Automorphic Galois representations and Langlands correspondences

I. Galois representations and automorphic forms: an introduction to the Langlands program

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Outline

Galois representations mod ℓ^n

What can you do with Galois theory?
You can compute the Galois group of the set of roots (in $\mathbb{C}$) of the cyclotomic polynomial

$$f_{\ell^n}(X) = X^{\ell^n} - 1 = 0.$$

Here ℓ is a prime number.
Denote the set of roots μ_{ℓ^n}; it is a cyclic group of order ℓ^n, and the group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the group by automorphisms:

$$\omega_{\ell,n} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to Aut(\mu_{\ell^n}) = (\mathbb{Z}/\ell^n\mathbb{Z})^{\times}.$$

Galois representations over $\mathbb{Q}_\ell$

A clever idea: act on all μ_{ℓ^n} together:

$$\omega_\ell : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to Aut(\bigcup_n \mu_{\ell^n}) = \mathbb{Z}_\ell^\times,$$

where $\mathbb{Z}_\ell$ is the compact topological ring $\varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z}$.
We get a 1-**dimensional representation**

$$\omega_\ell : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(1, \mathbb{Z}_\ell) \subset GL(1, \mathbb{Q}_\ell).$$

Elliptic curves

Similarly, if E is an elliptic curve over $\mathbb{Q}$:

$$y^2 = x^3 + ax + b$$

the set of complex solutions is an abelian group (a torus)

$$E(\mathbb{C}) \simeq \mathbb{C}/\Lambda, \ \ \Lambda \simeq \mathbb{Z}^2.$$

So the elements $E[\ell^n]$ of order ℓ^n form a group isomorphic to $(\mathbb{Z}/\ell^n\mathbb{Z})^2$ on which $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts, and we obtain

$$\rho_{E,p} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to Aut(\bigcup_n E[\ell^n]) = GL(2, \mathbb{Z}_\ell).$$

Galois representations on cohomology

More generally, let $X \subset \mathbb{P}^N$ be an algebraic variety defined by polynomials (in $N+1$ variables) with $\mathbb{Q}$-coefficients.
The set $X(\mathbb{C})$ of $\mathbb{C}$ points is a topological space, and for any positive integer i, the theory of étale cohomology provides an action

$$\rho^i_{X,\ell} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to Aut(H^i(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_\ell) \xrightarrow{\sim} GL(m, \mathbb{Q}_\ell).$$

for some m.
These Galois representations have special properties – they are *geometric* (to be defined below).

Two conjectures of Fontaine-Mazur

Conjecture (Vague version of one Fontaine-Mazur conjecture)

Any irreducible representation $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(m, \mathbb{Q}_\ell)$ *that is geometric occurs in the cohomology of some smooth projective variety.*

Conjecture (Vague version of another Fontaine-Mazur conjecture)

Any irreducible representation $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(m, \mathbb{Q}_\ell)$ *that is geometric is attached to an* automorphic representation *of* $GL(m, \mathbb{Q})$.

Galois groups of local fields

The group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is put together from the Galois groups of p-adic fields.
More precisely, any polynomial $f \in \mathbb{Q}[X]$ is also a polynomial in $\mathbb{Q}_p[X]$ for every prime p and thus $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on its roots.
The starting point of algebraic number theory is the observation that the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the roots of f can be reconstructed from that of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.
Likewise, the m-dimensional representations $\rho^i_{X,\ell}$ of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ are determined by representations of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on the same cohomology.

Structure of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$: the analogue on a complex curve

An m-dimensional representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is analogous to a local system L with $\mathbb{Q}$-coefficients on a punctured algebraic curve

$$X_0 = X \setminus \{x_1, \ldots, x_r\}$$

over $\mathbb{C}$. We obtain

$$\pi_1(X_0, x_0) \to Aut(L_{x_0}) = GL(m, \mathbb{Q})$$

If you draw a loop from x_0 that wraps once around x_i you get an element

$$\gamma_i \in Aut(L_{x_0}) = GL(m, \mathbb{Q}).$$

The "local Galois group" of X_0 at x_i is just $\mathbb{Z}$; or $\hat{\mathbb{Z}} = \varprojlim_i \mathbb{Z}/i\mathbb{Z}$ if you instead take local systems with finite or p-adic coefficients.

Structure of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$: the analogue on a curve over a finite field

If instead X_0 is a curve over a finite field k of characteristic p, the local Galois group has a three-step filtration.
Even at a point $x \in X_0(k')$, with k'/k finite, there is a local Galois group $Gal(\bar{k}/k')$ acting on local systems.
If $|k'| = \ell^r$, $Gal(\bar{k}/k')$ is isomorphic to $\hat{\mathbb{Z}}$ with canonical topological generator

$$Frob_x : t \mapsto t^{|k'|} = t^{\ell^r}.$$

At a puncture x_i defined over k', there are three non-trivial steps.

Structure of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$: the analogue on a curve over a finite field

At a puncture x_i defined over k', there are three non-trivial steps.

$$1 \to I_{x_i} \to \Gamma_{x_i} \to Gal(\bar{k}/k') = \hat{\mathbb{Z}} \to 1$$

And then the *inertia group* I_{x_i} itself has a two step filtration

$$1 \to I^{wild} \to I_{x_i} \to I^{tame} \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell \to 1.$$

The middle stage I^{tame} is exactly analogous to the loop that winds about the point x_i in the complex curve.
The top stage $Gal(\bar{k}/k')$ is there because k' is not algebraically closed.
The bottom stage I^{wild} is a pro-p group and is thus pronilpotent, but its structure does not have a simple description.

Structure of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

The group $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ has the same structure:

$$1 \to I_p \to Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p) = \hat{\mathbb{Z}} \to 1$$

$$1 \to I^{wild} \to I_p \to I^{tame} \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell \to 1.$$

The group $Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is topologically generated by

$$Frob_p : t \mapsto t^p.$$

The *local Langlands correspondence* classifies m-dimensional continuous representations of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ in terms of analysis on the locally compact topological group $GL(m, \mathbb{Q}_p)$.

Abelian representations

A 1-dimensional ℓ-adic representation of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is a homomorphism

$$Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{ab} \to GL(1, \mathbb{Q}_\ell).$$

Theorem (Local class field theory)

There is a canonical homomorphism

$${\mathbb{Q}_p}^\times \hookrightarrow Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{ab}$$

with dense image containing the inertia group I_p, such that $p \in {\mathbb{Q}_p}^\times$ maps to $Frob_p^{-1}$.

Abelian representations

Thus a representation $\rho : Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL(m, \mathbb{Q}_\ell)$ that factors through $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{ab}$ corresponds to an m-tuple of characters

$$\chi = (\chi_i : {\mathbb{Q}_p}^\times \to \mathbb{Q}_\ell^\times)\ i = 1, \ldots, m.$$

Define a representation $I(\chi)$ of $G = GL(m, \mathbb{Q}_p)$ as follows. Let $B \subset G$ be the upper triangular subgroup, $a = (a_1, \ldots, a_m) : B \to GL(1)^m$ the projection on the diagonal elements.
Define

$$\chi : B \to \mathbb{Q}_\ell^\times;\ \ \chi(a_1(b), \ldots, a_m(b)) = \prod_i \chi_i(a_i(b)).$$

$$I(\chi) = \{f : G \to \mathbb{Q}_\ell \mid f(bg) = \delta^{\frac{1}{2}}\chi(b)f(g)\forall b \in B, g \in G\}.$$

(Ignore the normalizing factor $\delta^{\frac{1}{2}}$.)

About coefficients

The Langlands correspondence for abelian representations assigns $I(\chi)$ to $\rho = \oplus_i \chi_i$.
It is more natural to replace the coefficient field $\mathbb{Q}_\ell$ by its algebraic closure $\bar{\mathbb{Q}}_\ell$.
An ℓ-adic representation is then a homomorphism to $GL(m, \overline{\mathbb{Q}}_\ell)$.
But we can also replace $\overline{\mathbb{Q}}_\ell$ by $\mathbb{C}$.

Since the theory is purely algebraic, the difference is inconsequential.

Irreducible representations

At the other extreme, we have

Theorem (H-Taylor, Henniart)

Suppose $\ell \neq p$. There is a canonical bijection between irreducible (continuous) m-dimensional ℓ-adic representations of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and irreducible supercuspidal representations *of $GL(m, \mathbb{Q}_p)$, preserving natural invariants of both sides.*

Remark

Supercuspidal representations are the building blocks of the theory.

The local Langlands correspondence

In between, there are irreducible representations of $GL(m,\mathbb{Q}_p)$ corresponding to any m-dimensional (continuous!) ℓ-adic representation of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, provided $\ell \neq p$.
The correspondence becomes a canonical bijection if one expands the class of representations of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to include more general objects (Weil-Deligne parameter).
Moreover, the theorem is proved when $\mathbb{Q}_p$ is replaced by any finite extension of $\mathbb{Q}_p$; or (Laumon-Rapoport-Stuhler) by the local field $\mathbb{F}_q((T))$ (any finite field $\mathbb{F}_q$).

Finally, there is a version of the correspondence when $\mathbb{Q}_p$ is replaced by $\mathbb{R}$ or $\mathbb{C}$.

Adelic representations

Now suppose we have a continuous representation

$$\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(m, \overline{\mathbb{Q}}_\ell).$$

Recall that $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is built out of the $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for primes p.
Concretely, we have embeddings

$$Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow Gal(\overline{\mathbb{Q}}/\mathbb{Q})$$

so by restriction ρ defines $\rho_p : Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to GL(m, \overline{\mathbb{Q}}_\ell)$.
For $p \neq \ell$ we have the local Langlands correspondence, hence an irreducible representation $\pi(\rho_p)$ of $GL(m, \mathbb{Q}_p)$.
For $p = \ell$ we need to assume ρ_ℓ is of *de Rham type* (Fontaine).

Geometric representations

Theorem (Fontaine)

If ρ_ℓ is of de Rham type then ℓ-adic Hodge theory attaches a Weil-Deligne parameter to ρ_ℓ, and thus an irreducible representation $\pi(\rho_\ell)$ of $GL(m, \mathbb{Q}_\ell)$.

Definition

The representation ρ is *geometric* if (a) ρ_ℓ is of de Rham type and (b) for all but finitely many p, ρ_p is trivial on the inertia group I_p.

Now a geometric ρ gives us a collection $\{\pi_p\}$ for all prime numbers p. Also (Fontaine) get a representation π_∞ of $GL(m, \mathbb{R})$.

Automorphic representations

Let $[\mathbf{S}(\mathbf{m})] = GL(m,\mathbb{Q})\backslash GL(m,\mathbb{R}) \times \prod'_p GL(m,\mathbb{Q}_p))$. It has simultaneous actions of $GL(m,\mathbb{R})$ and $GL(m,\mathbb{Q}_p)$ for all p.

Definition

The representation ρ is *automorphic* if the collection $(\{\pi_p\},\pi_\infty)$ occurs as a direct summand in the space

$$L_2([\mathbf{S}(\mathbf{m})]/\sim).$$

Conjecture (Fontaine-Mazur conjecture)

Any irreducible representation $Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(m,\mathbb{Q}_\ell)$ *that is geometric is automorphic.*

Fourier analysis on $GL(n)$

The promanifold $[\mathbf{S}(\mathbf{m})]/\sim$ is homogeneous for the action of $G(\mathbf{A}) = GL(m,\mathbb{R}) \times \prod'_p GL(m,\mathbb{Q}_p))$.

An analogue of the space of functions on (infinitely many copies of) the circle, a representation of the compact group $U(1)$.

Decomposition of the latter under the $U(1)$-action is the (easy part of) the theory of Fourier series.

Decomposition of $L_2([\mathbf{S}(\mathbf{m})]/\sim)$ under the action of the group $G(\mathbf{A})$ is analogous.

Langlands reciprocity, in two sentences

To each integer m we assign a promanifold $[\mathbf{S}(\mathbf{m})]/\sim$ that is homogenous for a very large group.
The theory of (geometric) Galois representations is – conjecturally – encompassed by non-abelian Fourier analysis on these homogeneous promanifolds.

(There is also a large part of the L_2 space that is irrelevant to Galois representations; we disregard all of this.)

Automorphic forms on function fields

Let $X = \mathbb{P}^1 \setminus \{x_1, \ldots, x_r\}$ be the punctured projective line over the finite field $\mathbb{F}_p$.
The field $k(X) = \mathbb{F}_p(T)$ is a global field. Choose $\ell \neq p$.
Replace $\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(m, \overline{\mathbb{Q}}_\ell)$ by an irreducible m-dimensional ℓ-adic local system Λ on X; in other words, a representation ρ of $Gal(\overline{\mathbb{F}_p(T)}/\mathbb{F}_p(T))$ with values in $GL(m, \overline{\mathbb{Q}}_\ell)$, ramified only at $\{x_1, \ldots, x_r\}$.
(The analogue for $\mathbb{P}^1(\mathbb{C}) \setminus \{x_1, \ldots, x_r\}$ is a linear system of differential equations with singularities only at the punctures.)

Langlands correspondence for function fields

As before, to such a Λ one attaches a representation π_x of $GL(m, k_x((T))$ for each $x \in \mathbb{P}^1(\bar{\mathbb{F}}_p)$ (including the singularities x_i) with coordinates in the finite field k_x.

The collection $\{\pi_x\}$ is *automorphic* if it occurs in $L_2(S(m,X))$ where

$$S(m,X) = GL(m, k(X))\backslash \prod_x{}' GL(m, k_x((T)))/\sim .$$

The space $S(m,X)$ is totally disconnected and in bijection with the set $Bun_m(X)(\mathbb{F}_p)$ of *rank m vector bundles* on X defined over $\mathbb{F}_p$ (with trivializations at all points).

Langlands correspondence for function fields

Theorem (Laurent Lafforgue)

Every such Λ is automorphic.

In this theorem the punctured projective line over $\mathbb{F}_p$ can be replaced by *any* algebraic curve over any finite field.

Langlands correspondence for function fields, general G

Replacing rank m vector bundles by principal G-bundles, where G is a reductive algebraic group, one gets an analogous space of automorphic forms

$$L_2(Bun_G(X)(k)); \;\; Bun_G(X)(k) = S(G,X) = G(k(X))\backslash \prod_x{}' G(k_x((T)))/\sim$$

This is the subject of the work of Vincent Lafforgue.
The m-dimensional local system Λ is replaced by a *Langlands parameter*: a homomorphism from the fundamental group to the *Langlands dual group*.