Automorphic Galois representations and Langlands correspondences

II. Attaching Galois representations to automorphic forms, and vice versa: recent progress

Michael Harris

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Outline

Fontaine-Mazur Conjecture over $\mathbb{Q}$

A geometric $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(m, \overline{\mathbb{Q}}_\ell)$ gives us a collection $\{\pi_p\}$ for all prime numbers p. Fontaine's theory: π_∞ of $GL(m, \mathbb{R})$.

Definition

The representation ρ is *automorphic* if the collection $(\{\pi_p\}, \pi_\infty)$ occurs as a direct summand in the space

$$L_2([\mathbf{S}(\mathbf{m})]/\sim).$$

Conjecture (Fontaine-Mazur conjecture)

Any irreducible representation $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(m, \mathbb{Q}_\ell)$ *that is geometric is automorphic.*

Fontaine-Mazur Conjecture over general number fields

Let $E/\mathbb{Q}$ be a finite extension. Let

$$\rho : Gal(\overline{\mathbb{Q}}/E) \to GL(m, \overline{\mathbb{Q}}_\ell)$$

be a continuous irreducible representation. For every embedding $v : E \to C_v$ where C_v is either $\overline{\mathbb{Q}}_p$, $\mathbb{R}$, or $\mathbb{C}$ the local Langlands correspondence provides an irreducible representation $\pi_v(\rho)$ of $GL(m, E_v)$ where E_v is the completion of E at v.

Fontaine-Mazur Conjecture over general number fields

Conjecture (Fontaine-Mazur conjecture)

If ρ is geometric then the collection $\{\pi_v(\rho)\}$ is automorphic.

Automorphic: occurs as a direct summand in

$$L_2([\mathbf{S}(\mathbf{m},\mathbf{E})])/\sim).$$

This is actually known in most (odd) cases for $E = \mathbb{Q}$ and was proved about ten years ago (Kisin, Emerton, Khare-Wintenberger). If E is *totally real* or a *CM field* (i.e., a totally imaginary quadratic extension of a totally real field) then a good deal is known.

Adelic symmetric spaces

Starting with a direct summand of $L_2(GL(m,E)\backslash \prod_v{}' GL(m,E_v))/\sim)$, how to construct a Galois representation?

Let $GL(m,E)_\infty = \prod_{E_v=\mathbb{R},\mathbb{C}} GL(m,E_v)$, X_E the symmetric space for this Lie group. Let

$$S_{m,E} = \coprod_{\alpha} \varprojlim_{\Gamma\subset GL(m,E)} \Gamma\backslash X_E.$$

Here Γ runs over arithmetic (congruence) subgroups and α runs over a profinite index set (a class group).

This is a projective limit of manifolds.

Galois representations and cohomology

Forget functions; consider

$$H^i_!(S_{m,E},\mathbb{C}) = image[H^i_c(S_{m,E},\mathbb{C}) \to H^i(S_{m,E},\mathbb{C})].$$

Fact

For each i there is a (more or less) canonical injection

$$H^i_!(S_{m,E},\mathbb{C}) \hookrightarrow L_2([\mathbf{S}(\mathbf{m},\mathbf{E})]/\sim).$$

Consider irreducible direct factors π of the image $L_2^{coh,i}(m,E) \subset L_2([\mathbf{S}(\mathbf{m},\mathbf{E})]v/\sim)$ that are representations for $GL(m,E_v)$ for all v with $E_v \neq \mathbb{R},\mathbb{C}$.

Galois representations for totally real or CM fields

Theorem (Many people)

If E is totally real or CM, then to every such π one can associate a (necessarily) automorphic Galois representation

$$\rho_{\pi,\ell} : Gal(\overline{\mathbb{Q}}/E) \to GL(m, \overline{\mathbb{Q}}_\ell)$$

for all ℓ; and the $\rho_{\pi,\ell}$ is geometric.

This starts with the work of Eichler and Shimura in the 1950s. In that case, $S_{2,\mathbb{Q}}$ is a (projective limit) of modular curves and the Galois representation is on the points of ℓ-power order on their Jacobians.

Galois representations for totally real or CM fields

In general, one uses harmonic analysis and geometry to relate $L_2^{coh,i}(m, E)$ to cohomology of *Shimura varieties* and obtain Galois representations on their ℓ-adic étale cohomology.
One then uses methods from p-adic geometry to extend the list. The most recent result of this type: MH, Lan, Taylor, Thorne (2011-2016).

Remark

Scholze extended and simplified the methods of [HLTT] and obtained a much stronger result: for cohomology $H^i_!(S_{m,E}, \mathbb{Z})$, including torsion classes.

Other groups

For a general connected reductive group G/E can define an adelic symmetric space $S_{G,E}$ and spaces $L_2^{coh,i}(G,E)$ of cohomological automorphic forms.
To a $\pi \subset L_2^{coh,i}(G,E)$ the Langlands reciprocity conjecture assigns a family of *Langlands parameters*

$$\rho_{\pi,\ell} : Gal(\overline{\mathbb{Q}}/E) \to {}^{C}G(\overline{\mathbb{Q}}_\ell) \sim {}^{L}G(\overline{\mathbb{Q}}_\ell).$$

In the simplest case, ${}^{L}G$ is the *Langlands dual group*, denoted $G^\vee$.

Langlands duality

Table: Langlands dual groups

type of G	type of $G^\vee$
A_n	A_n
$SL(n)$	$PGL(n)$
$PGL(n)$	$SL(n)$
B_n	C_n
C_n	B_n
D_n	D_n
E_n	E_n
F_4	F_4
G_2	G_2

Theorem of Kret-Shin

The next theorem concerns the red line, with G of type C_n, $G^\vee$ of type B_n.

Theorem (Kret-Shin, 2016)

Let $\pi \subset L_2^{coh,i}(G,E)$, with $G = GSp(2n)$, E totally real, i the middle dimension. Assume some (mild) technical hypotheses. Then for every ℓ there exists a Langlands parameter

$$\rho_{\pi,\ell} : Gal(\overline{\mathbb{Q}}/E) \to GSpin(\overline{\mathbb{Q}}_\ell)$$

for π.

Local Langlands duality

Question

What does it mean for $\rho_{\pi,\ell}$ to be a Langlands parameter?

Let v be a place of E, E_v a completion. As for $GL(n)$, for any (p-adic) place v of E, $\rho_{\pi,\ell}$ determines a local Langlands parameter:

$$\rho_{\pi,v} : Gal(\bar{E}_v/E_v) \to GSpin(\overline{\mathbb{Q}}_\ell).$$

Necessary condition: For every v, $\rho_{\pi,v}$ and π_v correspond under *local Langlands duality*.
For $G = GL(n)$, and (I believe) for the representations of Kret-Shin, this suffices to characterize $\rho_{\pi,\ell}$ up to isomorphism . For general groups, it does not even for $G = SL(3)$ [Blasius] and there is no precise conjecture.

Langlands correspondence for function fields, general G, review

X a complete curve over k finite; $D \subset X(\bar{k})$ an effective divisor.

$x \in X(\bar{k}) \rightsquigarrow LG_x = G(k_x((T)))$, $LG_x^+ = G(k_x[[T]])$ (*loop groups*).

Replacing rank m vector bundles by principal G-bundles, where G is a split semisimple algebraic group, consider

$$L_2(S(G,X)); \;\; S(G,X) = \varprojlim_D G(k(X)) \backslash \prod_x {}' LG_x / U(D).$$

Here $U(D) = \prod_x U(D)_x$, $U(D)_x = LG_x^+, x \notin |D|$.

Langlands correspondence for function fields, general G, review

VL: $\pi \subset L_2(S(G,X))$ (level D) $\rightsquigarrow$ its *Langlands parameter*: a (semisimple) homomorphism

$$\rho_{\pi,\ell} : \pi_1(X \setminus |D|, x_0) \to G^{\vee}(\overline{\mathbb{Q}}_\ell).$$

Pseudorepresentations of $GL(m)$

If $G = GL(m)$ (not semisimple ...) then $\rho = \rho_{\pi,\ell}$ is completely determined by its *character*:

$$g \mapsto tr(\rho(g)).$$

In fact, $\rho_{\pi,\ell}$ can be reconstructed from the function $tr(\rho)$ by geometric invariant theory.
Let Γ be a profinite topological group, A a topological ring.

Pseudorepresentations of $GL(d)$

Definition

A d-dimensional pseudocharacter of Γ with values in A is a continuous function $T : \Gamma \to A$ satisfying

(1) $T(1) = d$

(2) $T(gh) = T(hg)$

(3) The integer $d \geq 0$ is the smallest with the following property. Let $sgn : \mathfrak{S}_{d+1} \to \pm 1$ the sign character. Then for all $g_1, \ldots, g_{d+1} \in G$, the following sum equals zero:

$$\sum_{\sigma \in \mathfrak{S}_{d+1}} sgn(\sigma) T_\sigma(g_1, \ldots, g_{d+1}) = 0.$$

Pseudocharacters

Theorem (Taylor, Rouquier)

(a) Suppose ρ is a continuous d-dimensional representation. Then $Tr\rho$ is a pseudocharacter of dimension d.
(b) Conversely, if A is an algebraically closed field of characteristic 0 or of characteristic $> d$, then any d-dimensional pseudocharacter of G with values in A is the trace of a semisimple representation of dimension d.

VL applied results of Richardson to prove an analogue for any $G^{\vee}$.

Pseudocharacters of general G

For any finite set I let $X_I(G^\vee) = G^\vee \backslash G^{\vee,I} / G^\vee$ (GIT quotient) and $R_I = \mathcal{O}(X_I(G^\vee))$.
If $\zeta : I \to J$ we define a projection

$$i_\zeta : X_J(G^\vee) \to X_I(G^\vee)$$

and a pullback

$$\zeta^* : R_I \to R_J; \;\; f \mapsto f \circ i_\zeta$$

For $I = [n] = \{1, \ldots, n\}$, $J = [n+1]$, define

$$m_n : X_{[n+1]}(G) \to X_{[n]}(G); \;\; (g_1, \ldots, g_n, g_{n+1}) \mapsto (g_1, \ldots, g_n \cdot g_{n+1})$$

and

$$m_n^* : R_{[n]} \to R_{[n+1]}$$

Pseudocharacters of general G

A $G^\vee$-pseudocharacter Θ of Γ with values in A is the following data:

- For each I-tuple $(\gamma_i) \in \Gamma^I$, a homomorphism $\Theta((\gamma_i)) : R_I \to A$, such that $(\gamma_i) \mapsto \Theta((\gamma_i))(f)$ is continuous $\forall f \in R_I$.
- For each map $\zeta : I \to J$ identities

$$\Theta(\gamma_{\zeta(i)}) = \Theta((\gamma_j)) \circ \zeta^* : R_I \to A.$$

- For $n \geq 1$ identities

$$\Theta(\gamma_1, \ldots, \gamma_n, \gamma_{n+1}) \circ m_n^* = S(\gamma_1, \ldots, \gamma_n \gamma_{n+1}) : R_{[n]} \to A.$$

Formally, an A-valued point of $Map(B\Gamma^\bullet, B(G^\vee)^\bullet // Ad(G^\vee))$.

Automorphic pseudocharacters

Theorem (VL)

(i)[Easy] If $\rho : \Gamma \to G^{\vee}(A)$ is continuous then one canonically defines a $G^{\vee}$-pseudocharacter Θ_{ρ} with values in A.
(ii) Conversely, if A is an algebraically closed field then any $G^{\vee}$-pseudocharacter Θ with values in A is of the form Θ_{ρ} for a unique completely reducible ρ up to equivalence.

Theorem (VL)

(i) To each $\pi \subset L_2(S(G, X))$ of level D, there is a $G^{\vee}$-pseudocharacter $\Theta_{\pi,\ell}$ on $\Gamma = \pi_1(X \setminus |D|, x_0)$ with values in $\overline{\mathbb{Q}}_{\ell}$.
(ii) The $\rho = \rho_{\pi,\ell}$ such that $\Theta_{\pi,\ell} = \Theta_{\rho}$ is compatible with the local Langlands correspondence for $x \notin |D|$.

Unramified local Langlands correspondence

An irreducible representation σ of $G(k((t)))$ is *unramified* if $\sigma^{G(k[[t]])} \neq 0$.
If π as above is of level D then $\pi \xrightarrow{\sim} \prod_x{}' \pi_x$ and π_x is unramified for $x \notin |D|$.
Here is the explanation of Lafforgue's condition (ii):

Theorem (Satake)

The unramified representations of $G(k((t)))$ are in bijection with local Langlands parameters

$$\rho : Gal(\overline{k((t))}/k((t)))$$

trivial on the inertia group.

Open questions, 1. The local Langlands correspondence

V. Lafforgue's correspondence is compatible with the local Langlands correspondence at unramified places.
At ramified places, it *defines* a local correspondence (work of Genestier-Lafforgue).

Question

Is the Genestier-Lafforgue correspondence surjective?

Compare with other constructions: Scholze, Kaletha-Weinstein (in progress): no information about Galois parameters.
Gan-Lomelí (stability of Langlands-Shahidi γ-factors) , Kaletha (proposed partial local parametrization).

Open questions, 2. Reciprocity

We have seen that, when $G = GL(m)$, L. Lafforgue had already constructed the parametrization by very different methods and proved it defined a *bijective correspondence* between *cuspidal* $\pi \subset L_2(S(m,X))$ and *irreducible* Langlands parameters.

In other words, every irreducible $GL(m)$-pseudocharacter on Γ is automorphic.

Question

What about reciprocity for other groups?

Some answers

Theorem (Böckle, MH, Khare, Thorne)

Let G be a split semisimple group over X and let $\rho : \pi_1(X) \to G^\vee(\overline{\mathbb{Q}}_\ell)$ be a representation with Zariski dense image (and a few other conditions).
Then ρ is **potentially automorphic**. *That is, there are infinitely many Galois coverings X_i/X such that the pullback of ρ to $\pi_1(X_i)$ becomes automorphic.*

Some answers

There is also work in progress on local surjectivity but there are also serious obstacles.
For classical groups, there is the work of Arthur (for p-adic fields), plus Ganapathy-Varma (application of Deligne-Kazhdan theory of close local fields).