Automorphic Galois representations and Langlands correspondences

III. An idiosyncratic survey of open problems

Michael Harris

Bowen Lectures, Berkeley, February 2017

Outline

Two-dimensional Galois representations

Theorem (Allen-Calegari-Caraiani-Gee-Helm-Le Hung-Newton-Scholze-Taylor-Thorne)

Let F be a CM field and let $R = \{r_\lambda\}$ be a weakly compatible system of 2-dimensional ℓ-adic reps with Hodge-Tate numbers $\{0, 1\}$. Suppose that for each finite extension F'/F the restriction $r_\lambda|_{Gal_{\overline{\mathbb{Q}}/F'}}$ is absolutely irreducible for some λ. Then $Sym^m R$ is potentially automorphic for all $m \geq 1$.

Corollary

Let E be an elliptic curve over F as in the theorem. Then E is potentially automorphic and satisfies the Sato-Tate conjecture.

Sato-Tate conjecture explained, briefly

The **Sato-Tate Conjecture** (early 1960s) is the statement, now proved in many cases, that the number of points $E(\mathbb{F}_v)$ over finite residue fields of F, as v varies, is *equidistributed*. In other words, it is as random as possible, given geometric constraints.

It has analogues for general smooth projective algebraic varieties (due to Serre) which can best be understood in the framework of the conjectures on algebraic cycles (Hodge and Tate conjectures).

$G^\vee$-local systems

Now X is a curve over the finite field k.

Theorem (Böckle, MH, Khare, Thorne)

Let G be a split semisimple group over X and let $\rho : \pi_1(X) \to G^\vee(\overline{\mathbb{Q}}_\ell)$ (étale fundamental group) be a representation with Zariski dense image (and a few other conditions).
Then ρ is potentially automorphic.

Recall that this means that there are infinitely many Galois covers X'/X such that $\rho_{X'} = \rho|_{\pi_1(X')}$ is a parameter attached by V. Lafforgue's construction to a cuspidal automorphic representation of G.

Extensions

The assertion remains true if X is replaced by $X \setminus S$ for $S \subset X(\bar{k})$ finite, provided the monodromy at $s \in S$ is finite.

We have not considered the case of general tame monodromy, nor extensions to non-split groups or general reductive groups, but we expect these to be relatively straightforward.

Write

$$S(G,X)(D) = G(k(X)) \backslash \prod_x{}'(G(k_x((T)))/U(D).$$

Automorphic forms

Consider the space of cusp forms

$$L_2^0(S(G,X)(D)) \subset L_2(S(G,X)(D))$$

Lafforgue parametrization:

$$L_2^0(S(G,X)(D)) = \oplus_\sigma L_2^0(S(G,X)(D))_\sigma$$

where

$$\sigma = \{\sigma_\ell : \pi_1(X \setminus |D|) \to G^\vee(\overline{\mathbb{Q}}_\ell), \ell \neq p\}$$

Ramification at $x \in |D|$ limited by multiplicity of $x \in D$.

Hecke and excursion algebras

The space $L_2^0(S(G,X)(D)$ is finite-dimensional and has a canonical $\mathbb{Z}_\ell$-structure $H_\ell(G,X)(D)$.
Two finite rank $\mathbb{Z}_\ell$-algebras act on $H_\ell(G,X)(D)$:

$$T(D) \subset \mathcal{B}(D) \subset End(H_\ell(G,X)(D))$$

Points of $S(G,X)(D) \leftrightarrow$ principal G-bundles on X over k.
$T(D)$ is the *Hecke algebra* generated by modifications of a G-bundle at points not in D.
$\mathcal{B}(D)$ is the *excursion algebra* generated by viewing G-bundles as G-shtukas in various ways.
The Langlands parameters σ are indexed by homomorphisms

$$\nu : \mathcal{B}(D) \to \overline{\mathbb{Q}}_\ell.$$

Deformation rings

The parameter $\sigma_{\nu,\ell}$ takes values in $G^\vee(\mathcal{O})$ for an ℓ-adic integer ring $\mathcal{O}$with residue field $k_\mathcal{O}$.
Let $\bar{\sigma}_{\nu,\ell} : \pi_1(X \setminus |D|) \to G^\vee(k_\mathcal{O})$ reduction modulo maximal ideal of $\mathcal{O}$.
Assume *image*$(\bar{\sigma}_{\nu,\ell})$ is *abundant*. (For example, $\bar{\sigma}_{\nu,\ell}$ is surjective.) .
Then we can define the deformation ring $R_{\bar{\sigma}_{\nu,\ell}}$, a CNL (complete noetherian local) $\mathcal{O}$-algebra.
If A is a CNL $\mathcal{O}$-algebra with residue field $k_\mathcal{O}$, then

$$Hom_\mathcal{O}(R_{\bar{\sigma}_{\nu,\ell}}, A) \leftrightarrow \{ \text{ deformations of } \bar{\sigma}_{\nu,\ell} \text{ to } A\}.$$

Taylor-Wiles argument

If $\sigma = \sigma_{\nu,\ell}$, complete $\mathcal{B}(D)$ at $\mathfrak{m}_\nu := \ker \nu$.

Proposition

There is a surjective homomorphism

$$R_{\bar{\sigma}_{\nu,\ell}} \to \mathcal{B}(D)_{\mathfrak{m}_\nu}.$$

This requires generalizing an argument of Carayol-Serre to pseudocharacters.
The Taylor-Wiles type theorem is then

Theorem (BHKT)

Suppose $|D| = \emptyset$. Under reasonable hypotheses on image$(\bar{\sigma}_{\nu,\ell})$, the above homomorphism is an isomorphism.

Consequence of Taylor-Wiles argument

The TW method was introduced in order to derive:

Corollary

Let $\mathcal{O}$ be an ℓ-adic integer ring and let $\sigma : \pi_1(X) \to G^\vee(\mathcal{O})$ be a continuous homomorphism. Suppose

$$\bar{\sigma} : \pi_1(X) \to G^\vee(\mathcal{O}/\mathfrak{m}_\mathcal{O})$$

is isomorphic to $\bar{\sigma}_{\nu,\ell}$ for some ν.
Then (under reasonable hypotheses on $image(\bar{\sigma}_{\nu,\ell}) = image(\bar{\sigma})$)

$$\sigma \simeq \sigma_{\nu',\ell} \text{ for some } \nu'.$$

Potential automorphy

Starting with $\sigma : \pi_1(X) \to G^\vee(\mathcal{O})$,

1. Find a prime ℓ', an ℓ'-adic integer ring $\mathcal{O}'$, and a $\rho' : \pi_1(X) \to G^\vee(\mathcal{O}')$ that is *universally automorphic* with $G^\vee$-abundant reduction mod ℓ'. By *universally automorphic* we mean that ρ' remains automorphic after any base change.
2. Find Galois coverings X'/X as above and a compatible family σ of λ-adic representations of $\pi_1(X')$ with $\bar{\sigma}_\ell = \bar{\sigma}_{\nu,\ell}|_{\pi_1(X')}$ and $\bar{\sigma}_{\ell'} = \bar{\rho}'$.
3. (The last step uses an approximation argument of Moret-Bailly.)
4. Apply the TW argument to show that $\sigma_{\ell'}$ is automorphic, hence that $\bar{\sigma}_\ell|_{\pi_1(X')}$ has an automorphic lift.
5. Apply the TW argument again to show that $\sigma|_{\pi_1(X')}$ is automorphic.

Steps to potential automorphy

The method of potential automorphy was invented by Taylor for Hilbert modular forms extending an argument invented by Wiles.

Generalization to higher dimensions (MH, Shepherd-Barron, Taylor) using the Dwork family of CY hypersurfaces (to prove Sato-Tate Conjecture for elliptic curves over $\mathbb{Q}$) is now a staple in applications to number fields, including most recently the ACCGHLNSTT Theorem.

The paper [BHKT] develops the method over function fields.

1. Universally automorphic Galois parameters exist with appropriate properties.
2. Compatible families σ can be constructed as required.

Construction of universally automorphic parameters (p. 70)

Let L be a field. Call a continuous homomorphism $\varphi : \pi_1(X) \to \widehat{G}(L)$ a **Coxeter parameter** for the cyclic cover Y/X if it satisfies the following conditions:

1. The image $\varphi(\pi_1(Y))$ lies in $\widehat{T}(L)$ and has r-power order for some prime $r > h(G)$.
2. There exists $t \in \varphi(\pi_1(Y))$ such that $Z_{G^\vee}(t) = \widehat{T}$ and for all distinct roots $\alpha, \beta \in \Phi(G^\vee, \widehat{T})$, $\alpha(t) \neq \beta(t)$.
3. The image of $\varphi(Gal(Y/X))$ in $W = N_{\widehat{G}}(\widehat{T})/\widehat{T}$ is generated by a Coxeter element.

Theorem (Braverman-Gaitsgory)

Coxeter parameters exist (for appropriate Y) and are universally automorphic.

As the authors' names suggest, the proof is by the geometric Langlands correspondence.

Construction of compatible families

Very briefly:

- Using a theorem of Calegari one constructs a *surjective* $\bar{\rho} : \pi_1(X \setminus |D|) \to G^\vee(\mathcal{O}/m_{\mathcal{O}})$.
- By the method of Khare-Wintenberger, one lifts $\bar{\rho}$ to a $\tilde{\rho} : \pi_1(X \setminus |D|) \to G^\vee(\mathcal{O})$ that is also surjective provided ℓ is sufficiently large.
- Then this $\tilde{\rho}$ itself gives rise to a family of geometrically connected coverings of $X \setminus |D|$.

Local Langlands for classical groups

Arthur used the stable twisted trace formula to deduce a local Langlands correspondence for symplectic and special orthogonal groups.

Based on comparison with the correspondence for $GL(n)$.

Arthur's methods were generalized to unitary groups by Mok and by Kaletha-Mínguez-Shin-White.
There remain the exceptional groups: G_2, F_4, E_6, E_7, and E_8. There is no known way to use the trace formula to deduce anything about these groups from $GL(n)$.

Why bother with exceptional groups?

Why bother with exceptional groups?

Stable base change

Let F be a local field, G a semisimple group, π an irreducible admissible representation of $G(F)$.
Let E/F be a cyclic extension (preferably of prime order).
Let $\Gamma_F = Gal(\bar{F}/F)$. Suppose $\sigma : \Gamma_F \to {}^LG$ is the Langlands parameter attached (somehow) to π.
Stable base change assigns to π an irreducible representation π_E of $G(E)$ or a finite packet of such whose corresponding Langlands parameter is $\sigma \mid_{\Gamma_E}$.
This is well understood (Arthur-Clozel) for $G = GL(n)$. The relation between π and π_E is through their *distribution characters*.

Base change and descent (p. 100)

The general theory developed by Labesse was crucial in constructing Galois representations over number fields.

Problem

Describe the image of stable base change.

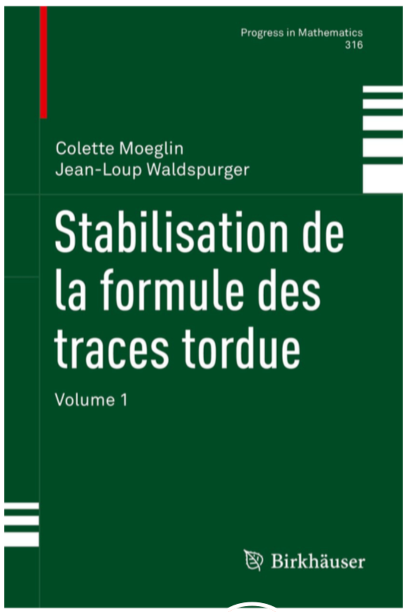
Expectation: the image consists in representations (or L-packets) fixed by $Gal(E/F)$.

If this is true, then it can be proved that the Genestier-Lafforgue local parametrization is *surjective*.

Globalization argument

- It is enough to treat *irreducible* parameters. (Definition of Serre.)
- Start with an irreducible local parameter $\sigma : \Gamma_F \to {}^L G$.
- Insert it in a global parameter $\rho : \pi_1(X) \to {}^L G$ for some curve X with F the completion of $k(X)$ at $x \in X$.
- Show that ρ becomes automorphic over a Galois covering X'/X.
- Descend in stages over a decomposition group D_x of X' over x.
- This is possible because D_x is solvable and the parameter is fixed by intermediate cyclic extensions.

A physical hurdle



1ère éd. 2016, XXVIII, 587 p.

un Birkhäuser Basel produit

C. Moeglin, J.-L. Waldspurger

Stabilisation de la formule des traces tordue

Volume 1

Series: Progress in Mathematics, Vol. 316

- **Donne une vue d'ensemble d'une méthode puissante, la formule des traces**
- **Reprend en un ouvrage unique toute une partie de la théorie jusqu'alors éparse dans la littérature**
- **Le résultat démontré est le point de départ d'applications en arithmétique et en géométrie**
- **Le livre complète les travaux de J. Arthur qui sont l'une des avancées majeures du domaine**

Ce travail en deux volumes donne la preuve de la stabilisation de la formule des trace tordue.

A physical hurdle

1ère éd. 2016, XXVIII, 587 p.

un Birkhäuser Basel produit

C. Moeglin, J.-L. Waldspurger

Stabilisation de la formule des traces tordue

Volume 1

Series: Progress in Mathematics, Vol. 316

- **Donne une vue d'ensemble d'une méthode puissante, la formule des traces**
- **Reprend en un ouvrage unique toute une partie de la théorie jusqu'alors éparse dans la littérature**
- **Le résultat démontré est le point de départ d'applications en arithmétique et en géométrie**
- **Le livre complète les travaux de J. Arthur qui sont l'une des avancées majeures du domaine**

Ce travail en deux volumes donne la preuve de la stabilisation de la formule des trace tordue.

Mathematical hurdles

In the first place, the known methods of the trace formula do not apply unchanged to fields of positive characteristic (but Bertrand Lemaire has made considerable progress).

Even when they can be applied, there are specific obstacles for exceptional groups that already arise over number fields.

In what follows, we assume stable cyclic base change can be obtained using the trace formula as in the book of Labesse.

Open questions

Question

Let π be a supercuspidal representation of $G(F)$ as above. Does there exist a finite sequence of cyclic extensions

$$F = F_0 \subset F_1 \subset \cdots \subset F_n$$

such that π_{F_n} has an Iwahori-fixed vector?

This is known for $GL(n)$ (proofs due to Henniart and Scholze) hence for classical groups by Arthur, Mok, KMSW.

Open questions

Question

Suppose π is a supercuspidal representation of $G(F)$. It its Langlands parameter non-trivial?

Again, known for $GL(n)$ and classical groups. For exceptional groups, one doesn't even know the following:

Question

Are there infinitely many supercuspidal representations of $G(F)$ with *trivial* Langlands parameter?

Open questions

Question (Discussions in progress)

Kaletha has proposed a candidate for the local correspondence for *regular tame supercuspidals*. This includes (almost) all parameters for exceptional groups if $p > 7$. (Maybe even $p > 5$.)
Does it coincide with the Genestier-Lafforgue correspondence?