

Manifolds with quadratic curvature decay and fast volume growth

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Abstract. We give sufficient conditions for a noncompact Riemannian manifold, which has quadratic curvature decay, to have finite topological type with ends that are cones over spherical space forms.

1. Introduction

Let M be a complete connected noncompact Riemannian manifold with a base-point $*$. A natural condition to put on M is that of quadratic curvature decay. To state this condition, for $m \in M$ and $r > 0$, let $B_r(m)$ denote the open distance ball around m of radius r and let $S_r(m) = \partial B_r(m)$ denote the distance sphere around m of radius r . If P is a 2-plane in $T_m M$, let $K(P)$ denote the sectional curvature of P . Then M has quadratic curvature decay if for some $C > 0$,

$$\limsup_{r \rightarrow \infty} \sup_{m \in S_r(*), P \subset T_m M} r^2 |K(P)| \leq C. \quad (1.1)$$

Note that (1.1) is scale-invariant, in that it is unchanged under a constant rescaling of the Riemannian metric.

In itself (1.1) does not impose any topological restrictions on M , as any smooth connected manifold admits a complete Riemannian metric satisfying (1.1) for some C [6, p. 96], [11, Lemma 2.1]. However, with additional assumptions one can obtain restrictions on M . For example, if

$$\limsup_{r \rightarrow \infty} \sup_{m \in S_r(*), P \subset T_m M} r^{2(1+\epsilon)} |K(P)| < \infty \quad (1.2)$$

for some $\epsilon > 0$ then Abresch showed that M has finite topological type, i.e. is homeomorphic to the interior of a compact manifold-with-boundary [1]. For other results on manifolds with faster-than-quadratic curvature decay, see [1], [4] and [14].

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If M has quadratic curvature decay and a volume growth which is slower than that of the Euclidean space of the same dimension then topological restrictions on the ends of M were obtained in a paper of the author with Zhongmin Shen [11]. Along these lines, we remark that a manifold with quadratic curvature decay and Euclidean volume growth can have infinite topological type [11, Section 2, Example 3]. Furthermore, even if we assume finite topological type, the interior of any connected compact manifold-with-boundary has a complete Riemannian metric with quadratic curvature decay and Euclidean volume growth [11, Section 2, Example 1]. Hence the assumptions of quadratic curvature decay and Euclidean volume growth do not in themselves give interesting topological restrictions.

In [11] the question was raised as to what one can say if one assumes that the constant C in (1.1) is small enough. In this paper we give some answers to this question. First, we show that if the constant C is small enough, if we have pinched Euclidean volume growth and if M is noncollapsed at infinity in a suitable sense then M has finite topological type, with ends that are cones over spherical space forms.

Theorem 1. *Given $n \in \mathbb{Z}^+$ and $c, c' \in \mathbb{R}^+$, there is a constant $\epsilon \equiv \epsilon(n, c, c') > 0$ so that if M is a complete connected n -dimensional Riemannian manifold with basepoint \star which satisfies*

$$\liminf_{r \rightarrow \infty} \inf_{m \in S_r(\star)} r^{-n} \text{vol}(B_{r/2}(m)) \geq c \quad (\text{noncollapsing}), \tag{1.3}$$

$$c' - \epsilon \leq \liminf_{r \rightarrow \infty} r^{-n} \text{vol}(B_r(\star)) \leq \limsup_{r \rightarrow \infty} r^{-n} \text{vol}(B_r(\star)) \leq c' + \epsilon \quad (\text{Eucl. vol. growth}) \tag{1.4}$$

and

$$\limsup_{r \rightarrow \infty} \sup_{m \in S_r(\star), P \subset T_m M} r^2 |K(P)| \leq \epsilon \quad (\text{quadratic curvature decay}) \tag{1.5}$$

then M has finite topological type with ends that are cones over spherical space forms. That is, for large R , $M - \overline{B_R(\star)}$ is homeomorphic to $(0, \infty) \times Y$ for some closed manifold Y which is a union of spherical space forms. Furthermore, Y has volume $n c'$ and the cone over Y satisfies (1.3). In particular, there is a finite number of topological possibilities for Y , with the number depending on c and c' .

Next, we show that there is a surface of infinite topological type which admits noncollapsing metrics of roughly Euclidean volume growth, and arbitrarily pinched quadratic curvature decay. The existence of such metrics was pointed out to me by Bruce Kleiner.

Theorem 2. *Given $\epsilon > 0$, there is a surface of infinite topological type, equipped with a complete Riemannian metric, along with constants $c, c'_1, c'_2 > 0$ such that*

$$\liminf_{r \rightarrow \infty} \inf_{m \in S_r(\star)} r^{-2} \text{vol}(B_{r/2}(m)) \geq c \quad (\text{noncollapsing}), \tag{1.6}$$

$$\begin{aligned}
 c'_1 &\leq \liminf_{r \rightarrow \infty} r^{-2} \operatorname{vol}(B_r(\star)) \leq \limsup_{r \rightarrow \infty} r^{-2} \operatorname{vol}(B_r(\star)) \\
 &\leq c'_2 \quad (\text{Euclidean volume growth})
 \end{aligned}
 \tag{1.7}$$

and

$$\limsup_{r \rightarrow \infty} \sup_{m \in S_r(\star), P \subset T_m M} r^2 |K(P)| \leq \epsilon. \quad (\text{quadratic curvature decay}) \tag{1.8}$$

Finally, we give a result in which the pinched Euclidean volume growth of Theorem 1 is replaced by a large-scale convexity assumption.

Definition 1. *A complete connected Riemannian manifold M with basepoint \star is large-scale pointed-convex if there is a constant $C' > 0$ such that*

1. *For any normalized minimizing geodesic $\gamma : [a, b] \rightarrow M$ and any $t \in [0, 1]$,*

$$d(\gamma(ta + (1-t)b), \star) \leq t d(\gamma(a), \star) + (1-t) d(\gamma(b), \star) + C' \tag{1.9}$$

and

2. *For any two normalized minimizing geodesics $\gamma_1, \gamma_2 : [0, b] \rightarrow M$ with $\gamma_1(0) = \gamma_2(0) = \star$ and any $t \in [0, 1]$,*

$$d(\gamma_1(tb), \gamma_2(tb)) \leq t d(\gamma_1(b), \gamma_2(b)) + C'. \tag{1.10}$$

Examples of large-scale pointed-convex manifolds are simply-connected manifolds of nonpositive curvature, and Riemannian manifolds whose underlying metric spaces are Gromov-hyperbolic [3, Chapitre 2, Pf. of Proposition 25].

Theorem 3. *Given $n > 2$ and $c \in \mathbb{R}^+$, there is a constant $\epsilon \equiv \epsilon(n, c) > 0$ with the following property. Suppose that M is a complete connected n -dimensional Riemannian manifold with basepoint which is large-scale pointed-convex and which satisfies*

$$\liminf_{r \rightarrow \infty} \inf_{m \in S_r(\star)} r^{-n} \operatorname{vol}(B_{r/2}(m)) \geq c \quad (\text{noncollapsing}) \tag{1.11}$$

and

$$\limsup_{r \rightarrow \infty} \sup_{m \in S_r(\star), P \subset T_m M} r^2 |K(P)| \leq \epsilon. \quad (\text{quadratic curvature decay}) \tag{1.12}$$

Then M has finite topological type, with ends that are cones over spherical space forms.

The method of proof of Theorem 1 is by contradiction. Here is the rough argument. Suppose that we have a sequence of n -dimensional Riemannian manifolds $\{M_i\}_{i=1}^\infty$ which together provide a counterexample to Theorem 1. Then each M_i has “bad” regions arbitrarily far away from the basepoint. By rescaling, we can assume that the unit sphere around the basepoint in each M_i intersects a bad region. We would like to take a convergent subsequence of the M_i ’s in order to argue by

contradiction. We may not be able to take a convergent subsequence in the pointed Gromov-Hausdorff sense, as the curvatures may not be uniformly bounded below at the basepoints. However, we can always take a pointed ultralimit (X_ω, \star_ω) (see Section 2). Then any ball in X_ω away from the basepoint will be the Gromov-Hausdorff limit of a subsequence of balls in the M_i 's. Under our assumptions, $X_\omega - \star_\omega$ will be n -dimensional and flat with volume growth $V(r) = c' r^n$. Then X_ω is a cone over a closed manifold Y which is a union of spherical space forms. It follows that for an infinite number of i 's, the "bad" region in M_i was actually good, which is a contradiction.

To prove Theorem 3 we again form an ultralimit X_ω , which will have a flat metric on $X_\omega - \star_\omega$ and which will be pointed-convex. If \mathcal{C} is a connected component of $X_\omega - \star_\omega$ then its developing map gives an isometric immersion of the universal cover $\tilde{\mathcal{C}}$ into \mathbb{R}^n . The convexity is used to show that the developing map is an embedding, with image $\mathbb{R}^n - \text{pt.}$, from which the theorem follows.

The structure of the paper is as follows. In Section 2 we recall some facts about ultralimits of metric spaces. In Section 3 we prove Theorem 1. In Section 4 we prove Theorem 2. In Section 5 we prove Theorem 3 and make some remarks about its hypotheses.

For background information about Gromov-Hausdorff limits and convergence results, we refer to [7] and [13].

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2. Ultralimits

If ω is a nonprincipal ultrafilter on \mathbb{Z}^+ and $\{X_i\}_{i=1}^\infty$ is a sequence of metric spaces, let X_ω be the ω -limit of the X_i 's (see, for example, [7, Section 3.29], [9, Chapter 9] and [10, Section 2.4] for background material). It is a complete metric space. An element of X_ω has a representative $\{x_i\} \in \prod_{i=1}^\infty X_i$. Two such sequences $\{x_i\}$ and $\{x'_i\}$ are equivalent if $\lim_\omega d_{X_i}(x_i, x'_i) = 0$. The metric on X_ω is

$$d_{X_\omega}(\{x_i\}, \{x'_i\}) = \lim_\omega d_{X_i}(x_i, x'_i). \tag{2.1}$$

If $\{(X_i, \star_i)\}_{i=1}^\infty$ are pointed metric spaces then the pointed limit (X_ω, \star_ω) is the subset of X_ω given by representatives $\{x_i\}$ such that $\{d_{X_i}(x_i, \star_i)\}_{i=1}^\infty$ is a bounded sequence. The basepoint \star_ω in X_ω has representative $\{\star_i\}$. If each X_i is a length space then X_ω is a length space and minimizing geodesic segments in X_ω are ultralimits of minimizing geodesic segments in $\{X_i\}_{i=1}^\infty$ [9, Proposition 9.4].

If X is a metric space then we let $\text{cone}(X)$ denote the cone on X , a pointed metric space.

Example. Fix $\alpha > 1$. Take $(X_i, \star_i) = (\mathbb{R}^2, 0)$ with Riemannian metric

$$g_i = i^{-2} (dr^2 + r^{2\alpha} d\theta^2) \tag{2.2}$$

on $\mathbb{R}^2 - 0 \cong \mathbb{R}^+ \times \frac{\mathbb{R}}{2\pi\mathbb{Z}}$. Then by definition, (X_ω, \star_ω) is the asymptotic cone of X_1 . To describe it, first, by a change of radial coordinate, g_i is equivalent to $dr^2 + i^{2\alpha-2} r^{2\alpha} d\theta^2$. Then by a change of angular coordinate, X_i consists of $\mathbb{R}^+ \times \frac{\mathbb{R}}{i^{\alpha-1} 2\pi\mathbb{Z}}$ equipped with the metric $dr^2 + r^{2\alpha} d\theta^2$, along with the basepoint \star_i . Put $Y_\omega = \lim_\omega \frac{\mathbb{R}}{i^{\alpha-1} 2\pi\mathbb{Z}}$ (an unpointed limit), which is an infinite disjoint union of real lines. (Two points in Y_ω , represented by sequences $\{y_i\}$ and $\{y'_i\}$, lie in the same connected component of Y_ω if and only if $\lim_\omega d_{Y_i}(y_i, y'_i) < \infty$.) Then X_ω consists of $\mathbb{R}^+ \times Y$ with the metric $dr^2 + r^{2\alpha} g_{Y_\omega}$, along with the basepoint \star_ω . The manifolds $\{X_i\}_{i=1}^\infty$ have uniform quadratic curvature decay. Clearly the sequence $\{(X_i, \star_i)\}_{i=1}^\infty$ is not precompact in the pointed Gromov-Hausdorff topology. Nevertheless, in a sense it has well-defined Gromov-Hausdorff limits away from the basepoint.

For a related relevant example, take $(X_i, \star_i) = (\mathbb{R}^2, 0)$ with Riemannian metric

$$g_i = dr^2 + i^2 r^2 d\theta^2 \tag{2.3}$$

on $\mathbb{R}^2 - 0 \cong \mathbb{R}^+ \times \frac{\mathbb{R}}{2\pi\mathbb{Z}}$. Put $Y_\omega = \lim_\omega \frac{\mathbb{R}}{i 2\pi\mathbb{Z}}$. Then $X_\omega = \text{cone}(Y_\omega)$. There is a flat Riemannian metric on $X_\omega - \star_\omega$.

3. Proof of Theorem 1

Suppose that the theorem is not true. Then there is a sequence of pointed complete connected n -dimensional Riemannian manifolds $\{(M_i, \star_i)\}_{i=1}^\infty$ such that :

1. Condition (1.3) is satisfied for each M_i .
2. On M_i , we have

$$c' - \frac{1}{i} \leq \liminf_{r \rightarrow \infty} r^{-n} \text{vol}(B_r(\star_i)) \leq \limsup_{r \rightarrow \infty} r^{-n} \text{vol}(B_r(\star_i)) \leq c' + \frac{1}{i}. \tag{3.1}$$

3. On M_i , we have

$$\limsup_{r \rightarrow \infty} \sup_{m_i \in S_r(\star_i), P_i \subset T_{m_i} M_i} r^2 |K(P_i)| \leq \frac{1}{i}. \tag{3.2}$$

- 4.a. M_i has infinite topological type or
- 4.b. M_i has an end which has no neighborhood homeomorphic to $(0, \infty) \times N$ for any closed manifold N which is a union of spherical space forms.

Define $\rho_i \in C^0(M_i)$ by $\rho_i(m_i) = d(m_i, \star_i)$.

Lemma 1. *For each i , there is a sequence $\{r_{i,j}\}_{j=1}^\infty$ of numbers tending toward infinity such that for each j , there is a connected component $C_{i,j}$ of $\overline{B_{4r_{i,j}}(\star_i)} - B_{r_{i,j}}(\star_i)$ with the property that it is not true that the map $C_{i,j} \rightarrow [r_{i,j}, 4r_{i,j}]$, given by restriction of ρ_i , defines a topological fiber bundle whose fiber is a spherical space form.*

Proof. Fix i . If the lemma is false then there is a number $R > 0$ so that for all $r > R$ and for each connected component C of $\overline{B_{4r}(\star_i)} - B_r(\star_i)$, the map $\rho_i|_C : C \rightarrow [r, 4r]$ defines a topological fiber bundle whose fiber is a spherical space form. In particular, C is homeomorphic to $[r, 4r] \times N$ for some spherical space form N .

Put $s_1 = R + 1$. Then $\overline{B_{4s_1}(\star_i)} - B_{s_1}(\star_i)$ is homeomorphic to $[s_1, 4s_1] \times \coprod_{k \in K} N_k$, where K is an indexing set and each N_k is a spherical space form. The restriction of ρ_i to $\overline{B_{4s_1}(\star_i)} - B_{s_1}(\star_i)$ is given by projection onto the first factor of $[s_1, 4s_1] \times \coprod_{k \in K} N_k$. As $\overline{B_{3s_1}(\star_i)} - B_{2s_1}(\star_i)$ is compact, K must be a finite set. Let C_k be the connected component of $\overline{B_{4s_1}(\star_i)} - B_{s_1}(\star_i)$ corresponding to $[s_1, 4s_1] \times N_k$. Put $s_2 = 3s_1$. There is a connected component C'_k of $\overline{B_{4s_2}(\star_i)} - B_{s_2}(\star_i)$ which intersects C_k . We know that it is homeomorphic to $[s_2, 4s_2] \times N'$ for some spherical space form N' , with the restriction of ρ_i to C'_k given by projection onto the first factor of $[s_2, 4s_2] \times N'$. Then $N' = N_k$. Thus $C_k \cup C'_k$ is homeomorphic to $[s_1, 4s_2] \times N_k$ and extends C_k . As each connected component of $\overline{B_{4s_2}(\star_i)} - B_{s_2}(\star_i)$ intersects $\overline{B_{4s_1}(\star_i)} - B_{s_1}(\star_i)$, we see that $\overline{B_{4s_2}(\star_i)} - B_{s_1}(\star_i)$ is homeomorphic to $[s_1, 4s_2] \times \coprod_{k \in K} N_k$. Taking $s_3 = 3s_2$ and continuing the process, we obtain that $M_i - \overline{B_{R+1}(\star_i)}$ is homeomorphic to $(0, \infty) \times \coprod_{k \in K} N_k$. □

With reference to Lemma 1, (1.3), (3.1) and (3.2), we can find a sequence $R_i = r_{i,j(i)}$ tending towards infinity such that

1. For $r > \frac{1}{i}$,

$$\inf_{m_i \in \mathcal{S}_{R_i r}(\star_i)} (R_i r)^{-n} \text{vol}(B_{R_i r/2}(m_i)) \geq c - \frac{1}{i}, \tag{3.3}$$

$$c' - \frac{2}{i} \leq (R_i r)^{-n} \text{vol}(B_{R_i r}(\star_i)) \leq c' + \frac{2}{i}, \tag{3.4}$$

and

$$\sup_{m_i \in \mathcal{S}_{R_i r}(\star_i), P_i \subset T_{m_i} M_i} (R_i r)^2 |K(P_i)| \leq \frac{2}{i}. \tag{3.5}$$

2. There is a connected component C_i of $\overline{B_{4R_i}(\star_i)} - B_{R_i}(\star_i) \subset M_i$ with the property that it is not true that the map $C_i \rightarrow [R_i, 4R_i]$, given by restriction of ρ_i , defines a topological fiber bundle whose fiber is a spherical space form.

Let X_i be M_i with the rescaled metric $g_{X_i} = (2R_i)^{-2} g_{M_i}$. Define $\mu_i \in C^0(X_i)$ by $\mu_i(x_i) = d(x_i, \star_i)$. Let (X_ω, \star_ω) be the ω -limit of $\{(X_i, \star_i)\}_{i=1}^\infty$.

Lemma 2. $X_\omega - \star_\omega$ is a flat n -dimensional manifold.

Proof. Given $x_\omega \in X_\omega - \star_\omega$, put $D = d(x_\omega, \star_\omega)$. Then $D > 0$. Choose a representative $\{x_i\} \in \prod_{i=1}^\infty X_i$ of x_ω . For any $\epsilon > 0$, there is a subset $W \subset \mathbb{Z}^+$ of full ω -measure such that for all $i \in W$,

$$|d(x_i, \star_i) - D| < \epsilon. \tag{3.6}$$

If $i \in W$ put $y_i = x_i$ and if $i \notin W$, choose $y_i \in S_D(\star_i) \subset X_i$. Then $\lim_\omega d_{X_i}(x_i, y_i) = 0$ and so $\{y_i\}$ also represents x_ω . Thus in replacing $\{x_i\}$ by $\{y_i\}$, we may assume that $d(x_i, \star_i) \in (D - \epsilon, D + \epsilon)$ for all $i \in \mathbb{Z}^+$. Take $\epsilon \in (0, \frac{D}{10})$.

Due to the rescaling used to define X_i , for all $r > \frac{1}{i}$,

$$\inf_{x_i \in S_r(\star_i)} r^{-n} \text{vol}(B_{r/2}(x_i)) \geq c - \frac{1}{i} \tag{3.7}$$

$$c' - \frac{2}{i} \leq r^{-n} \text{vol}(B_r(\star_i)) \leq c' + \frac{2}{i}, \tag{3.8}$$

and

$$\sup_{x_i \in S_r(\star_i), P_i \subset T_{x_i} X_i} r^2 |K(P_i)| \leq \frac{2}{i}. \tag{3.9}$$

Equation (3.9) gives a uniform lower bound on the sectional curvatures of $\{\overline{B_{4D/5}(x_i)}\}_{i=1}^\infty$. It follows that the closed balls $\{\overline{B_{3D/4}(x_i)}\}_{i=1}^\infty$ are precompact in the pointed Gromov-Hausdorff topology [13, Theorem 2.2, Fact 4]. To be precise, [13, Theorem 2.2, Fact 4] deals with pointed Gromov-Hausdorff precompactness in the case of complete manifolds. However, in view of the definition of pointed Gromov-Hausdorff precompactness, the same argument applies to the distance balls.

Sublemma 1. $\overline{B_{3D/4}(x_\omega)}$ is a limit point of $\{\overline{B_{3D/4}(x_i)}\}_{i=1}^\infty$ in the pointed Gromov-Hausdorff topology.

Proof. The proof is similar to that of [10, Lemma 2.4.3]. By precompactness, for any $\delta > 0$ there is a number J such that for each i , there is a δ -net $\{x_{i,j}\}_{j=1}^J$ in $\overline{B_{3D/4}(x_i)}$, with $x_{i,1} = x_i$. Let $x_{\omega,j} \in X_\omega$ be represented by the sequence $\{x_{i,j}\}$. In particular, $x_{\omega,1} = x_\omega$. We claim that $\{x_{\omega,j}\}_{j=1}^J$ is a δ -net in $\overline{B_{3D/4}(x_\omega)}$. First,

$$d_{X_\omega}(x_{\omega,j}, x_\omega) = \lim_\omega d_{X_i}(x_{i,j}, x_i) \leq 3D/4, \tag{3.10}$$

so $x_{\omega,j} \in \overline{B_{3D/4}(x_\omega)}$. Next, given $y_\omega = \{y_i\} \in \overline{B_{3D/4}(x_\omega)}$, for $j \in \{1, \dots, J\}$ put

$$U_j = \{i : d_{X_i}(x_{i,j}, y_i) \leq \delta\}. \tag{3.11}$$

As $\mathbb{Z}^+ = \bigcup_{j=1}^J U_j$, there is some j so that U_j has full ω -measure. Then for this j , $d_{X_\omega}(x_{\omega,j}, y_\omega) = \lim_\omega d_{X_i}(x_{i,j}, y_i) \leq \delta$. Thus $\{x_{\omega,j}\}_{j=1}^J$ is a δ -net in $\overline{B_{3D/4}(x_\omega)}$.

From the definition of d_{X_ω} , there is a subset $W \subset \mathbb{Z}^+$ of full ω -measure such that for all $i \in W$ and $j, k \in \{1, \dots, J\}$,

$$|d_{X_\omega}(x_{\omega,j}, x_{\omega,k}) - d_{X_i}(x_{i,j}, x_{i,k})| < \delta. \tag{3.12}$$

For any $i \in W$, it follows as in the proof of [7, Proposition 3.5(b)] that the pointed Gromov-Hausdorff distance between $\overline{B_{3D/4}(x_\omega)} \subset X_\omega$ and $\overline{B_{3D/4}(x_i)} \subset X_i$ is at most 2δ . This proves the sublemma. \square

From (3.7), for i sufficiently large,

$$\text{vol}(B_{3D/5}(x_i)) \geq \text{vol}(B_{d(x_i, \star_i)/2}(x_i)) \geq \frac{1}{2} c d(x_i, \star_i)^n \geq \frac{1}{2} c (9D/10)^n. \tag{3.13}$$

Hence we are in the noncollapsing situation and so from [13, Corollary 2.3, Lemma 3.4 and Theorem 4.1], $X_\omega - \star_\omega$ has a flat n -dimensional Riemannian metric. \square

From Sublemma 1 and [13, Theorem 2.2], there is an infinite subset $S \subset \mathbb{Z}^+$ such that $\overline{B_{3D/5}(x_\omega)}$ is actually the limit of $\{\overline{B_{3D/5}(x_i)}\}_{i \in S}$ in the $C^{1,\sigma}$ -topology for any $\sigma \in (0, 1)$. Given $\alpha > 1$ and $r > 0$, put $A_i(\alpha r, r) = \overline{B_{\alpha r}(\star_i)} - B_r(\star_i) \subset X_i$ and $A_\omega(\alpha r, r) = \overline{B_{\alpha r}(\star_\omega)} - B_r(\star_\omega) \subset X_\omega$. From (3.8),

$$\lim_{i \rightarrow \infty} \frac{\text{vol}(A_i(\alpha r, r))}{r^n} = (\alpha^n - 1) c'. \tag{3.14}$$

By abuse of notation, we write $\text{vol}(B_r(\star_\omega))$ for $\text{vol}(B_r(\star_\omega) - \star_\omega)$.

Lemma 3. For all $r > 0$,

$$\text{vol}(A_\omega(\alpha r, r)) = (\alpha^n - 1) c' r^n \tag{3.15}$$

and

$$\inf_{x_\omega \in S_r(\star_\omega)} r^{-n} \text{vol}(B_{r/2}(x_\omega)) \geq c. \tag{3.16}$$

Proof. Given $x_\omega \in A_\omega(\alpha r, r)$, let $\{x_i\}_{i=1}^\infty$ be as in the proof of Lemma 2. For $\epsilon > 0$ sufficiently small, the method of proof of Sublemma 1 shows that $\overline{B_\epsilon(x_\omega)}$ is the pointed Gromov-Hausdorff limit of a sequence of ϵ -balls $\{\overline{B_\epsilon(x_i)}\}_{i \in S}$. From the Vitali covering theorem [12, Theorem 2.8], if $\text{vol}(A_\omega(\alpha r, r)) < \infty$ then for any $\delta > 0$ there is a finite number of disjoint closed metric balls $\{\overline{B(x_{\omega,j}, r_j)}\}_{j=1}^J$ contained in $A_\omega(\alpha r, r)$ such that

$$\sum_{j=1}^J \text{vol}(\overline{B(x_{\omega,j}, r_j)}) \geq \text{vol}(A_\omega(\alpha r, r)) - \delta, \tag{3.17}$$

while if $\text{vol}(A_\omega(\alpha r, r)) = \infty$ then for any $\Delta > 0$, there is a finite number of disjoint closed metric balls $\{\overline{B(x_{\omega,j}, r_j)}\}_{j=1}^J$ contained in $A_\omega(\alpha r, r)$ such that

$$\sum_{j=1}^J \text{vol}(\overline{B(x_{\omega,j}, r_j)}) \geq \Delta. \tag{3.18}$$

(Note that $A_\omega(\alpha r, r)$ could *a priori* have an infinite number of connected components.) The $C^{1,\sigma}$ metric convergence implies that for any $\epsilon > 0$ and for an infinite number of i 's, there are disjoint closed metric balls $\{\overline{B}(x_{i,j}, r_j)\}_{j=1}^J$ contained in $A_i(\alpha r, r)$ with

$$\sum_{j=1}^J \text{vol}(\overline{B}(x_{\omega,j}, r_j)) \leq \sum_{j=1}^J \text{vol}(\overline{B}(x_{i,j}, r_j)) + \epsilon \leq \text{vol}(A_i(\alpha r, r)) + \epsilon. \tag{3.19}$$

Equations (3.14), (3.18) and (3.19) imply that in fact $\text{vol}(A_\omega(\alpha r, r)) < \infty$. Then equations (3.14), (3.17) and (3.19) imply that $\text{vol}(A_\omega(\alpha r, r)) \leq (\alpha^n - 1) c' r^n + \delta + \epsilon$. As δ and ϵ are arbitrary, we obtain that

$$\text{vol}(A_\omega(\alpha r, r)) \leq (\alpha^n - 1) c' r^n. \tag{3.20}$$

From (3.7), the lower curvature bound and the Bishop-Gromov inequality [7, Lemma 5.3.bis], for large i we obtain a lower bound on $\text{vol}(B_\epsilon(x_i))$ in terms of ϵ, α, r and c . Using the $C^{1,\sigma}$ metric convergence, we obtain a lower bound on $\text{vol}(B_\epsilon(x_\omega))$ in terms of ϵ, α, r and c . We then obtain an upper bound on the number of elements in a maximal 2ϵ -separated net in $A_\omega(\alpha r, r)$. As the 4ϵ -balls with centers at the netpoints cover $A_\omega(\alpha r, r)$, it follows that $A_\omega(\alpha r, r)$ is compact. Then $A_\omega(\alpha r, r)$ is the Gromov-Hausdorff limit of a subsequence of $\{A_i(\alpha r, r)\}_{i=1}^\infty$. It follows from the $C^{1,\sigma}$ metric convergence that

$$\text{vol}(A_\omega(\alpha r, r)) = \lim_{i \rightarrow \infty} \text{vol}(A_i(\alpha r, r)) = (\alpha^n - 1) c' r^n. \tag{3.21}$$

Equation (3.16) follows from (3.7) and the $C^{1,\sigma}$ metric convergence. □

Hence $\text{vol}(B_r(\star_\omega)) = c' r^n$. As $A_\omega(\alpha r, r)$ is compact, we can now use the analysis of manifolds that are flat outside of a compact set, as given in [2]. For simplicity suppose that $X_\omega - \star_\omega$ is connected; the general case is similar. Suppose that $n > 2$. From [2], the complement of some bounded set in X_ω is isometric to the complement of a bounded set in \mathbb{R}^n/F , for some finite group $F \subset O(n)$ that acts freely on S^{n-1} . For r_0 large, we identify $S_{r_0}(\star_\omega)$ with a hypersurface in \mathbb{R}^n/F . Then for $r < r_0$, $S_r(\star_\omega)$ is the result of (possibly) making identifications on the equidistant set with signed distance $r - r_0$ from $S_{r_0}(\star_\omega)$. We know that

$$\text{Area}(S_r(\star_\omega)) = n c' r^{n-1}. \tag{3.22}$$

As this is analytic in r , it follows that there are in fact no identifications made, and $S_{r_0}(\star_\omega)$ is convex when lifted to \mathbb{R}^n . If r_0 is large enough, we may assume that $S_{r_0}(\star_\omega)$ is C^1 -smooth with measurable principal curvature functions $\{h_j\}_{j=1}^{n-1}$. For r near r_0 , the tube formula gives

$$n c' r^{n-1} = \int_{S_{r_0}(\star_\omega)} \prod_{j=1}^{n-1} (1 + h_j(r - r_0)) \, d\text{vol}. \tag{3.23}$$

By analyticity, (3.23) is true for all r . As in the proof of the Bishop-Gromov inequality, for $r \in (0, r_0)$,

$$\text{Area}(S_r(\star_\omega)) = \int_{S_{r_0}(\star_\omega)} \chi_r \prod_{j=1}^{n-1} (1 + h_j(r - r_0)) \, d\text{vol}, \tag{3.24}$$

where χ_r is the characteristic function of the set of points on $S_{r_0}(\star_\omega)$ whose normal rays are distance-minimizing down to $S_r(\star_\omega)$. It follows from (3.22), (3.23) and (3.24) that for all $r \in (0, r_0)$, $\chi_r = 1$ and $1 + h_j(r - r_0) > 0$ for all j . Equation (3.23), for small r , now implies that for all j , $h_j = \frac{1}{r_0}$. Then for all $r > 0$, $S_r(\star_\omega)$ can be identified with the sphere of distance r from the vertex of \mathbb{R}^n/F . Hence $X_\omega - \star_\omega$ is a cone over a spherical space form. If $n = 2$ then one can apply a similar argument, using the results of [2] in this case.

Hence X_ω is a cone over a finite union Y of spherical space forms with total volume $n c'$. Let $C_i \subset X_i$ be as above. Choose $c_i \in C_i \cap S_1(\star_i)$. Let $c_\omega \in X_\omega$ be the point represented by $\{c_i\}$. Consider the connected component \mathcal{C} of $X_\omega - \star_\omega$ which contains c_ω . Define $v_\omega \in C^0(\mathcal{C})$ by $v_\omega(x_\omega) = d_{X_\omega}(x_\omega, \star_\omega)$.

Consider the closed annulus $A = \overline{B_4(\star_\omega)} - B_{\frac{1}{4}}(\star_\omega)$ in \mathcal{C} . It is compact. Given $\epsilon \in (0, \frac{1}{100})$, choose a finite ϵ -net $\mathcal{N} = \{a_{\omega,j}\}_{j=1}^J$ in A , with $a_{\omega,1} = c_\omega$. For each j , choose a sequence $\{a_{i,j}\}$ which represents $a_{\omega,j}$, with $a_{i,j} \in X_i$ and $a_{i,1} = c_i$. As in the proof of Lemma 2, we may assume that $a_{i,j} \in \overline{B_{4+\epsilon}(\star_i)} - B_{\frac{1}{4}-\epsilon}(\star_i)$. By the definition of d_{X_ω} , there is a subset $S_0 \subset \mathbb{Z}^+$ of full ω -measure such that if $i \in S_0$ then for all $j \in \{1, \dots, J\}$,

$$|d_{X_i}(a_{i,j}, \star_i) - d_{X_\omega}(a_{\omega,j}, \star_\omega)| < \epsilon. \tag{3.25}$$

Consider the closed subsets $\left\{ \bigcup_{j=1}^J \overline{B_{1/8}(a_{i,j})} \right\}_{i \in S_0}$ of $\{X_i\}_{i \in S_0}$. From (3.9), they form a precompact set in the multipointed Gromov-Hausdorff topology, where the multibasepoint of $\bigcup_{j=1}^J \overline{B_{1/8}(a_{i,j})}$ is the ordered set $\{a_{i,j}\}_{j=1}^J$ and by ‘‘multipointed Gromov-Hausdorff topology’’ we mean the analog of the pointed Gromov-Hausdorff topology, in which all of the maps in the definitions respect the multibasepoints. Put

$$F = \bigcup_{j=1}^J \overline{B_{1/8}(a_{\omega,j})} \subset \overline{B_{10}(\star_\omega)} - B_{\frac{1}{10}}(\star_\omega) \subset \mathcal{C}. \tag{3.26}$$

As in the proof of Sublemma 1, F is a limit point of $\left\{ \bigcup_{j=1}^J \overline{B_{1/8}(a_{i,j})} \right\}_{i \in S_0}$ in the multipointed Gromov-Hausdorff topology. Then there is a subsequence of $\left\{ \bigcup_{j=1}^J \overline{B_{1/8}(a_{i,j})} \right\}_{i \in S_0}$ which converges in the multipointed $C^{1,\sigma}$ -topology to F . In particular, there is an infinite subset $S_1 \subset S_0$ such that if $i \in S_1$ then there is a $C^{2,\sigma}$ -regular diffeomorphism $\pi_i : \bigcup_{j=1}^J \overline{B_{1/8}(a_{i,j})} \rightarrow F$ with $\pi_i(a_{i,j}) = a_{\omega,j}$.

For $i \in S_1$, let g_i denote the corresponding Riemannian metric on F , pulled back from X_i via π_i^{-1} . Then $\{g_i\}_{i \in S_1}$ converges to the $g_\omega|_F$ in the $C^{1,\sigma}$ -topology. Taking $\epsilon \in \frac{\mathbb{Z}^+}{100}$ and doing a diagonal argument, we obtain a sequence parametrized by $k \in \mathbb{Z}^+$ of

1. $\frac{1}{100k}$ -nets $\mathcal{N}_k = \{a_{\omega,j}\}_{j=1}^{J_k}$ in A ,
2. Sets $F_k = \bigcup_{j=1}^{J_k} \overline{B_{1/8}(a_{\omega,j})}$ and
3. $C^{2,\sigma}$ -regular diffeomorphisms $\pi_k : \bigcup_{j=1}^{J_k} \overline{B_{1/8}(a_{i_k,j})} \rightarrow F_k$ with $\pi_k(a_{i_k,j}) = a_{\omega,j}$

such that

4. $\lim_{k \rightarrow \infty} g_k|_A = g_\omega|_A$ in the $C^{1,\sigma}$ -topology and
- 5.

$$\sup_{y \in \mathcal{N}_k} |v_k(y) - v_\omega(y)| < \frac{1}{k}, \tag{3.27}$$

where $v_k = \mu_{i_k} \circ \pi_k^{-1} \in C^0(F_k)$. Here g_k is the pullback of the Riemannian metric from X_{i_k} , using π_k^{-1} . By the Arzela-Ascoli theorem, it follows from 4. that there is a subsequence of $\{v_k|_A\}_{k=1}^\infty$ which converges in the Lipschitz topology. Relabelling this subsequence as $\{v_k|_A\}_{k=1}^\infty$, it follows from (3.27) that $\lim_{k \rightarrow \infty} v_k|_A = v_\omega|_A$. For large k , we will identify C_{i_k} with the connected component of $v_k^{-1}([\frac{1}{2}, 2]) \subset A$ containing c_ω .

Let r be the coordinate on \mathcal{C} given by the distance from \star_ω and let $Z = -\frac{d}{dr}$ be the corresponding (smooth) vector field on \mathcal{C} . Clearly Z is transversal to v_ω in the sense of [5]. Then for large k , $Z|_A$ is transversal to $v_k|_A$. By flowing along Z from $v_k^{-1}(2)$ to $v_k^{-1}(\frac{1}{2})$ and using the arguments of [5], it follows that the map $v_k : v_k^{-1}([\frac{1}{2}, 2]) \rightarrow [\frac{1}{2}, 2]$ defines a topological fiber bundle. By further flowing along Z down to $S_{1/100}(\star_\omega) \subset \mathcal{C}$, it follows that the fiber of the bundle is homeomorphic to a connected component of Y . Then C_{i_k} is the total space of this fiber bundle, which contradicts the construction of $\{C_i\}_{i=1}^\infty$.

For given c and c' , Y has volume $n c'$ and $\text{cone}(Y)$ satisfies the lower local volume bound (3.16). It follows that there is an upper bound in terms of c and c' on the number of components of Y , and a finite number of possible diffeomorphism types for each component. □

4. Proof of Theorem 2

The underlying basis for the result is the fact that for $\beta \in (0, 1)$, there is a flat 2-dimensional cone surface with one cone point, of total angle $2\pi(1 + \beta)$, and one open end, with cone angle $2\pi(1 - \beta)$. Because of this fact, it is plausible that one can construct a sequence of surfaces as in the statement of the theorem with the property that when one takes an ultralimit as $\epsilon \rightarrow 0$, one obtains this flat cone surface.

This suggests constructing the surface of the theorem to have a self-similar structure of the form

$$M = D^2 \cup_{S^1} P \cup_{S^1} (C \cdot P) \cup_{S^1} \dots \cup_{S^1} (C^k \cdot P) \cup_{S^1} \dots \tag{4.1}$$

Here P , the basic building block, is the gluing $N_1 \cup_{S^1} N_2$ of two compact surfaces-with-boundary N_1 and N_2 along a circle. The surface N_1 will be the above cone surface truncated both near the cone point and near infinity. Topologically N_1 will be a torus with two balls removed, equipped with a flat metric. Then the surface N_2 will be an annulus that attaches N_1 and a rescaled version $C \cdot N_1$, for an appropriate constant C .

To write this in detail, let T^2 denote the 2-torus equipped with an arbitrary but fixed complex structure with local complex coordinate z , and flat Riemannian metric $|dz|^2$. Let f be a meromorphic function on T^2 with one zero, at $p_0 \in T^2$, and one pole, at $p_\infty \in T^2$. Fix $\beta \in (0, 1)$ and put $g = |f(z)|^{2\beta} |dz|^2$, a Riemannian metric on $T^2 - \{p_0, p_\infty\}$. In general, a metric $e^{2\phi} |dz|^2$ has Gaussian curvature $-e^{-2\phi} (\partial_x^2 + \partial_y^2)\phi$. As $\ln |f|$ is harmonic, it follows that g is flat. As a metric on T^2 , it has a cone point at p_0 with total angle $2\pi(1 + \beta)$ (i.e. angle excess $2\pi\beta$) and an open cone near p_∞ with cone angle $2\pi(1 - \beta)$. The end of $T^2 - \{p_0, p_\infty\}$ approaching p_0 has a neighborhood U_0 with the metric $ds^2 + (1 + \beta)^2 s^2 d\theta^2$ for $s \in (0, \delta_0)$, and the end of $T^2 - \{p_0, p_\infty\}$ approaching p_∞ has a neighborhood U_∞ with the metric $dt^2 + (1 - \beta)^2 t^2 d\theta^2$ for $t \in (\delta_\infty, \infty)$. We take $\delta_\infty > 1$. Put $N_1 = (T^2 - \{p_0, p_\infty\}) - U_0 - U_\infty$. It is a compact surface-with-boundary whose boundary circles $\partial_0(N_1)$ and $\partial_\infty(N_1)$ have lengths $2\pi(1 + \beta)\delta_0$ and $2\pi(1 - \beta)\delta_\infty$, respectively. If C is a positive constant, we denote by $C \cdot N_1$ the Riemannian manifold obtained by rescaling the Riemannian metric on N_1 by C^2 , i.e. multiplying the lengths by C .

For ϵ a small positive number, we now wish to construct a metric

$$ds^2 = dr^2 + f^2(r) d\theta^2 \tag{4.2}$$

on an annulus $N_2 = [0, R] \times S^1$ with

$$f(r) = c_1 (1 + r)^{-\epsilon} + c_2 (1 + r)^{1+\epsilon}, \tag{4.3}$$

so that N_1 glues isometrically to N_2 to first order, with $\{0\} \times S^1$ gluing to $\partial_\infty(N_1)$, and N_2 glues isometrically to $C \cdot N_1$ to first order, with $\{R\} \times S^1$ gluing to $C \cdot \partial_0(N_1)$, for some $C > 1$. These conditions become

$$\begin{aligned} c_1 + c_2 &= (1 - \beta) \delta_\infty, \\ -\epsilon c_1 + (1 + \epsilon) c_2 &= 1 - \beta, \\ c_1 (1 + R)^{-\epsilon} + c_2 (1 + R)^{1+\epsilon} &= C (1 + \beta) \delta_0, \\ -\epsilon c_1 (1 + R)^{-\epsilon-1} + (1 + \epsilon) c_2 (1 + R)^\epsilon &= 1 + \beta. \end{aligned} \tag{4.4}$$

(The third equation in (4.4) says that the sizes of the circles $\{R\} \times S^1$ and $C \cdot \partial_0(N_1)$ are the same, while the fourth equation in (4.4) says that the cone angles along the circles are the same.) The solution to the first two equations in (4.4) is

$$\begin{aligned} c_1 &= \frac{1 - \beta}{1 + 2\epsilon} ((1 + \epsilon) \delta_\infty - 1), \\ c_2 &= \frac{1 - \beta}{1 + 2\epsilon} (\epsilon \delta_\infty + 1). \end{aligned} \tag{4.5}$$

For small ϵ and large R , the dominant term on the left-hand-side of the last equation in (4.4) is $(1 + \epsilon) c_2 (1 + R)^\epsilon$. Hence for small ϵ , there is a solution for R with the asymptotics

$$R \sim \left(\frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{\epsilon}}. \tag{4.6}$$

Substituting into the third equation of (4.4) gives

$$C \sim \delta_0^{-1} \left(\frac{1 + \beta}{1 - \beta} \right)^{\frac{1}{\epsilon}}. \tag{4.7}$$

Put $P = N_1 \cup_{S^1} N_2$, where the gluing identifies $\partial_\infty N_1$ with $\{0\} \times S^1 \subset N_2$. Then P has a C^1 -smooth Riemannian metric which is flat on N_1 and has curvature $-\frac{f''}{f} = -\frac{\epsilon(1+\epsilon)}{(1+r)^2}$ on N_2 . By smoothing the metric on P and slightly moving the boundary curve between N_1 and N_2 into N_2 , we can construct a Riemannian metric on P which is flat on N_1 , which satisfies $|K| \leq \frac{2\epsilon(1+\epsilon)}{(1+r)^2}$ on N_2 and for which P glues isometrically onto $C \cdot P$ by identifying $\{R\} \times S^1 \subset P$ with $C \cdot \partial_0 N_1 \subset C \cdot P$. Let D^2 be a 2-disk which caps P at $\partial_0 N_1$. Put

$$M = D^2 \cup_{S^1} P \cup_{S^1} (C \cdot P) \cup_{S^1} \dots \cup_{S^1} (C^k \cdot P) \cup_{S^1} \dots, \tag{4.8}$$

with basepoint $\star \in D^2$. There is an obvious Riemannian metric on $M - D^2$, which we extend over M . We claim that this Riemannian metric satisfies the conditions of the theorem. First, M has infinite topological type. By the self-similar nature of the Riemannian metric, equations (1.6) and (1.7) are satisfied for some $c, c'_1, c'_2 > 0$. In order to check (1.8) on $(C^k \cdot P) \subset M$, we can use the scale invariance to instead check it on the subset P of

$$C^{-k} \cdot M = (C^{-k} \cdot D^2) \cup_{S^1} (C^{-k} \cdot P) \cup_{S^1} (C^{-k+1} \cdot P) \cup_{S^1} \dots \cup_{S^1} P \cup_{S^1} \dots \tag{4.9}$$

As the metric is flat on $N_1 \subset P$, it is enough to just consider a point $m \in N_2$, say with coordinates $(r, \theta) \in [0, R] \times S^1$. Put

$$a_1 = \max_{z_1 \in \partial_0 N_1, z_2 \in \partial_\infty N_1} d(z_1, z_2) \tag{4.10}$$

and

$$a_2 = \max_{z \in \partial D^2} d(\star, z). \tag{4.11}$$

Then we can construct a path from \star to m with length at most

$$C^{-k} a_2 + C^{-k}(a_1 + R) + \dots + C^{-1}(a_1 + R) + a_1 + r. \tag{4.12}$$

Thus

$$d(m, \star) \leq a_2 + \frac{a_1 + R}{C - 1} + a_1 + r \leq r + \text{const.}, \tag{4.13}$$

where const. is independent of ϵ . It follows that

$$|K(m)| \cdot d(m, \star)^2 \leq 2\epsilon(1 + \epsilon) \left(\frac{r + \text{const.}}{r + 1} \right)^2, \tag{4.14}$$

which proves the theorem.

Remark. It should be fairly clear that by using building blocks consisting of appropriate (rescaled) flat metrics on $T^2 - (D^2 \cup D^2)$, $S^2 - (D^2 \cup D^2 \cup D^2)$ and $\mathbb{R}P^2 - (D^2 \cup D^2)$, along with the classification of surfaces in [15], we can construct a complete Riemannian metric on *any* connected surface so as to satisfy (1.6), (1.7) and (1.8) for any $\epsilon > 0$ and for some c, c'_1 and c'_2 .

5. Proof of Theorem 3

We follow the method of proof of Theorem 1, which is a proof by contradiction. Hence we obtain a pointed length space (X_ω, \star_ω) along with a flat n -dimensional Riemannian metric on $X_\omega - \star_\omega$. By using appropriate rescalings in the construction of X_ω , we obtain the analog of equations (1.9) and (1.10) for X_ω , but with $C' = 0$. That is, the distance function $d_{X_\omega}(\cdot, \star_\omega)$ is convex on X_ω and for any two normalized minimizing geodesics $\gamma_1, \gamma_2 : [0, b] \rightarrow X_\omega$ with $\gamma_1(0) = \gamma_2(0) = \star_\omega$ and any $t \in [0, 1]$,

$$d_{X_\omega}(\gamma_1(tb), \gamma_2(tb)) \leq t d_{X_\omega}(\gamma_1(b), \gamma_2(b)) \tag{5.1}$$

Let $c_\omega \in X_\omega$ and $\mathcal{C} \subset X_\omega - \star_\omega$ be as in the proof of Theorem 1. Let $\tilde{\mathcal{C}}$ denote the universal cover of \mathcal{C} , defined with the basepoint c_ω , with projection $\pi : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$. As \mathcal{C} is flat, there is a developing map $D : \tilde{\mathcal{C}} \rightarrow \mathbb{R}^n$ and a homomorphism $\pi_1(\mathcal{C}, c_\omega) \rightarrow \text{Isom}(\mathbb{R}^n)$ with respect to which D is equivariant.

From the convexity of $d(\cdot, \star_\omega)$, for any $r > 0$ the ball $B_r(\star_\omega)$ is geodesically convex in \mathcal{C} . Then $S_r(\star_\omega)$ is locally convex in the sense that for each $x_\omega \in S_r(\star_\omega)$, there is a neighborhood of x_ω in $S_r(\star_\omega)$ which is contained in the boundary of a convex set. Given $\tilde{x}_\omega \in \pi^{-1}(S_r(\star_\omega))$, using a local isometry between a neighborhood of \tilde{x}_ω and a neighborhood of $\pi(\tilde{x}_\omega)$, it follows that there is a neighborhood of \tilde{x}_ω in $\pi^{-1}(S_r(\star_\omega))$ which is contained in the boundary of a convex set. That is, $\pi^{-1}(S_r(\star_\omega))$ is locally convex. From [8], for each $r > 0$,

1. $\pi^{-1}(S_r(\star_\omega))$ is embedded by D as the boundary of a convex subset of \mathbb{R}^n , or
2. $\pi^{-1}(S_r(\star_\omega))$ is isometric to $S^1 \times \mathbb{R}^{n-2}$ and D is the product $\alpha \times \text{Id}_{\mathbb{R}^{n-2}}$ of an

immersed convex curve $\alpha : S^1 \rightarrow \mathbb{R}^2$ with the identity map on \mathbb{R}^{n-2} , or
 3. $\pi^{-1}(S_r(\star_\omega))$ is isometric to $\mathbb{R} \times \mathbb{R}^{n-2}$ and D is the product $\alpha \times \text{Id}_{\mathbb{R}^{n-2}}$ of an immersed convex curve $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ with the identity map on \mathbb{R}^{n-2} .

Suppose first that for each $r > 0$, D embeds $\pi^{-1}(S_r(\star_\omega))$ into \mathbb{R}^n as the boundary of a convex subset. Then D is an embedding of $\tilde{\mathcal{C}}$ into \mathbb{R}^n . Identifying $\tilde{\mathcal{C}}$ with its image under D , convexity implies that $\tilde{\mathcal{C}}$ is the complement of a closed convex subset $Z \subset \mathbb{R}^n$. Letting \mathbb{R}^n/Z denote the collapsing of Z to a point, there is a continuous map $\mathbb{R}^n \rightarrow \mathbb{R}^n/Z \rightarrow \bar{\mathcal{C}}$ which sends Z to \star_ω . Now Z is invariant under the isometric action of $\pi_1(\mathcal{C}, c_\omega)$ on \mathbb{R}^n . Given $x_\omega \in \mathcal{C}_\omega$ and a lift $\tilde{x}_\omega \in \pi^{-1}(x_\omega)$, the convexity of Z implies that there is a unique minimizing geodesic from x_ω to \star_ω , which coincides with the projection of the minimizing segment from \tilde{x}_ω to Z .

Suppose that Z contains more than one point. Then we can find two distinct points $\{z_i\}_{i=1,2}$ in ∂Z and support planes H_i containing z_i so that the normalized rays $\{\tilde{\gamma}_i\}_{i=1,2}$ from z_i orthogonal to H_i , which point away from Z , have the property that $\tilde{\gamma}_1$ eventually lies on the same side of H_2 as $\tilde{\gamma}_2$, and $\tilde{\gamma}_2$ eventually lies on the same side of H_1 as $\tilde{\gamma}_1$. Put $\gamma_i = \pi \circ \tilde{\gamma}_i$. For t sufficiently small, we will have $d_{X_\omega}(\gamma_1(t), \gamma_2(t)) = 2t$, as the shortest way to get from $\tilde{\gamma}_1(t)$ to $\tilde{\gamma}_2(t)$ in \mathbb{R}^n/Z will be to follow $\tilde{\gamma}_1$ from $\tilde{\gamma}_1(t)$ to z_1 and then follow $\tilde{\gamma}_2$ from z_2 to $\tilde{\gamma}_2(t)$. (Note that Z gets collapsed to \star_ω .) Then from (5.1), it follows that $d_{X_\omega}(\gamma_1(t), \gamma_2(t)) = 2t$ for all $t > 0$. Thus $d(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)) = 2t$ for all $t > 0$, where the distance is measured in the length metric on \mathbb{R}^n/Z , which is a contradiction to the construction of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$.

Thus Z is a point, which we can assume without loss of generality to be the origin in \mathbb{R}^n . Then $\pi_1(\mathcal{C}, c_\omega)$ acts on $\mathbb{R}^n - \{0\}$ by elements of $O(n)$ and \mathcal{C} is a cone over a spherical space form. The rest of the proof proceeds as in the proof of Theorem 1.

Now suppose that for some $r_0 > 0$, D immerses $\pi^{-1}(S_{r_0}(\star_\omega))$ as $\alpha_{r_0} \times \text{Id}_{\mathbb{R}^{n-2}}$, where α_{r_0} is an immersed convex curve $\alpha_{r_0} : S^1 \rightarrow \mathbb{R}^2$. Then for all $r > 0$, D immerses $\pi^{-1}(S_r(\star_\omega))$ as $\alpha_r \times \text{Id}_{\mathbb{R}^{n-2}}$, where α_r is an immersed convex curve $\alpha_r : S^1 \rightarrow \mathbb{R}^2$ which is the curve of distance $r - r_0$ from α_{r_0} . (Recall that D is a local isometry.) It follows that $\tilde{\mathcal{C}} = (0, \infty) \times S^1 \times \mathbb{R}^{n-2}$, which contradicts the fact that $\tilde{\mathcal{C}}$ is simply-connected.

Finally, suppose that for some $r_0 > 0$, D immerses $\pi^{-1}(S_{r_0}(\star_\omega))$ as $\alpha_{r_0} \times \text{Id}_{\mathbb{R}^{n-2}}$, where α_{r_0} is an immersed convex curve $\alpha_{r_0} : \mathbb{R} \rightarrow \mathbb{R}^2$. Then for all $r > 0$, D immerses $\pi^{-1}(S_r(\star_\omega))$ as $\alpha_r \times \text{Id}_{\mathbb{R}^{n-2}}$, where α_r is an immersed convex curve $\alpha_r : \mathbb{R} \rightarrow \mathbb{R}^2$ which is the curve of distance $r - r_0$ from α_{r_0} . In particular, $\tilde{\mathcal{C}}$ splits isometrically as a product $\mathcal{A} \times \mathbb{R}^{n-2}$, where \mathcal{A} is diffeomorphic to $(0, \infty) \times \mathbb{R}$, with $\tilde{\mathcal{C}}$ having the flat metric which pulls back from D . Put $\bar{\mathcal{A}} = ([0, \infty) \times \mathbb{R})/(\{0\} \times \mathbb{R})$, the union of \mathcal{A} with a point. Similarly, put $\bar{\tilde{\mathcal{C}}} = ([0, \infty) \times \mathbb{R} \times \mathbb{R}^{n-2})/(\{0\} \times \mathbb{R} \times \mathbb{R}^{n-2})$, the union of $\tilde{\mathcal{C}}$ with a point. There is a continuous map from $\bar{\tilde{\mathcal{C}}}$ to $\bar{\mathcal{C}}$ which restricts to the covering map on

$\widetilde{\mathcal{C}}$, and an obvious embedding $\overline{\mathcal{A}} \rightarrow \widetilde{\mathcal{C}}$. Let $\widetilde{\gamma} : [0, \infty) \rightarrow \overline{\mathcal{A}}$ be a normalized ray. Choose distinct points $b_1, b_2 \in \mathbb{R}^{n-2}$. Then $(t \in \mathbb{R}^+) \rightarrow \widetilde{\gamma}(t) \times \{b_1\}$ and $(t \in \mathbb{R}^+) \rightarrow \widetilde{\gamma}(t) \times \{b_2\}$ extend to rays $r_i : [0, \infty) \rightarrow \widetilde{\mathcal{C}}$, with $r_i(0)$ being the basepoint. As before, we have $d(r_1(t), r_2(t)) = 2t$ for t small, where d is the length metric on $\widetilde{\mathcal{C}}$. Then (5.1) implies that $d(r_1(t), r_2(t)) = 2t$ for all t , which is a contradiction.

Remark. To see where the hypotheses of Theorem 3 enter into the proof, note that the method of proof is to show that $\overline{\mathcal{C}}$ is a cone over a spherical space form. If $n = 2$ then $\overline{\mathcal{C}}$ could *a priori* be a cone over \mathbb{R} , as in the example of Section 2. To see where the assumption of large-scale pointed-convexity enters, let M_i be the effect of attaching a wormhole between two points of distance $2i$ in \mathbb{R}^n . More precisely, give $[-1/2, 1/2] \times S^{n-1}$ a metric whose restrictions to $[-1/2, -1/4] \times S^{n-1}$ and $[1/4, 1/2] \times S^{n-1}$ are isometric to $\overline{B_{1/2}(0)} - B_{1/4}(0) \subset \mathbb{R}^n$. Put $M_i = (\mathbb{R}^n - B_{1/2}(p_1) - B_{1/2}(p_2)) \cup_{S^{n-1} \cup S^{n-1}} [-1/2, 1/2] \times S^{n-1}$, where $p_1, p_2 \in \mathbb{R}^n$ have distance $2i$. It is flat outside of a compact set. Put the basepoint of M_i somewhere on the component $[-1/2, 1/2] \times S^{n-1}$. Then the limit space $X_\omega = \lim_{\omega \rightarrow \infty} \frac{1}{\omega} \cdot M_i$ is the result of identifying two points in \mathbb{R}^n of distance 2, with its basepoint at the identification point. Clearly X_ω is not a cone. Without the assumption of large-scale pointed-convexity, or some such assumption, it could *a priori* arise in a rescaling limit as in the proof of Theorem 3. One can find similar examples with $X_\omega = \mathbb{R}^n / K$, where K is any closed subset of \mathbb{R}^n . The large-scale pointed-convexity assumption is used to show first that K is convex and then to show that K is a point.

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