b B$ and let $\{\overline{e}_\alpha\}$ be the horizontal lift of $\{e_\alpha\}$ along $F_b$. Then

$$(2.11) \qquad \frac{\left(e_\alpha \phi^B\right)^2}{\phi^B} = (\phi^B)^{-1} \left(\int_{F_b} \left(\frac{\overline{e}_\alpha \phi^M}{\phi^M} - (\overline{e}_\alpha, N)\right) \phi^M \, \mathrm{dvol}_F\right)^2$$

$$= \int_{F_b} \left(\frac{\overline{e}_\alpha \phi^M}{\phi^M} - (\overline{e}_\alpha, N)\right)^2 \phi^M \, \mathrm{dvol}_F,$$

and so

$$(2.12) \qquad \frac{\left|\nabla_B \phi^B\right|^2}{\phi^B} = \int_{F_b} \left|\frac{\nabla^M_{hor}\phi^M}{\phi^M} - N\right|^2 \phi^M \, \mathrm{dvol}_F \,.$$

From (2.9) and (2.12),

(2.13)

$$\phi^B(b)\, R_\infty^B(b) \;=\; \int_{F_b}\left[\left(R^M \,-\, R^F \,+\, |A|^2 \,+\, |T|^2 \,+\, |N|^2\right)\phi^M\right.$$

$$\left.-2\,\nabla_{hor}^2\phi^M \,+\, 2\,\frac{\left|\nabla_{hor}\phi^M\right|^2}{\phi^M} \,-\, \left|\frac{\nabla_{hor}\phi^M}{\phi^M} \,-\, N\right|^2\phi^M\right]\mathrm{dvol}_F$$

$$=\; \int_{F_b}\left[\left(R^M \,-\, R^F \,+\, |A|^2 \,+\, |T|^2\right)\phi^M \,-\, 2\,\nabla_{hor}^2\phi^M\right.$$

$$\left.+\frac{\left|\nabla_{hor}\phi^M\right|^2}{\phi^M} \,+\, 2\left(\nabla_{hor}\phi^M, N\right)\right]\mathrm{dvol}_F\,.$$

As

(2.14)
$$\left|\nabla_M\phi^M\right|^2 \;=\; \left|\nabla_{hor}\phi^M\right|^2 \,+\, \left|\nabla_F\phi^M\right|^2,$$

we can combine (2.13) with (2.2) and (2.14) to obtain

(2.15)
$$\phi^B(b)\, R_\infty^B(b) \;=\; \int_{F_b}\left(R_\infty^M \,-\, R_\infty^F \,+\, |A|^2 \,+\, |T|^2\right)\phi^M\,\mathrm{dvol}_F$$

$$\geq\; \int_{F_b}\left(R_\infty^M \,-\, R_\infty^F\right)\phi^M\,\mathrm{dvol}_F\,.$$

This proves Theorem 1.1.1.

Now suppose that $\phi^M \,=\, 1$. From (2.12), (2.15) and the definition of $R_q$ from (1.2),

(2.16)
$$\phi^B(b)\, R_q^B(b) \;=\; \phi^B(b)\, R_\infty^B(b) \,-\, \frac{1}{q}\,\frac{|\nabla_B\phi^B|^2}{\phi^B}$$

$$=\; \int_{F_b}\left(R^M \,-\, R^F \,+\, |A|^2 \,+\, |T|^2 \,-\, \frac{1}{q}\,|N|^2\right)\mathrm{dvol}_F\,.$$

As $(\overline{e}_\alpha, N) \,=\, -\,\mathrm{Tr}\,(T\overline{e}_\alpha)$, we know that $(T\overline{e}_\alpha, T\overline{e}_\alpha) \,-\, \frac{1}{q}\,(\overline{e}_\alpha, N)^2 \geq 0$. Summing over $\alpha$ gives $|T|^2 \,-\, \frac{1}{q}\,|N|^2 \,\geq\, 0$. This proves Theorem 1.1.2.

## 3. Proof of Theorem 1.2

Let $\{M_i, g_i\}_{i=1}^\infty$ be a sequence as in the statement of Theorem 1.2. We may assume that $\lim_{i\to\infty}(M_i, g_i, \mathrm{dvol}_i) \,=\, (X, \mu)$ in the measured Gromov-Hausdorff topology. If $q \,=\, 0$, then $X$ is a smooth manifold with a $W^{2,p}$-regular metric $g_X$; after taking a subsequence and applying diffeomorphisms, we may assume that $(M_i, g_i)$ converges to $(X, g_X)$ in the $W^{2,p}$-topology (see, for example, [5, 9, 10]). In particular, $\lim_{i\to\infty} R(g_i) \,=\, R(g_X)$ in $L^p(X)$. Thus $R^X \,\geq\, r$.

Suppose that $q \,>\, 0$. By saying that $X$ is a manifold, we mean that in the construction of $X$ as a quotient space $\widehat{X}/O(N)$ [4], the action of $O(N)$ on the manifold $\widehat{X}$ has a single orbit type. Then $X$ has the structure of a smooth manifold with a $W^{2,p}$-regular pair $(g_X, \phi^X)$.

For any $\epsilon > 0$, we can apply smoothing results of Abresch and others [2, Theorem 1.12] to obtain new metrics $g_i(\epsilon)$ with

$$(3.1) \qquad\qquad e^{-\epsilon} g_i \ \leq \ g_i(\epsilon) \ \leq \ e^{\epsilon} g_i,$$

$$|\nabla_{g_i} \ - \ \nabla_{g_i(\epsilon)}| \ \leq \ \epsilon,$$

$$|\nabla_{g_i(\epsilon)}^k \ \mathrm{Riem}(M_i, g_i(\epsilon))| \ \leq \ C_k(N, \epsilon, \Lambda),$$

where the constants are uniform. We can also assume that $R(M_i, g_i(\epsilon)) \geq r - \epsilon$; compare [3, Remark 2, p. 51] and [11, Theorem 2.1]. For small $\epsilon$, let $B(\epsilon)$ be a Gromov-Hausdorff limit of a subsequence of $\{(M_i, g_i(\epsilon))\}_{i=1}^{\infty}$. We relabel the subsequence as $\{(M_i, g_i(\epsilon))\}_{i=1}^{\infty}$. From [2, Proposition 4.9], for large $i$, there is a small $C^2$-perturbation $g_i'(\epsilon)$ of $g_i(\epsilon)$ which is invariant with respect to a *Nil*-structure. In particular, we may assume that $R(M_i, g_i'(\epsilon)) \geq r - 2\epsilon$. Now $(M_i, g_i'(\epsilon))$ is the total space of a Riemannian submersion $M_i \to B(\epsilon)$ with infranil fibers and affine holonomy. Let $\left(g_i^{B(\epsilon)}, \phi_i^{B(\epsilon)}\right)$ denote the induced metric and measure on $B(\epsilon)$. As the fiber transport of the Riemannian submersion preserves the affine-parallel volume forms of the fibers up to multiplicative constants, and the infranil fibers have nonpositive scalar curvature, Theorem 1.1.2 implies that $R_q \left(B(\epsilon), g_i^{B(\epsilon)}, \phi_i^{B(\epsilon)}\right) \geq r - 2\epsilon$. Varying $i$ and $\epsilon$, we can extract a subsequence of $\left\{\left(B(\epsilon), g_i^{B(\epsilon)}, \phi_i^{B(\epsilon)}\right)\right\}$ with $i \to \infty$ and $\epsilon \to 0$ that converges in the $W^{2,p}$-topology to $(X, g_X, \phi^X)$. Thus $R_q^X \geq r$.

Finally,

$$(3.2) \qquad R_q^X \ = \ R^X \ - \ 2\, \nabla^2 \ln(\phi^X) \ - \ \left(1 \ + \ \frac{1}{q}\right) \ |\nabla \ln(\phi^X)|^2.$$

Then

$$(3.3) \qquad\qquad r \ \leq \ \frac{\int_X R_q^X \ \mathrm{dvol}_X}{\mathrm{vol}(X)} \ \leq \ \frac{\int_X R^X \ \mathrm{dvol}_X}{\mathrm{vol}(X)}.$$

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