DUKE MATHEMATICAL JOURNAL
Vol. 114, No. 2, © 2002

COLLAPSING AND THE DIFFERENTIAL FORM LAPLACIAN: THE CASE OF A SMOOTH LIMIT SPACE

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Abstract

We analyze the limit of the p-form Laplacian under a collapse, with bounded sectional curvature and bounded diameter, to a smooth limit space. As an application, we characterize when the p-form Laplacian has small positive eigenvalues in a collapsing sequence.

1. Introduction

A central problem in geometric analysis is to estimate the spectrum of the Laplacian on a compact Riemannian manifold M in terms of geometric invariants. In the case of the Laplacian on functions, a major result is J. Cheeger's lower bound on the smallest positive eigenvalue in terms of an isoperimetric constant (see [11]). The problem of extending his lower bound to the case of the p-form Laplacian was posed in [11]. There has been little progress on this problem. We address the more general question of estimating the eigenvalues $\{\lambda_{p,j}(M)\}_{j=1}^{\infty}$ of the p-form Laplacian $\triangle_p$ (counted with multiplicity) in terms of geometric invariants of M.

A basic fact, due to Cheeger and J. Dodziuk, is that $\lambda_{p,j}(M)$ depends continuously on the Riemannian metric g^{TM} in the C^0-topology (see [17]). Then an immediate consequence of the C^α-compactness theorem of M. Anderson and Cheeger [1] is that for any $n \in \mathbb{Z}^+$, $r \in \mathbb{R}$, and $D, i_0 > 0$, there are uniform bounds on $\lambda_{p,j}(M)$ among connected closed n-dimensional Riemannian manifolds M with $\mathrm{Ric}(M) \geq r$, $\mathrm{diam}(M) \leq D$, and $\mathrm{inj}(M) > i_0$ (cf. [10, Theorem 1.3], [14, Theorem 0.4]). In particular, there is a uniform positive lower bound on the smallest positive eigenvalue of the p-form Laplacian under these geometric assumptions.

The question, then, is what happens when $\mathrm{inj}(M) \to 0$. For technical reasons, in this paper we assume uniform bounds on the Riemannian curvature R^M. Then we wish to study how the spectrum of $\triangle_p$ behaves in the collapsing limit. By collapsing

DUKE MATHEMATICAL JOURNAL
Vol. 114, No. 2, © 2002
Received 20 May 2000. Revision received 20 August 2001.
2000 *Mathematics Subject Classification.* Primary 58J50; Secondary 35P15.
Author's research supported by National Science Foundation grant number DMS-9704633.

we mean the phenomenon of a sequence of Riemannian manifolds converging in the Gromov-Hausdorff topology to a lower-dimensional space. We refer to [23, Chapters 1 and 3] for basic information about collapsing and to [12, Section I], [13], [21], and [23, Chapter 6] for information about bounded curvature collapsing. In this paper we analyze the behavior of the spectrum of $\triangle_p$ under collapse, with bounded sectional curvature and bounded diameter, to a smooth limit space. The answer is in terms of a type of Laplacian on the limit space. As an application, we characterize when the p-form Laplacian has small positive eigenvalues in a collapsing sequence. In a subsequent paper we will extend the results to the case of singular limit space and give additional applications.

From Hodge theory, $\dim(\mathrm{Ker}(\triangle_p)) = \mathrm{b}_p(M)$, the pth Betti number of M. Given $K \geq 0$, let $\mathscr{M}(M, K)$ be the set of Riemannian metrics g on M with $\|R^M\|_\infty \leq K$ and $\mathrm{diam}(M, g) \leq 1$. We say that M has small positive eigenvalues of the p-form Laplacian if

$$\inf_{g \in \mathscr{M}(M,K)} \lambda_{p,j}(M, g) = 0 \tag{1.1}$$

for some $j > \mathrm{b}_p(M)$ and some $K > 0$. If this is the case, then we say that M has (at least) j small eigenvalues. Note that this is a statement about the (smooth) topological type of M.

There are no small positive eigenvalues of the Laplacian on functions on M (see, e.g., [3]). B. Colbois and G. Courtois gave examples of manifolds with small positive eigenvalues of the p-form Laplacian for $p > 0$ (see [14]). Their examples were manifolds M with free isometric T^k-actions, which one shrinks in the direction of the T^k-orbits. In terms of the fiber bundle $M \to M/T^k$, this sort of collapsing is a case of the so-called adiabatic limit. The asymptotic behavior of the small eigenvalues of the p-form Laplacian in the adiabatic limit was related to the Leray spectral sequence of the fiber bundle in [5], [16], [18], and [26].

In another direction, K. Fukaya considered the behavior of the Laplacian on functions in the case of a sequence of manifolds that converge in the Gromov-Hausdorff metric d_{GH} to a lower-dimensional limit space X, the collapsing assumed to be with bounded sectional curvature and bounded diameter (see [19]). He found that in order to get limits, one needs to widen the class of spaces being considered by adding a Borel measure and to consider measured metric spaces. This is the case even if X happens to be a smooth manifold. He defined a Laplacian acting on functions on the measured limit space and proved a convergence theorem for the spectrum of the Laplacian on functions, under the geometric assumption of convergence in the measured Gromov-Hausdorff topology.

We consider the behavior of the spectrum of $\triangle_p$ under collapse with bounded sectional curvature and bounded diameter. We find that we need a somewhat more refined structure on the limit space, namely, a superconnection as introduced by D. Quillen

[27]. More precisely, we need a flat degree-1 superconnection in the sense of [8]. Suppose that B is a smooth connected closed manifold and that $E = \bigoplus_{j=0}^{m} E^j$ is a $\mathbb{Z}$-graded real vector bundle on B. The degree-1 superconnections A' that we need are of the form

$$A' = A'_{[0]} + A'_{[1]} + A'_{[2]}, \tag{1.2}$$

where

- $A'_{[0]} \in C^\infty(B; \mathrm{Hom}(E^*, E^{*+1}))$,
- $A'_{[1]}$ is a grading-preserving connection ∇^E on E, and
- $A'_{[2]} \in \Omega^2(B; \mathrm{Hom}(E^*, E^{*-1}))$.

The superconnection extends by Leibniz's rule to an operator A' on the E-valued differential forms $\Omega(B; E)$. The flatness condition $(A')^2 = 0$ becomes

- $(A'_{[0]})^2 = (A'_{[2]})^2 = 0$,
- $\nabla^E A'_{[0]} = \nabla^E A'_{[2]} = 0$, and
- $(\nabla^E)^2 + A'_{[0]}A'_{[2]} + A'_{[2]}A'_{[0]} = 0$.

In particular, $A'_{[0]}$ defines a differential complex on the fibers of E. Let g^{TB} be a Riemannian metric on B, and let h^E be a graded Euclidean inner product on E, meaning that E^i is orthogonal to E^j if $i \neq j$. Then there are an adjoint $(A')^*$ to A' and a Laplacian $\triangle^E = A'(A')^* + (A')^*A'$ on $\Omega(B; E)$. Let $\triangle_p^E$ be the restriction of $\triangle^E$ to $\bigoplus_{a+b=p} \Omega^a(B; E^b)$.

Using the C^0-continuity of the spectrum and the geometric results of Cheeger, Fukaya, and M. Gromov [12], we can reduce our study of collapsing to certain special fiber bundles. As is recalled in Section 3, an infranilmanifold Z has a canonical flat linear connection ∇^{aff}. Let $\mathrm{Aff}(Z)$ be the group of diffeomorphisms of Z which preserve ∇^{aff}.

Definition 1

An *affine fiber bundle* is a smooth fiber bundle $M \to B$ whose fiber Z is an infranilmanifold and whose structure group is reduced from $\mathrm{Diff}(Z)$ to $\mathrm{Aff}(Z)$. A *Riemannian affine fiber bundle* is an affine fiber bundle with the following:

- a horizontal distribution $T^H M$ on M whose holonomy lies in $\mathrm{Aff}(Z)$,
- a family g^{TZ} of vertical Riemannian metrics that are parallel with respect to the flat affine connections on the fibers Z_b, and
- a Riemannian metric g^{TB} on B.

Fix a smooth connected closed Riemannian manifold B. Fukaya showed that any manifold M that collapses to B, with bounded sectional curvature, is the total space of an affine fiber bundle over B (see [20]). If $M \to B$ is an affine fiber bundle, let $T^H M$ be a horizontal distribution on M as above. Let $T \in \Omega^2(M; TZ)$ be the curvature of $T^H M$. There is a $\mathbb{Z}$-graded real vector bundle E on B whose fiber over

$b \in B$ is isomorphic to the differential forms on the fiber Z_b which are parallel with respect to the flat affine connection on Z_b. The exterior derivative d^M induces a flat degree-1 superconnection A' on E. If $M \to B$ is in addition a Riemannian affine fiber bundle, then we obtain a Riemannian metric g^{TM} on M constructed from g^{TZ}, g^{TB}, and $T^H M$. There is an induced L^2 inner product h^E on E. Define $\triangle^E$ as above. Let $\mathrm{diam}(Z)$ denote the maximum diameter of the fibers $\{Z_b\}_{b\in B}$ in the intrinsic metric, and let Π denote the second fundamental forms of the fibers $\{Z_b\}_{b\in B}$. Our first result says that the spectrum $\sigma(\triangle_p^E)$ of $\triangle_p^E$ contains all of the spectrum of the p-form Laplacian $\triangle_p^M$ which stays bounded as $d_{GH}(M, B) \to 0$.

THEOREM 1
There are positive constants A, A', and C which depend only on $\dim(M)$ *such that if* $\|R^Z\|_\infty \mathrm{diam}(Z)^2 \leq A'$, *then for all* $0 \leq p \leq \dim(M)$,

$$\sigma(\triangle_p^M) \cap \left[0, A\,\mathrm{diam}(Z)^{-2} - C(\|R^M\|_\infty + \|\Pi\|_\infty^2 + \|T\|_\infty^2)\right) = \sigma(\triangle_p^E) \cap \left[0, A\,\mathrm{diam}(Z)^{-2} - C(\|R^M\|_\infty + \|\Pi\|_\infty^2 + \|T\|_\infty^2)\right). \quad (1.3)$$

When Z is flat, there is some intersection between Theorem 1 and the adiabatic limit results of [5], [16], [18], and [26]. However, there is the important difference that we need estimates that are uniform with respect to $d_{GH}(M, B)$, whereas the adiabatic limit results concern the asymptotics of the eigenvalues under the collapse of a given Riemannian fiber bundle coming from a constant rescaling of its fibers.

We apply Theorem 1 to estimate the eigenvalues of a general Riemannian manifold M which is Gromov-Hausdorff close to B, assuming sectional curvature bounds on M. Of course, we cannot say precisely what $\sigma(\triangle^M)$ is, but we can use Theorem 1 to approximate it to a given precision $\epsilon > 0$. We say that two nonnegative numbers λ_1 and λ_2 are ϵ-close if $e^{-\epsilon}\lambda_2 \leq \lambda_1 \leq e^{\epsilon}\lambda_2$. We show that for a given $\epsilon > 0$, if $d_{GH}(M, B)$ is sufficiently small, then there is a flat degree-1 superconnection A' on B whose Laplacian $\triangle_p^E$ has a spectrum that is ϵ-close to that of $\triangle_p^M$, at least up to a high level.

THEOREM 2
Let B be a fixed smooth connected closed Riemannian manifold. Given $n \in \mathbb{Z}^+$, $\epsilon > 0$, and $K \geq 0$, there are positive constants $A(n, \epsilon, K)$, $A'(n, \epsilon, K)$, and $C(n, \epsilon, K)$ with the following property: if M^n is an n-dimensional connected closed Riemannian manifold with $\|R^M\|_\infty \leq K$ and $d_{GH}(M, B) \leq A'(n, \epsilon, K)$, then there are

(1) *a $\mathbb{Z}$-graded real vector bundle E on B,*
(2) *a flat degree-1 superconnection A' on E, and*
(3) *a Euclidean inner product h^E on E,*

such that if $\lambda_{p,j}(M)$ is the jth eigenvalue of the p-form Laplacian on M, $\lambda_{p,j}(B; E)$ is the jth eigenvalue of $\triangle_p^E$, and

$$\min\left(\lambda_{p,j}(M), \lambda_{p,j}(B; E)\right) \leq A(n, \epsilon, K)\, d_{GH}(M, B)^{-2} - C(n, \epsilon, K), \qquad (1.4)$$

then $\lambda_{p,j}(M)$ is ϵ-close to $\lambda_{p,j}(B; E)$.

Using [2], one can also show that the eigenspaces of $\triangle_p^E$ are L^∞-close to those of $\triangle_p^M$, with respect to the embedding $\Omega(B; E) \to \Omega(M)$.

In the case of the Laplacian on functions, only E^0 is relevant. Although E^0 is the trivial $\mathbb{R}$-bundle on B with a trivial connection, its Euclidean inner product h^{E^0} need not be trivial and corresponds exactly to the measure in Fukaya's work.

In order to apply Theorem 2, we prove a compactness result for the superconnection and the Euclidean metric.

Definition 2
Let $\mathscr{S}_E$ be the space of degree-1 superconnections on E, let $\mathscr{G}_E$ be the group of smooth grading-preserving $\mathrm{GL}(E)$ gauge transformations on E, and let $\mathscr{H}_E$ be the space of graded Euclidean inner products on E. We equip $\mathscr{S}_E$ and $\mathscr{H}_E$ with the C^∞-topology. Give $(\mathscr{S}_E \times \mathscr{H}_E)/\mathscr{G}_E$ the quotient topology.

THEOREM 3
In Theorem 2, we may assume that E is one of a finite number of isomorphism classes of real $\mathbb{Z}$-graded topological vector bundles $\{E_i\}$ on B. Furthermore, there are compact subsets $D_{E_i} \subset (\mathscr{S}_{E_i} \times \mathscr{H}_{E_i})/\mathscr{G}_{E_i}$ depending on n, ϵ, and K, such that we may assume that the gauge equivalence class of the pair (A', h^E) lies in D_E.

We remark that there may well be a sequence of topologically distinct Riemannian manifolds of a given dimension, with uniformly bounded sectional curvatures, which converge to B in the Gromov-Hausdorff topology (see Example 3). This contrasts with the finiteness statement in Theorem 3.

The eigenvalues of $\triangle_p^E$ are continuous with respect to $[(A', h^E)] \in (\mathscr{S}_E \times \mathscr{H}_E)/\mathscr{G}_E$. One application of Theorem 3 is the following relationship between the spectra of $\triangle_p^M$ and the ordinary differential form Laplacian on B.

THEOREM 4
Under the hypotheses of Theorem 2, let $\lambda'_{p,j}(B)$ be the jth eigenvalue of the Laplacian on $\bigoplus_r \Omega^r(B) \otimes \mathbb{R}^{\dim(E^{p-r})}$. Then there is a positive constant $D(n, \epsilon, K)$ such that

$$e^{-\epsilon/2}\lambda'_{p,j}(B)^{1/2} - D(n, \epsilon, K) \leq \lambda_{p,j}(M)^{1/2} \leq e^{\epsilon/2}\lambda'_{p,j}(B)^{1/2} + D(n, \epsilon, K). \quad (1.5)$$

Now consider a flat degree-1 superconnection A' on a real $\mathbb{Z}$-graded vector bundle E over a smooth manifold B. As $(A')^2 = 0$, there is a cohomology $\mathrm{H}^*(A')$ for the action of A' on $\Omega(B; E)$, the latter having the total grading. There is a flat $\mathbb{Z}$-graded "cohomology" vector bundle $\mathrm{H}^*(A'_{[0]})$ on B. Furthermore, there is a spectral sequence to compute $\mathrm{H}^*(A')$, with E_2-term $\mathrm{H}^*(B; \mathrm{H}^*(A'_{[0]}))$.

Suppose that M is a connected closed manifold with at least j small eigenvalues of $\triangle_p$ for $j > \mathrm{b}_p(M)$. Consider a sequence of Riemannian metrics $\{g_i\}_{i=1}^{\infty}$ in $\mathscr{M}(M, K)$ with $\lim_{i\to\infty} \lambda_{p,j}(M, g_i) = 0$. There must be a subsequence of $\{(M, g_i)\}_{i=1}^{\infty}$ which converges to a lower-dimensional limit space X. That is, we are in the collapsing situation. Suppose that the limit space is a smooth manifold B. From Theorems 2 and 3, we can take a further subsequence of $\{(M, g_i)\}_{i=1}^{\infty}$ to obtain a single vector bundle E on B, equipped with a sequence $\{(A'_i, h_i^E)\}_{i=1}^{\infty}$ of superconnections and Euclidean inner products. Using the compactness result in Theorem 3, we can take a convergent subsequence of these pairs, modulo gauge transformations, to obtain a superconnection A'_{∞} on E with $\dim \mathrm{Ker}(\triangle_p^E) \geq j$. Then $\dim(\mathrm{H}^p(A'_{\infty})) \geq j$. It is no longer true that $\mathrm{H}^*(A'_{\infty}) \cong \mathrm{H}^*(M; \mathbb{R})$ for this limit superconnection. However, we can analyze $\mathrm{H}^*(A'_{\infty})$ using the spectral sequence. We obtain

$$j \leq \sum_{a+b=p} \dim\left(\mathrm{H}^a(B; \mathrm{H}^b(A'_{\infty,[0]}))\right). \tag{1.6}$$

This formula has some immediate consequences. The first one is a bound on the number of small eigenvalues of the 1-form Laplacian.

COROLLARY 1

Suppose that M has j small eigenvalues of the 1-form Laplacian with $j > \mathrm{b}_1(M)$. Let X be the limit space coming from the above argument. Suppose that X is a smooth manifold B. Then

$$j \leq \mathrm{b}_1(B) + \dim(M) - \dim(B) \leq \mathrm{b}_1(M) + \dim(M). \tag{1.7}$$

The second consequence is a bound on the number of small eigenvalues of the p-form Laplacian for a manifold that is Gromov-Hausdorff close to a codimension-1 manifold.

COROLLARY 2

Let B be a connected closed $(n-1)$-dimensional Riemannian manifold. Then for any $K \geq 0$, there are $\delta, c > 0$ with the following property: suppose that M is a connected closed smooth n-dimensional Riemannian manifold with $\|R^M\|_{\infty} \leq K$ and $d_{GH}(M, B) < \delta$. First, M is the total space of a circle bundle over B. Let $\mathscr{O}$ be the orientation bundle of $M \to B$, a flat real line bundle on B. Then $\lambda_{p,j}(M, g) > c$ for $j = \mathrm{b}_p(B) + \mathrm{b}_{p-1}(B; \mathscr{O}) + 1$.

The rest of our results concern small eigenvalues in collapsing sequences.

Definition 3
If $M \to B$ is an affine fiber bundle, a collapsing sequence associated to the affine fiber bundle is a sequence of metrics $\{g_i\}_{i=1}^{\infty} \in \mathscr{M}(M, K)$ for some $K \geq 0$ such that $\lim_{i\to\infty}(M, g_i) = B$ in the Gromov-Hausdorff topology, and for some $\epsilon > 0$, each (M, g_i) is ϵ-bi-Lipschitz to a Riemannian affine fiber bundle structure on $M \to B$.

We show that there are three mechanisms to make small positive eigenvalues of the differential form Laplacian on M in a collapsing sequence. Either the differential form Laplacian on the fiber admits small positive eigenvalues, the holonomy of the flat "cohomology" bundle on B fails to be semisimple, or the Leray spectral sequence of $M \to B$ does not degenerate at the E_2-term.

THEOREM 5
Let $\{(M, g_i)\}_{i=1}^{\infty}$ be a collapsing sequence associated to an affine fiber bundle $M \to B$. Suppose that $\lim_{i\to\infty} \lambda_{p,j}(M, g_i) = 0$ for some $j > \mathrm{b}_p(M)$. Write the fiber Z of the affine fiber bundle as the quotient of a nilmanifold $\widehat{Z} = \widehat{\Gamma}\backslash N$ by a finite group F. Then

(1) *for some $q \in [0, p]$, $b_q(Z) < \dim(\Lambda^q(\mathfrak{n}^*)^F)$; or*
(2) *for all $q \in [0, p]$, $b_q(Z) = \dim(\Lambda^q(\mathfrak{n}^*)^F)$, and for some $q \in [0, p]$, the holonomy representation of the flat vector bundle $\mathrm{H}^q(Z; \mathbb{R})$ on B fails to be semisimple; or*
(3) *for all $q \in [0, p]$, $b_q(Z) = \dim(\Lambda^q(\mathfrak{n}^*)^F)$ and the holonomy representation of the flat vector bundle $\mathrm{H}^q(Z; \mathbb{R})$ on B is semisimple, and the Leray spectral sequence to compute $\mathrm{H}^p(M; \mathbb{R})$ does not degenerate at the E_2 term.*

Examples show that small positive eigenvalues can occur in each of the three cases in Theorem 5.

Theorem 5 has some immediate consequences. The first is a characterization of when the 1-form Laplacian has small positive eigenvalues in a collapsing sequence.

COROLLARY 3
Let $\{(M, g_i)\}_{i=1}^{\infty}$ be a collapsing sequence associated to an affine fiber bundle $M \to B$. Suppose that $\lim_{i\to\infty} \lambda_{1,j}(M, g_i) = 0$ for some $j > \mathrm{b}_1(M)$. Then

(1) *the differential $d_2 : \mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R})) \to \mathrm{H}^2(B; \mathbb{R})$ in the Leray spectral sequence for $\mathrm{H}^*(M; \mathbb{R})$ is nonzero; or*
(2) *the holonomy representation of the flat vector bundle $\mathrm{H}^1(Z; \mathbb{R})$ on B has a nontrivial unipotent subrepresentation; or*

(3) *Z is almost flat but not flat and there is a nonzero covariantly constant section of the flat vector bundle $(H^1(A'_{\infty,[0]}))/(H^1(Z;\mathbb{R})_\infty)$.*

The differential $d_2 : \mathrm{H}^0(B;\mathrm{H}^1(Z;\mathbb{R})) \to \mathrm{H}^2(B;\mathbb{R})$ can be considered to be a type of Euler class; in the case of an oriented circle bundle over a smooth base, it gives exactly the Euler class.

The second consequence is a characterization of when the p-form Laplacian has small positive eigenvalues in a collapsing sequence over a circle.

COROLLARY 4
Let $\{(M, g_i)\}_{i=1}^\infty$ be a collapsing sequence associated to an affine fiber bundle $M \to S^1$. Suppose that $\lim_{i\to\infty} \lambda_{p,j}(M, g_i) = 0$ for some $j > \mathrm{b}_p(M)$. Write the fiber Z of the affine fiber bundle as in Theorem 5. Then

(1) *for some $q \in \{p-1, p\}$, $b_q(Z) < \dim(\Lambda^q(\mathfrak{n}^*)^F)$; or*
(2) *for $q \in \{p-1, p\}$, $b_q(Z) = \dim(\Lambda^q(\mathfrak{n}^*)^F)$, and if $\Phi^* \in \mathrm{Aut}(\mathrm{H}^*(Z;\mathbb{R}))$ denotes the holonomy action on the fiber cohomology, then Φ^p or Φ^{p-1} has a nontrivial unipotent factor in its Jordan normal form.*

The third consequence is a characterization of when the p-form Laplacian has small positive eigenvalues in a collapsing sequence over a codimension-1 manifold.

COROLLARY 5
Let $\{(M, g_i)\}_{i=1}^\infty$ be a collapsing sequence associated to an affine fiber bundle $M \to B$ with $\dim(B) = \dim(M) - 1$. Suppose that $\lim_{i\to\infty} \lambda_{p,j}(M, g_i) = 0$ for some $j > \mathrm{b}_p(M)$. Let $\mathscr{O}$ be the orientation bundle of $M \to B$, a flat real line bundle on B. Let $\chi \in \mathrm{H}^2(B;\mathscr{O})$ be the Euler class of the circle bundle $M \to B$. Let $\mathscr{M}_\chi$ be multiplication by χ. Then $\mathscr{M}_\chi : \mathrm{H}^{p-1}(B;\mathscr{O}) \to \mathrm{H}^{p+1}(B;\mathbb{R})$ is nonzero or $\mathscr{M}_\chi : \mathrm{H}^{p-2}(B;\mathscr{O}) \to \mathrm{H}^p(B;\mathbb{R})$ is nonzero.

Finally, we give a class of examples for which the inequality in (1.6) is an equality.

THEOREM 6
Suppose that $M \to B$ is an affine fiber bundle with a smooth base B and fiber $Z = \widehat{Z}/F$, where $\widehat{Z}$ is a nilmanifold $\widehat{\Gamma}\backslash N$ and F is a finite group. Let

$$\mathfrak{n} = \mathfrak{n}'_{[0]} \supset \mathfrak{n}'_{[1]} \supset \cdots \supset \mathfrak{n}'_{[S]} \supset 0 \tag{1.8}$$

be the lower central series of the Lie algebra $\mathfrak{n}$. Let $\mathfrak{c}(\mathfrak{n})$ be the center of $\mathfrak{n}$. For $0 \leq k \leq S$, put

$$\mathfrak{n}_{[k]} = \mathfrak{n}'_{[k]} + \mathfrak{c}(\mathfrak{n}) \tag{1.9}$$

and put $\mathfrak{r}_{[k]} = \mathfrak{n}_{[k]}/\mathfrak{n}_{[k+1]}$. *Let* P *be the principal* $\mathrm{Aff}(Z)$*-bundle such that* $M = P \times_{\mathrm{Aff}(Z)} Z$. *Let* $G = \bigoplus_b G^b$ *be the* $\mathbb{Z}$*-graded flat vector bundle on* B *with*

$$G^b = P \times_{\mathrm{Aff}(Z)} \left(\Lambda^b \Big(\bigoplus_{k=0}^{S} \mathfrak{r}_{[k]}^* \Big) \right)^F. \tag{1.10}$$

Then for any $0 \le p \le \dim(M)$, M *has* $\sum_{a+b=p} \dim(\mathrm{H}^a(B; G^b))$ *small eigenvalues of the* p*-form Laplacian.*

The structure of the paper is as follows. In Section 2 we give examples of collapsing which show that the superconnection formalism is necessary. In Section 3 we give some background information about infranilmanifolds Z and show that the orthogonal projection onto the parallel forms of Z is independent of the choice of parallel Riemannian metric. In Section 4 we give a detailed analysis of the spectrum of the differential form Laplacian on an infranilmanifold. In Section 5 we show that the eigenvalues of the superconnection Laplacian are continuous with respect to the superconnection, the Riemannian metric, and the Euclidean inner product. We then analyze the differential form Laplacian on a Riemannian affine fiber bundle and prove Theorem 1. In Section 6 we consider manifolds M that are Gromov-Hausdorff close to a smooth manifold B and prove Theorems 2, 3, and 4. Section 7 uses the compactness results to prove Theorem 5 and Corollaries 1 – 5. We then prove Theorem 6. More detailed descriptions appear at the beginnings of the sections.

2. Examples

For notation in this paper, if G is a group that acts on a set X, we let X^G denote the set of fixed points. If B is a smooth manifold and E is a smooth vector bundle on B, we let $\Omega(B; E)$ denote the smooth E-valued differential forms on B. If $\mathfrak{n}$ is a nilpotent Lie algebra on which a finite group F acts by automorphisms, then $\mathfrak{n}^*$ denotes the dual space, $\Lambda^*(\mathfrak{n}^*)$ denotes the exterior algebra of the dual space, and $\Lambda^*(\mathfrak{n}^*)^F$ denotes the F-invariant subspace of the exterior algebra.

Example 1

Let N be a simply connected, connected nilpotent Lie group, such as the 3-dimensional Heisenberg group. Let $\mathfrak{n}$ be its Lie algebra of left-invariant vector fields, let g^{TN} be a left-invariant Riemannian metric on N, and let $\triangle^N$ be the corresponding Laplacian on $\Omega^*(N)$. (For simplicity of notation, we omit reference to the form degree p.) The left-invariant differential forms $\Lambda^*(\mathfrak{n}^*)$ form a subcomplex of $\Omega^*(N)$ with differential $d^{\mathfrak{n}}$, on which $\triangle^N$ restricts to a finite-dimensional operator $\triangle^{\mathfrak{n}}$. If Γ is a lattice in N, then the left-invariant forms on N push down to forms on $Z = \Gamma\backslash N$, giving a subcomplex of $\Omega^*(Z)$ which is isomorphic to $\Lambda^*(\mathfrak{n}^*)$. One knows that $\mathrm{H}^*(Z; \mathbb{R})$

is isomorphic to the cohomology of this subcomplex (see [28, Corollary 7.28]). We see that the spectrum $\sigma(\triangle^{\mathfrak{n}})$ of $\triangle^{\mathfrak{n}}$ is contained in the spectrum $\sigma(\triangle^{Z})$ of the differential form Laplacian on $\Omega^*(Z)$.

Suppose that $\{\Gamma_i\}_{i=1}^{\infty}$ is a sequence of lattices in N with quotients $Z_i = \Gamma_i \backslash N$ such that $\lim_{i\to\infty} \mathrm{diam}(Z_i) = 0$. Then $\{Z_i\}_{i=1}^{\infty}$ obviously converges to a point, with bounded sectional curvature in the collapse. We see that there are eigenvalues of $\triangle^{Z_i}$ which are constant in i, namely, those that come from $\sigma(\triangle^{\mathfrak{n}})$. By Proposition 2, the other eigenvalues go to infinity as $i \to \infty$. If N is nonabelian, then there are positive eigenvalues of $\triangle^{Z_i}$ which are constant in i.

In terms of Theorem 2, B is a point, $E^* = \Lambda^*(\mathfrak{n}^*)$, and $A' = A'_{[0]} = d^{\mathfrak{n}}$. This shows that the term $A'_{[0]}$ does appear in examples. In fact, $A'_{[0]} = 0$ if and only if N is abelian.

By choosing different left-invariant metrics on N, we can make $\sigma(\triangle^{\mathfrak{n}})$ arbitrarily close to zero while keeping the sectional curvature bounded. (In fact, the sectional curvature goes to zero.) This is a special case of Theorem 6. We see that, in general, there is no nontrivial lower bound on the first positive eigenvalue of $\triangle^{Z}$ under the assumptions of bounded sectional curvature and bounded diameter.

Example 2

Let M be a compact manifold with a free T^k-action. Let g^{TM} be a T^k-invariant Riemannian metric on M. Then for $\epsilon > 0$, there is a Riemannian metric g_{ϵ}^{TM} obtained by multiplying g^{TM} in the direction of the T^k-orbit by ϵ. Clearly, $\lim_{\epsilon\to 0}(M, g_{\epsilon}^{TM}) = M/T^k$, the collapse being with bounded sectional curvature (see [13]). This collapsing is an example of the so-called adiabatic limit, for which the eigenvalues of the differential form Laplacian have been studied in [5], [16], [18], and [26]. Let E be the flat "cohomology" vector bundle on M/T^k with fiber $\mathrm{H}^*(T^k; \mathbb{R})$; in fact, it is a trivial bundle. The results of the cited references imply that as $\epsilon \to 0$, the eigenvalues of $\triangle^{M}$ which remain finite approach those of the Laplacian on $\Omega^*(M/T^k; E)$. In particular, the number of eigenvalues of the p-form Laplacian which go to zero as $\epsilon \to 0$ is $\sum_{a+b=p} \dim\left(\mathrm{H}^a(M/T^k; E^b)\right)$, which is also the dimension of the E_2-term of the Leray spectral sequence for computing $\mathrm{H}^p(M; \mathbb{R})$. This is consistent with Theorems 5 and 6. Let A'_{ϵ} be the superconnection on E coming from Theorem 2, using g_{ϵ}^{TM}. Then $\lim_{\epsilon\to 0} A'_{\epsilon} = \nabla^E$.

Example 3

Suppose that M is the total space of an oriented circle bundle, with an S^1-invariant Riemannian metric. For $k \in \mathbb{Z}^+$, consider the subgroup $\mathbb{Z}_k \subset S^1$. Then $\lim_{k\to\infty} M/\mathbb{Z}_k = M/S^1$, the collapse obviously being with bounded sectional curvature. By Fourier analysis, one finds that as $k \to \infty$, the spectrum of $\triangle^{M/\mathbb{Z}_k}$ ap-

proaches the spectrum of the Laplacian on S^1-invariant (not necessarily basic) differential forms on M. In terms of Theorem 2, $B = M/S^1$ and E is the direct sum of two trivial $\mathbb{R}$-bundles on B. Let T be the curvature 2-form of the fiber bundle $M \to M/S^1$. Then one finds that the Laplacian acting on S^1-invariant forms on M is isomorphic to the Laplacian $\triangle^E = A'\left(A'\right)^* + \left(A'\right)^* A'$, where A' is the extension of the superconnection on $C^\infty(B; E) = C^\infty(B) \oplus C^\infty(B)$ given by

$$A' = \begin{pmatrix} \nabla^{E^0} & T \\ 0 & \nabla^{E^1} \end{pmatrix}. \tag{2.1}$$

Here ∇^{E^0} and ∇^{E^1} are product connections. This shows that the term $A'_{[2]}$ does appear in examples. Note that if M is simply connected, then $\{M/\mathbb{Z}_k\}_{k=1}^\infty$ are mutually nondiffeomorphic.

3. Infranilmanifolds

In this section we first recall some basic facts about infranilmanifolds. Then in Proposition 1 we show that the orthogonal projection onto the parallel differential forms of Z comes from an averaging technique and so is independent of the choice of parallel metric on Z, a result that is crucial in what follows.

Let N be a simply connected, connected nilpotent Lie group. Following [12], when N acts on a manifold on the left, we denote it by N_L, and when it acts on a manifold on the right, we denote it by N_R. As in [12], let us recall the elementary but confusing point that the right action of N on N generates left-invariant vector fields, while the left action of N on N generates right-invariant vector fields.

There is a flat linear connection ∇^{aff} on N which is characterized by the fact that left-invariant vector fields are parallel. The group $\text{Aff}(N)$ of diffeomorphisms of N which preserve ∇^{aff} is isomorphic to $N_L \tilde{\times} \text{Aut}(N)$.

Suppose that Γ is a discrete subgroup of $\text{Aff}(N)$ which acts freely and cocompactly on N, with $\Gamma \cap N_L$ of finite index in Γ. Then the quotient space $Z = \Gamma\backslash N$ is an infranilmanifold modeled on N. We have the short exact sequences

$$1 \longrightarrow N_L \longrightarrow \text{Aff}(N) \xrightarrow{p} \text{Aut}(N) \longrightarrow 1 \tag{3.1}$$

and

$$1 \longrightarrow \Gamma \cap N_L \longrightarrow \Gamma \xrightarrow{p} p(\Gamma) \longrightarrow 1. \tag{3.2}$$

Put $\widehat{\Gamma} = \Gamma \cap N_L$ and $F = p(\Gamma)$. Then F is a finite group. There is a normal cover $\widehat{Z} = \widehat{\Gamma}\backslash N$ of Z with covering group F.

The connection ∇^{aff} descends to a flat connection on TZ, which we again denote by ∇^{aff}. Let $\text{Aff}(Z)$ denote the affine group of Z, let $\text{Aff}_0(Z)$ denote the connected component of the identity in $\text{Aff}(Z)$, and let $\text{aff}(Z)$ denote the affine Lie algebra of Z. Any element of $\text{Aff}(Z)$ can be lifted to an element of $\text{Aff}(N)$. That is,

$\mathrm{Aff}(Z) = \Gamma\backslash(N_\Gamma \mathrm{Aff}(N))$, where $N_\Gamma \mathrm{Aff}(N)$ is the normalizer of Γ in $\mathrm{Aff}(N)$. Similarly, $\mathrm{Aff}_0(Z) = C(\Gamma)\backslash(C_\Gamma \mathrm{Aff}(N))$, where $C_\Gamma \mathrm{Aff}(N)$ is the centralizer of Γ in $\mathrm{Aff}(N)$ and $C(\Gamma)$ is the center of Γ. There is a short exact sequence

$$1 \longrightarrow \mathrm{Aff}_0(Z) \longrightarrow \mathrm{Aff}(Z) \longrightarrow \mathrm{Out}(\Gamma) \longrightarrow 1. \tag{3.3}$$

As affine vector fields on Z can be lifted to F-invariant affine vector fields on $\widehat{Z}$, we have $\mathrm{aff}(Z) = \mathrm{aff}(\widehat{Z})^F$. If $C(N)$ denotes the center of N, then $\mathrm{Aff}_0(\widehat{Z}) = (\widehat{\Gamma} \cap C(N_R))\backslash N_R$. In particular, if $\mathfrak{n}$ is the Lie algebra of N, then F acts by automorphisms on $\mathfrak{n}$ and $\mathrm{aff}(Z) = \mathfrak{n}_R^F$.

The F-invariant subspace $\Lambda^*(\mathfrak{n}^*)^F$ of $\Lambda^*(\mathfrak{n}^*)$ is isomorphic to the vector space of differential forms on Z which are parallel with respect to ∇^{aff} or, equivalently, to the $(N_L \widetilde{\times} F)$-invariant subspace of $\Omega^*(N)$.

Let g^{TZ} be a Riemannian metric on Z which is parallel with respect to ∇^{aff}. Such metrics correspond to F-invariant inner products on $\mathfrak{n}$. Let $\mathrm{diam}(Z)$ denote the diameter of Z, let ∇^Z denote the Levi-Civita connection of Z, and let R^Z denote the Riemann curvature tensor of Z.

Let $\mathscr{P} : \Omega^*(Z) \to \Lambda^*(\mathfrak{n}^*)^F$ be the orthogonal projection onto parallel differential forms.

PROPOSITION 1
The orthogonal projection $\mathscr{P}$ is independent of the parallel metric g^{TZ}.

Proof
We first consider the case when $F = \{e\}$, so that Z is a nilmanifold $\Gamma\backslash N$. As N is nilpotent, it has a bi-invariant Haar measure μ. We normalize μ so that $\int_{\Gamma\backslash N} d\mu = 1$. Given $\omega \in \Omega^*(Z)$, let $\widetilde{\omega} \in \Omega^*(N)$ be its pullback to N. If L_g denotes the left action of $g \in N_L$ on N, then for all $\gamma \in \Gamma$,

$$L_{\gamma g}^* \widetilde{\omega} = L_g^* L_\gamma^* \widetilde{\omega} = L_g^* \widetilde{\omega}. \tag{3.4}$$

Hence it makes sense to define $\overline{\widetilde{\omega}} \in \Omega^*(N)$ by

$$\overline{\widetilde{\omega}} = \int_{\Gamma\backslash N_L} (L_g^* \widetilde{\omega})\, d\mu(g). \tag{3.5}$$

For $h \in N_L$,

$$\begin{aligned} L_h^* \overline{\widetilde{\omega}} &= \int_{\Gamma\backslash N_L} (L_h^* L_g^* \widetilde{\omega})\, d\mu(g) = \int_{\Gamma\backslash N_L} (L_{gh}^* \widetilde{\omega})\, d\mu(g) \\ &= \int_{\Gamma\backslash N_L} (L_g^* \widetilde{\omega})\, d\mu(gh^{-1}) = \int_{\Gamma\backslash N_L} (L_g^* \widetilde{\omega})\, d\mu(g) = \overline{\widetilde{\omega}}. \end{aligned} \tag{3.6}$$

Thus $\overline{\widetilde{\omega}}$ is N_L-invariant and, in particular, descends to a form $\overline{\omega} \in \Omega^*(Z)$. Put $P(\omega) = \overline{\omega}$. Then P is idempotent, with $\mathrm{Im}(P)$ being the parallel differential forms. By construction, P is independent of the choice of g^{TZ}. It remains to show that P is self-adjoint. Given $\eta \in \Omega^*(Z)$, let $\widetilde{\eta}$ be its lift to N. Consider the function $f : N \times N \to \mathbb{R}$ given by

$$f(g, n) = \langle \widetilde{\eta}, L_g^* \widetilde{\omega} \rangle_n = \big\langle \widetilde{\eta}(n), \widetilde{\omega}(gn) \big\rangle_n. \tag{3.7}$$

For $\gamma \in \Gamma$, we have $f(\gamma g, n) = f(g\gamma^{-1}, \gamma n) = f(g, n)$. It follows that we can write

$$\langle \eta, P\omega \rangle_Z = \int_{(\Gamma\times\Gamma)\backslash(N\times N)} \langle \widetilde{\eta}, L_g^* \widetilde{\omega} \rangle_n \, d\mu(g) \, d\mu(n), \tag{3.8}$$

where the action of $\Gamma \times \Gamma$ on $N \times N$ is $(\gamma_1, \gamma_2) \cdot (g, n) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 n)$. Changing variable to $g' = gn$, we have

$$\begin{aligned}
\langle \eta, P\omega \rangle_Z &= \int_{\Gamma\backslash N_L} \int_{\Gamma\backslash N_L} \langle \widetilde{\eta}, L^*_{g' n^{-1}} \widetilde{\omega} \rangle_n \, d\mu(g' n^{-1}) \, d\mu(n) \\
&= \int_{\Gamma\backslash N_L} \int_{\Gamma\backslash N_L} \langle \widetilde{\eta}, L^*_{n^{-1}} L^*_{g'} \widetilde{\omega} \rangle_n \, d\mu(g') \, d\mu(n) \\
&= \int_{\Gamma\backslash N_L} \int_{\Gamma\backslash N_L} \langle L_n^* \widetilde{\eta}, L^*_{g'} \widetilde{\omega} \rangle_e \, d\mu(g') \, d\mu(n) \\
&= \int_{\Gamma\backslash N_L} \int_{\Gamma\backslash N_L} \langle L^*_{g'} \widetilde{\omega}, L_n^* \widetilde{\eta} \rangle_e \, d\mu(n) \, d\mu(g') \\
&= \langle \omega, P\eta \rangle_Z = \langle P\eta, \omega \rangle_Z.
\end{aligned} \tag{3.9}$$

Thus P is self-adjoint.

In the case of general F, we can apply the above argument equivariantly on $\widehat{Z}$ with respect to F. As F acts isometrically on $\widehat{Z}$, it commutes with the orthogonal projection $\mathscr{P}$ on $\widehat{Z}$. As F preserves μ, it also commutes with the averaging operator P on $\widehat{Z}$. The proposition follows. □

4. Eigenvalue estimates on infranilmanifolds

In this section we show in Proposition 2 that if an infranilmanifold Z has bounded sectional curvature and a diameter that goes to zero, then all of the eigenvalues of $\triangle^Z$ go to infinity, except for those that correspond to eigenforms which are parallel on Z.

Let N be a simply connected, connected n-dimensional nilpotent Lie group with a left-invariant Riemannian metric. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $\mathfrak{n}$. Define the structure constants of $\mathfrak{n}$ by $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$. Take the corresponding left-invariant basis $\{e_i\}_{i=1}^n$ of TN, with dual basis of 1-forms $\{\tau^i\}_{i=1}^n$. Then the components $\omega^i_{\ j} = \sum_{k=1}^n \omega^i_{\ jk} \tau^k$ of the Levi-Civita connection 1-form $\omega = \sum_k \omega_k \tau^k$ are

the constant matrices

$$\omega^i_{jk} = -\frac{1}{2}(c^i_{jk} - c^j_{ik} - c^k_{ij}). \tag{4.1}$$

LEMMA 1

Let κ denote the scalar curvature of Z. Then

$$\sum_{i,j,k=1}^{n} (c^i_{jk})^2 = -4\kappa. \tag{4.2}$$

Proof

As $\sum_{i,j,k=1}^{n}(c^i_{jk})^2$ is independent of the choice of orthonormal basis, we compute it using a special orthonormal basis. Recall the definition of $\mathfrak{n}_{[k]}$ from (1.9). In particular, $\mathfrak{n}_{[S]} = \mathfrak{c}(\mathfrak{n})$. Following the notation of [20, §6], we take an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathfrak{n}$ such that $e_i \in \mathfrak{n}_{[O(i)]}$ for some nondecreasing function

$$O : \{1, \ldots, n\} \to \{0, \ldots, S\}, \tag{4.3}$$

and $e_i \perp \mathfrak{n}_{[O(i)+1]}$.

For a general Riemannian manifold, we have the structure equations

$$\begin{aligned} d\tau^i &= -\sum_j \omega^i_j \wedge \tau^j, \\ \Omega^i_j &= d\omega^i_j + \sum_m \omega^i_m \wedge \omega^m_j. \end{aligned} \tag{4.4}$$

Then

$$\begin{aligned} \Omega^i_j &= d\sum_l \omega^i_{jl}\tau^l + \sum_m \omega^i_m \wedge \omega^m_j \\ &= \sum_{k,l}(e_k\omega^i_{jl})\tau^k \wedge \tau^l + \sum_m \omega^i_{jm} d\tau^m + \sum_{k,l,m} \omega^i_{mk}\omega^m_{jl}\tau^k \wedge \tau^l. \end{aligned} \tag{4.5}$$

This gives the Riemann curvature tensor as

$$R^i_{jkl} = e_k\omega^i_{jl} - e_l\omega^i_{jk} + \sum_m [-\omega^i_{jm}\omega^m_{lk} + \omega^i_{jm}\omega^m_{kl} + \omega^i_{mk}\omega^m_{jl} - \omega^i_{ml}\omega^m_{jk}]. \tag{4.6}$$

Then

$$\begin{aligned} \kappa &= \sum_{i,j}(e_i\omega^i_{jj} - e_j\omega^i_{ji}) + \sum_{i,j,m}[-\omega^i_{jm}\omega^m_{ji} + \omega^i_{jm}\omega^m_{ij} + \omega^i_{mi}\omega^m_{jj} - \omega^i_{mj}\omega^m_{ji}] \\ &= \sum_{i,j}(e_i\omega^i_{jj} - e_j\omega^i_{ji}) + \sum_{i,j,m}[\omega^i_{jm}\omega^m_{ij} + \omega^i_{mi}\omega^m_{jj}]. \end{aligned} \tag{4.7}$$

In our case, the components of the connection matrix are constant. Also, as $\mathfrak{n}$ is nilpotent,

$$\omega^i{}_{jj} = c^j_{ij} = 0. \tag{4.8}$$

Then one obtains

$$\kappa = -\sum_{i,j,k=1}^{n} \omega^i{}_{jk}\omega^i{}_{kj}. \tag{4.9}$$

Separating $\omega^i{}_{jk}$ into components that are symmetric or antisymmetric in j and k, and using (4.1), we obtain

$$\begin{aligned}\kappa &= -\frac{1}{4}\sum_{i,j,k=1}^{n}(c^j_{ik}+c^k_{ij})^2 + \frac{1}{4}\sum_{i,j,k=1}^{n}(c^i{}_{jk})^2 \\ &= -\sum_{i,j,k=1}^{n}\Big[\frac{1}{2}c^j_{ik}c^k_{ij} + \frac{1}{4}(c^i{}_{jk})^2\Big].\end{aligned} \tag{4.10}$$

As $\mathfrak{n}$ is nilpotent, it follows that $c^j_{ik}c^k_{ij} = 0$. This proves the lemma. □

Let Z be an infranilmanifold with an affine-parallel metric. Let $\triangle^Z$ denote the Laplacian acting on $\Omega^*(Z)$. Let $\triangle^{\mathrm{inv}}$ be the finite-dimensional Laplacian acting on $\Lambda^*(\mathfrak{n}^*)^F$.

PROPOSITION 2
There are positive constants A and A', depending only on $\dim(Z)$, *such that if* $\|R^Z\|_\infty \operatorname{diam}(Z)^2 \le A'$, *then the spectrum* $\sigma(\triangle^Z)$ *of* $\triangle^Z$ *satisfies*

$$\sigma(\triangle^Z) \cap \big[0, A\operatorname{diam}(Z)^{-2}\big) = \sigma(\triangle^{\mathrm{inv}}) \cap \big[0, A\operatorname{diam}(Z)^{-2}\big). \tag{4.11}$$

Proof
Recall the definition of $\mathscr{P}$ from Proposition 1. It is enough to show that under the hypotheses of the present proposition, the spectrum of $\triangle^Z$ on $\operatorname{Ker}(\mathscr{P})$ is bounded below by $A\operatorname{diam}(Z)^{-2}$.

As $\widehat{Z}$ isometrically covers Z with covering group F, the spectrum of $\triangle^Z$ on $\operatorname{Ker}(\mathscr{P}) \subset \Omega^*(Z)$ is contained in the spectrum of $\triangle^{\widehat{Z}}$ on $\operatorname{Ker}(\mathscr{P}) \subset \Omega^*(\widehat{Z})$.

LEMMA 2
There is a function $\eta : \mathbb{N} \to \mathbb{N}$ *such that*

$$\operatorname{diam}(\widehat{Z}) \le \eta\big(|F|\big)\operatorname{diam}(Z). \tag{4.12}$$

Proof
Let $\widehat{z}_1, \widehat{z}_2 \in \widehat{Z}$ be such that $\mathrm{diam}(\widehat{Z}) = d(\widehat{z}_1, \widehat{z}_2)$. It is easy to see that $d(\widehat{z}_1, F \cdot \widehat{z}_2) \leq \mathrm{diam}(Z)$. Let $z_2 \in Z$ be the projection of $\widehat{z}_2 \in \widehat{Z}$. Then it is enough to bound $\| \cdot \|_{\mathrm{geo}}$ from above on $\pi_1(Z, z_2) \cong F$, that is, to bound the minimal lengths of curves in the classes of $\pi_1(Z, z_2)$. From [23, Proposition 3.22], there is a set of generators of $\pi_1(Z, z_2)$ on which $\| \cdot \|_{\mathrm{geo}}$ is bounded above by $2\,\mathrm{diam}(Z)$. Given $r \in \mathbb{N}$, there is a finite number of groups of order r, up to isomorphism, and each of these groups has a finite number of generating sets. The lemma follows. □

Furthermore, there is a universal bound $|F| \leq \mathrm{const.}(\dim(Z))$ (see [9]). Hence, without loss of generality, we may assume that $F = \{e\}$, so that Z is a nilmanifold $\Gamma \backslash N$.

Let E^i denote exterior multiplication on $\Omega^*(Z)$ by τ^i, and let I^i denote interior multiplication by e_i. From the Bochner formula, if $\eta \in \Omega^*(Z)$, then

$$\langle \eta, \triangle^Z \eta \rangle_Z = \langle \nabla^Z \eta, \nabla^Z \eta \rangle_Z + \sum_{ijkl} \int_Z R^Z_{ijkl} \langle E^i I^j \eta, E^k I^l \eta \rangle \, d\,\mathrm{vol}_Z \,. \tag{4.13}$$

Using the left-invariant vector fields on N, there is an isometric isomorphism

$$\Omega^*(Z) \cong C^\infty(Z) \otimes \Lambda^*(\mathfrak{n}^*). \tag{4.14}$$

With respect to this isomorphism,

$$\nabla^{TZ}_{e_i} = (e_i \otimes \mathrm{Id}) + \Big(\mathrm{Id} \otimes \sum_{j,k} \omega^j_{ki} E^j I^k \Big), \tag{4.15}$$

where E^j and I^k now act on $\Lambda^*(\mathfrak{n}^*)$. It follows that

$$\langle \nabla^Z \eta, \nabla^Z \eta \rangle_Z \geq \sum_i \big\langle (e_i \otimes \mathrm{Id})\eta, (e_i \otimes \mathrm{Id})\eta \big\rangle_Z - \sum_i \Big| \sum_{j,k} \omega^j_{ki} E^j I^k \eta \Big|^2_Z. \tag{4.16}$$

Let $\triangle^Z_0$ be the ordinary Laplacian on $C^\infty(Z)$. With respect to (4.14), consider the operator $\triangle^Z_0 \otimes \mathrm{Id}$. We have

$$\big\langle \eta, (\triangle^Z_0 \otimes \mathrm{Id})\eta \big\rangle_Z = \sum_i \big\langle (e_i \otimes \mathrm{Id})\eta, (e_i \otimes \mathrm{Id})\eta \big\rangle_Z. \tag{4.17}$$

Using (4.1), (4.13), (4.16), (4.17), and Lemma 1, we obtain

$$\langle \eta, \triangle^Z \eta \rangle_Z \geq \big\langle \eta, (\triangle^Z_0 \otimes \mathrm{Id})\eta \big\rangle_Z - \mathrm{const.}\, \|R^Z\|_\infty |\eta|^2_Z. \tag{4.18}$$

In terms of (4.14), $\mathrm{Ker}(\mathscr{P}) \cong 1^\perp \otimes \Lambda^*(\mathfrak{n}^*)$, where 1 denotes the constant function on Z. Thus if $\eta \in \mathrm{Ker}(\mathscr{P})$, then $\langle \eta, (\triangle^Z_0 \otimes \mathrm{Id})\eta \rangle_Z \geq \lambda_{0,2} |\eta|^2_Z$, where $\lambda_{0,2}$ is the first positive eigenvalue of the function Laplacian on Z. There is a lower bound

$$\lambda_{0,2} \geq \mathrm{diam}(Z)^{-2} \, f\big(\|R^Z\|_\infty \, \mathrm{diam}(Z)^2 \big) \tag{4.19}$$

for some smooth function f with $f(0) > 0$ (see [3]). Thus the spectrum of $\triangle^Z$ on $\mathrm{Ker}(\mathscr{P})$ is bounded below by

$$\mathrm{diam}(Z)^{-2}\big[\, f\big(\|R^Z\|_\infty \,\mathrm{diam}(Z)^2\big) - \text{const.}\, \|R^Z\|_\infty \,\mathrm{diam}(Z)^2\big]. \tag{4.20}$$

Taking $A = (3/4)\, f(0)$, the proposition follows. □

5. Affine fiber bundles

In this section we first show that the eigenvalues of a superconnection Laplacian are continuous with respect to the superconnection, the Riemannian metric, and the Euclidean inner product. We then construct the superconnection A' associated to an affine fiber bundle $M \to B$ and prove Theorem 1.

Let B be a smooth connected closed Riemannian manifold. Let $E = \bigoplus_{j=0}^m E^j$ be a $\mathbb{Z}$-graded real vector bundle on B. For background information about superconnections, we refer to [4, Chapter 1.4], [7], [8], and [27]. Let A' be a degree-1 superconnection on E. That is, A' is an $\mathbb{R}$-linear map from $C^\infty(B; E)$ to $\Omega(B; E)$ with a decomposition

$$A' = \sum_{k=0}^{\dim(B)} A'_{[k]}, \tag{5.1}$$

where

- $A'_{[1]}$ is a connection ∇^E on E which preserves the $\mathbb{Z}$-grading;
- for $k \neq 1$, $A'_{[k]} \in \Omega^k(B; \mathrm{Hom}(E^*, E^{*+1-k}))$.

We can extend A' to an $\mathbb{R}$-linear map on $\Omega(B; E)$ using the Leibniz rule. We assume that A' is flat in the sense that

$$(A')^2 = 0. \tag{5.2}$$

Let h^E be a Euclidean inner product on E such that E^j is orthogonal to $E^{j'}$ if $j \neq j'$. Let $(A')^*$ be the adjoint superconnection with respect to h^E, and put

$$\triangle^E = A'(A')^* + (A')^* A'. \tag{5.3}$$

Then $\triangle^E$ preserves the total $\mathbb{Z}$-grading on $\Omega(B; E)$ and decomposes with respect to the grading as $\triangle^E = \bigoplus_p \triangle^E_p$. By elliptic theory, $\triangle^E_p$ has a discrete spectrum.

If g_1^{TB} and g_2^{TB} are two Riemannian metrics on B and $\epsilon \geq 0$, we say that g_1^{TB} and g_2^{TB} are ϵ-close if

$$e^{-\epsilon} g_2^{TB} \leq g_1^{TB} \leq e^{\epsilon} g_2^{TB}. \tag{5.4}$$

Similarly, if h_1^E and h_2^E are two Euclidean inner products on E, we say that h_1^E and h_2^E are ϵ-close if

$$e^{-\epsilon} h_2^E \leq h_1^E \leq e^{\epsilon} h_2^E. \tag{5.5}$$

If $S_1 = \{\lambda_{1,j}\}$ and $S_2 = \{\lambda_{2,j}\}$ are two countable nondecreasing ordered sets of nonnegative real numbers, then we say that S_1 and S_2 are ϵ-close if for all j,

$$e^{-\epsilon}\lambda_{2,j} \leq \lambda_{1,j} \leq e^{\epsilon}\lambda_{2,j}. \tag{5.6}$$

For simplicity we omit the subscript p, the form degree, in this section when its role is obvious.

LEMMA 3
There is an integer $J = J(\dim(B)) > 0$ such that if g_1^{TB} and g_2^{TB} are ϵ-close, and h_1^E and h_2^E are ϵ-close, then the corresponding Laplacians $\triangle_1^E$ and $\triangle_2^E$ have spectra that are $J\epsilon$-close.

Proof
As in [17, Proposition 3.1], using a trick apparently first due to Cheeger, we can write the spectrum of $\triangle^E$ on $\mathrm{Im}((A')^*)$ as

$$\lambda_j = \inf_V \sup_{\eta \in V - \{0\}} \sup_{\theta \in \Omega(B;E)} \left\{ \frac{\langle \eta, \eta \rangle}{\langle \theta, \theta \rangle} : \eta = A'\theta \right\}, \tag{5.7}$$

where V ranges over j-dimensional subspaces of $\mathrm{Im}(A')$. As the Riemannian metric and Euclidean inner product enter only in defining $\langle \cdot, \cdot \rangle$, the lemma follows as in [17]. □

We also need a result about how the spectrum of $\triangle^E$ depends on the superconnection A'. Given $X \in \Omega(B; \mathrm{End}(E))$, let $\|X\|$ be the operator norm for the action of X on the L^2-completion of $\Omega(B; E)$. If A'_1 and A'_2 are two superconnections as above, then $A'_1 - A'_2 \in \Omega(B; \mathrm{End}(E))$. Fix g^{TB} and h^E.

LEMMA 4
For all $j \in \mathbb{Z}^+$,

$$\left|\lambda_j (A'_1)^{1/2} - \lambda_j (A'_2)^{1/2}\right| \leq (2 + \sqrt{2})\|A'_1 - A'_2\|. \tag{5.8}$$

Proof
Put $x = \|A'_1 - A'_2\|$. If $\omega \in \Omega(B; E)$ is nonzero, then

$$\left| \frac{|A'_1\omega|}{|\omega|} - \frac{|A'_2\omega|}{|\omega|} \right| \leq \frac{|(A'_1 - A'_2)\omega|}{|\omega|} \leq x \tag{5.9}$$

and

$$\left| \frac{|(A'_1)^*\omega|}{|\omega|} - \frac{|(A'_2)^*\omega|}{|\omega|} \right| \leq \frac{|((A'_1)^* - (A'_2)^*)\omega|}{|\omega|} \leq x. \tag{5.10}$$

Define $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ by

$$\vec{v}_i = \Big(\frac{|A_i'\omega|}{|\omega|}, \frac{|(A_i')^*\omega|}{|\omega|}\Big). \tag{5.11}$$

Then (5.10) and (5.11) imply that

$$\big|(\|\vec{v}_2\| - \|\vec{v}_1\|)\big| \leq \|\vec{v}_2 - \vec{v}_1\| \leq \sqrt{2}x. \tag{5.12}$$

Hence

$$\|\vec{v}_2\|^2 - \|\vec{v}_1\|^2 = (\|\vec{v}_2\| - \|\vec{v}_1\|) \cdot (\|\vec{v}_2\| + \|\vec{v}_1\|) \leq \sqrt{2}x(2\|\vec{v}_2\| + \sqrt{2}x), \tag{5.13}$$

so

$$\begin{aligned} \|\vec{v}_1\|^2 &\geq \|\vec{v}_2\|^2 - 2\sqrt{2}x\|\vec{v}_2\| - 2x^2 \\ &= (\|\vec{v}_2\| - \sqrt{2}x)^2 - 4x^2. \end{aligned} \tag{5.14}$$

Thus

$$\|\vec{v}_1\|^2 \geq \max\big(0, (\|\vec{v}_2\| - \sqrt{2}x)^2 - 4x^2\big), \tag{5.15}$$

or equivalently,

$$\frac{\langle\omega, \triangle_{A_1'}\omega\rangle}{\langle\omega, \omega\rangle} \geq \max\left(0, \left(\left(\frac{\langle\omega, \triangle_{A_2'}\omega\rangle}{\langle\omega, \omega\rangle}\right)^{1/2} - \sqrt{2}x\right)^2 - 4x^2\right). \tag{5.16}$$

The minmax characterization of eigenvalues

$$\lambda_j(A') = \inf_V \sup_{\omega \in V - \{0\}} \left\{\frac{\langle\omega, \triangle_{A'}\omega\rangle}{\langle\omega, \omega\rangle}\right\}, \tag{5.17}$$

where V ranges over j-dimensional subspaces of $\Omega(B; E)$, implies

$$\lambda_j(A_1') \geq \max\big(0, (\lambda_j^{1/2}(A_2') - \sqrt{2}x)^2 - 4x^2\big). \tag{5.18}$$

An elementary calculation then gives

$$\lambda_j(A_1')^{1/2} - \lambda_j(A_2')^{1/2} \geq -(2 + \sqrt{2})x. \tag{5.19}$$

Symmetrizing in A_1' and A_2', the proposition follows. □

Let M be a closed manifold that is the total space of an affine fiber bundle, as in Definition 1. Let $T^H M$ be a horizontal distribution on M so that the corresponding holonomy on B lies in $\mathrm{Aff}(Z)$. If $m \in Z_b$, then using $T^H M$, we can write $\Lambda^*(T_m^*M) \cong \Lambda^*(T_b^*B)\widehat{\otimes}\Lambda^*(T_m^*Z_b)$. That is, we can compose differential forms on M into their horizontal and vertical components. Correspondingly, there is an infinite-dimensional $\mathbb{Z}$-graded real vector bundle W on B such that $\Omega^*(M) \cong \Omega(B; W)$ (see

[8, Section III(a)]). A fiber W_b of W is isomorphic to $\Omega^*(Z_b)$. We call $C^\infty(B; W)$ the vertical differential forms. The exterior derivative $d^M : \Omega^*(M) \to \Omega^*(M)$, when considered to be an operator $d^M : \Omega(B; W) \to \Omega(B; W)$, is the extension to $\Omega(B; W)$ of a flat degree-1 superconnection on W. From [8, Proposition 3.4], we can write the superconnection as

$$d^Z + \nabla^W + i_T, \tag{5.20}$$

where

- $d^Z \in C^\infty(B; \mathrm{Hom}(W^*; W^{*+1}))$ is vertical differentiation,
- $\nabla^W : C^\infty(B; W) \to \Omega^1(B; W)$ comes from Lie differentiation in the horizontal direction, and
- $i_T \in \Omega^2(B; \mathrm{Hom}(W^*; W^{*-1}))$ is interior multiplication by the curvature 2-form $T \in \Omega^2(M; TZ)$ of $T^H M$.

Acting on $\Omega^*(M)$, we have

$$d^M = d^Z + d^W + i_T, \tag{5.21}$$

where $d^W : \Omega^*(B; W) \to \Omega^{*+1}(B; W)$ is exterior differentiation on B using ∇^W.

Let E be the finite-dimensional subbundle of W such that E_b consists of the elements of $\Omega^*(Z_b)$ which are parallel on Z_b. The fibers of E are isomorphic to $\Lambda^*(\mathfrak{n}^*)^F$, and $C^\infty(B; E)$ is isomorphic to the vertical differential forms on M whose restrictions to the fibers are parallel. Furthermore, the superconnection (5.20) restricts to a flat degree-1 superconnection A' on E, as exterior differentiation on M preserves the space of fiberwise-parallel differential forms. From (5.20),

$$A' = d^{\mathfrak{n}} + \nabla^E + i_T, \tag{5.22}$$

where $d^{\mathfrak{n}}$ is the differential on $\Lambda^*(\mathfrak{n}^*)^F$ and ∇^E comes from $T^H M$ through the action of $\mathrm{Aff}(Z)$ on $\Lambda^*(\mathfrak{n}^*)^F$. Acting on $\Omega(B; E)$, we have

$$A' = d^{\mathfrak{n}} + d^E + i_T, \tag{5.23}$$

where d^E is exterior differentiation on $\Omega(B; E)$ using ∇^E.

Remark. The connection ∇^E is generally not flat. As A' is flat, we have

$$(\nabla^E)^2 = -(d^{\mathfrak{n}} i_T + i_T d^{\mathfrak{n}}). \tag{5.24}$$

Thus the curvature of ∇^E is given by Lie differentiation with respect to the (negative of the) curvature 2-form T. More geometrically, given $b \in B$, let γ be a loop in B starting from b, and let $h(\gamma) \in \mathrm{Aff}(Z_b)$ be the holonomy of the connection $T^H M$ around γ. Then the holonomy of ∇^E around γ is the action of $h(\gamma)$ on the fiber E_b.

In particular, the infinitesimal holonomy of ∇^E lies in the image of the Lie algebra $\mathrm{aff}(Z)$ in $\mathrm{End}(E_b)$. From the discussion after (3.3), $\mathrm{aff}(Z)$ lies in $\mathfrak{n}_R$. As the elements of E_b are N_L-invariant forms on N, they are generally not annihilated by $\mathrm{aff}(Z)$.

Suppose in addition that M is a Riemannian affine fiber bundle, as in Definition 1. Then g^{TZ} induces an L^2 inner product h^W on W and a Euclidean inner product h^E on E. Let $\mathrm{diam}(Z)$ denote the maximum diameter of the fibers $\{Z_b\}_{b\in B}$ in the intrinsic metric, and let Π denote the second fundamental forms of the fibers. From g^{TZ}, T^HM, and g^{TB}, we obtain a Riemannian metric g^{TM} on M. Let $\triangle^M$ denote the Laplacian acting on $\Omega^*(M)$, and define $\triangle^E$, acting on $\Omega(B;E)$, as in (5.3). Let R^M denote the Riemann curvature tensor of g^{TM}.

Let $\mathscr{P}^{\mathrm{fib}}$ be fiberwise orthogonal projection from $\Omega(B;W)$ to $\Omega(B;E)$. We claim that $\mathscr{P}^{\mathrm{fib}}$ commutes with d^M. Looking at (3.5), $\mathscr{P}^{\mathrm{fib}}$ clearly commutes with d^Z. Using the fact that the holonomy of T^HM lies in $\mathrm{Aff}(Z)$, it follows from (3.5) and the proof of Proposition 1 that $\mathscr{P}^{\mathrm{fib}}$ commutes with ∇^W. As T takes values in parallel vector fields on Z, it follows from (3.5) and the proof of Proposition 1 that $\mathscr{P}^{\mathrm{fib}}$ commutes with i_T. Thus $\mathscr{P}^{\mathrm{fib}}$ commutes with d^M. As the fiberwise metrics are parallel on the fibers, it follows that $\mathscr{P}^{\mathrm{fib}}$ also commutes with $(d^M)^*$.

Then with respect to the decomposition $\Omega^*(M) = \mathrm{Im}(\mathscr{P}^{\mathrm{fib}}) \oplus \mathrm{Ker}(\mathscr{P}^{\mathrm{fib}})$, $\triangle^M$ is isomorphic to $\triangle^E \oplus \triangle^M\big|_{\mathrm{Ker}(\mathscr{P}^{\mathrm{fib}})}$.

Proof of Theorem 1

From Proposition 2, there is a constant $A > 0$ such that for all $b \in B$, the spectrum of $\triangle^{Z_b}\big|_{\mathrm{Ker}(\mathscr{P})}$ is bounded below by $A \cdot \mathrm{diam}(Z_b)^{-2}$. It suffices to show that there is a constant C as in the statement of the theorem such that

$$\sigma\big(\triangle^M\big|_{\mathrm{Ker}(\mathscr{P}^{\mathrm{fib}})}\big) \subset \big[A\,\mathrm{diam}(Z)^{-2} - C(\|R^M\|_\infty + \|\Pi\|_\infty^2 + \|T\|_\infty^2), \infty\big). \quad (5.25)$$

We use the notation of [8, Section III(c)] to describe the geometry of the fiber bundle M. In particular, lowercase Greek indices refer to horizontal directions, lowercase italic indices refer to vertical directions, and uppercase italic indices refer to either. Let $\{\tau^i\}_{i=1}^{\dim(Z)}$ and $\{\tau^\alpha\}_{\alpha=1}^{\dim(B)}$ be a local orthonormal basis of 1-forms as in [8, Section III(c)], with dual basis $\{e_i\}_{i=1}^{\dim(Z)}$ and $\{e_\alpha\}_{\alpha=1}^{\dim(B)}$. Let E^J be exterior multiplication by τ^J, and let I^J be interior multiplication by e_J. The tensors Π and T are parts of the connection 1-form component $\omega^i_\alpha = \sum_j \omega^i_{\alpha j}\tau^j + \sum_\beta \omega^i_{\alpha\beta}\tau^\beta$ with symmetries

$$\begin{aligned}
&\omega_{\alpha kj} = \omega_{\alpha jk} = -\omega_{j\alpha k} = -\omega_{k\alpha j},\\
&\omega_{\beta\alpha j} = -\omega_{\alpha\beta j} = -\omega_{\alpha j\beta} = \omega_{j\alpha\beta} = \omega_{\beta j\alpha} = -\omega_{j\beta\alpha}.
\end{aligned} \quad (5.26)$$

Given $\eta \in \Omega^*(M)$, the Bochner formula gives

$$\langle \eta, \triangle^M \eta \rangle_M = \langle \nabla^M \eta, \nabla^M \eta \rangle_M + \sum_{PQRS} \int_M R^M_{PQRS} \langle E^P I^Q \eta, E^R I^S \eta \rangle \, d \, \mathrm{vol}_M \, . \tag{5.27}$$

Here

$$\nabla^M : C^\infty(M; \Lambda^* T^* M) \to C^\infty(M; T^* M \otimes \Lambda^* T^* M) \tag{5.28}$$

is, of course, the Levi-Civita connection on M. We can write $\nabla^M = \nabla^V + \nabla^H$, where

$$\nabla^V : C^\infty(M; \Lambda^* T^* M) \to C^\infty(M; T^{\mathrm{vert}} M \otimes \Lambda^* T^* M) \tag{5.29}$$

denotes covariant differentiation in the vertical direction and

$$\nabla^H : C^\infty(M; \Lambda^* T^* M) \to C^\infty(M; T^{\mathrm{hor}} M \otimes \Lambda^* T^* M) \tag{5.30}$$

denotes covariant differentiation in the horizontal direction. Then

$$\begin{aligned} \langle \eta, \triangle^M \eta \rangle_M &= \langle \nabla^V \eta, \nabla^V \eta \rangle_M \\ &\quad + \langle \nabla^H \eta, \nabla^H \eta \rangle_M + \int_M \sum_{PQRS} R^M_{PQRS} \langle E^P I^Q \eta, E^R I^S \eta \rangle \, d \, \mathrm{vol}_M \\ &\geq \langle \nabla^V \eta, \nabla^V \eta \rangle_M - \text{const.} \, \| R^M \|_\infty \langle \eta, \eta \rangle_M \\ &= \int_B \int_{Z_b} \left[|\nabla^V \eta|^2(z) - \text{const.} \, \| R^M \|_\infty |\eta(z)|^2 \right] d \, \mathrm{vol}_{Z_b} \, d \, \mathrm{vol}_B \, . \end{aligned} \tag{5.31}$$

Let

$$\nabla^{TZ} : C^\infty\big(M; \Lambda^*(T^{*,\mathrm{vert}} M)\big) \to C^\infty\big(M; T^* M \otimes \Lambda^*(T^{*,\mathrm{vert}} M)\big) \tag{5.32}$$

denote the Bismut connection acting on $\Lambda^*(T^{*,\mathrm{vert}} M)$ (see [4, Proposition 10.2], [7, Definition 1.6]). On a given fiber Z_b, there is a canonical flat connection on $T^{\mathrm{hor}} M\big|_{Z_b}$. Hence we can use ∇^{TZ} to vertically differentiate sections of $\Lambda^*(T^* M) = \Lambda^*(T^{*,\mathrm{vert}} M) \widehat{\otimes} \Lambda^*(T^{*,\mathrm{hor}} M)$. That is, we can define

$$\nabla^{TZ} : C^\infty\big(M; \Lambda^*(T^* M)\big) \to C^\infty\big(M; T^{*,\mathrm{vert}} M \otimes \Lambda^*(T^* M)\big). \tag{5.33}$$

Explicitly, with respect to a local framing,

$$\nabla^{TZ}_{e_i} \eta = e_i \eta + \sum_{j,k} \omega^j_{ki} E^j I^k \eta \tag{5.34}$$

and

$$\nabla^V_{e_i} \eta = \nabla^{TZ}_{e_i} \eta + \sum_{j\alpha} \omega^j_{\alpha i} E^j I^\alpha \eta + \sum_{\alpha k} \omega^\alpha_{ki} E^\alpha I^k \eta + \sum_{\alpha\beta} \omega^\alpha_{\beta i} E^\alpha I^\beta \eta. \tag{5.35}$$

Then from (5.34) and (5.35),

$$\langle \nabla^V \eta, \nabla^V \eta \rangle_M \geq \int_B \int_{Z_b} \left[|\nabla^{TZ} \eta|^2(z) - \text{const.}(\|T_b\|^2 + \|\Pi_b\|^2)|\eta|^2(z) \right] d\,\text{vol}_{Z_b}\, d\,\text{vol}_B. \quad (5.36)$$

On a given fiber Z_b, for $\eta_{Z_b} \in \Omega^*(Z_b)$, we have

$$\langle \eta_{Z_b}, \triangle^{Z_b} \eta_{Z_b} \rangle_{Z_b} = \int_{Z_b} |\nabla^{TZ_b} \eta_{Z_b}|^2(z)\, d\,\text{vol}_{Z_b} + \sum_{ijkl} \int_{Z_b} R^{Z_b}_{ijkl} \langle E^i I^j \eta, E^k I^l \eta \rangle\, d\,\text{vol}_{Z_b}. \quad (5.37)$$

If $\eta_{Z_b} \in \text{Ker}(\mathscr{P})$, then

$$\langle \eta, \triangle^{Z_b} \eta \rangle_{Z_b} \geq A\, \text{diam}(Z_b)^{-2} \langle \eta, \eta \rangle_{Z_b}. \quad (5.38)$$

Hence

$$\int_{Z_b} |\nabla^{TZ_b} \eta_{Z_b}|^2(z)\, d\,\text{vol}_{Z_b} \geq \left(A\, \text{diam}(Z_b)^{-2} - \text{const.}\, \|R^{Z_b}\|_\infty \right) \langle \eta, \eta \rangle_{Z_b}. \quad (5.39)$$

From (5.31), (5.36), and (5.39), if $\eta \in \text{Ker}(\mathscr{P}^{\text{fib}})$, then

$$\langle \eta, \triangle^M \eta \rangle_M \geq \left(A\, \text{diam}(Z)^{-2} - \text{const.}(\|R^M\|_\infty + \|T\|^2_\infty + \|\Pi\|^2_\infty + \|R^Z\|_\infty) \right) \langle \eta, \eta \rangle_M. \quad (5.40)$$

Using the Gauss-Codazzi equation, we can estimate $\|R^Z\|_\infty$ in terms of $\|R^M\|_\infty$ and $\|\Pi\|^2_\infty$. The theorem follows. □

6. Collapsing to a smooth base

In this section we prove Theorem 2, concerning the spectrum of the Laplacian $\triangle^M$ on a manifold M which is Gromov-Hausdorff close to a smooth manifold B. We prove Theorem 3, showing that the pairs (A', h^E) that appear in the conclusion of Theorem 2 satisfy a compactness property. We then prove Theorem 4, relating the spectrum of $\triangle^M$ to the spectrum of the differential form Laplacian on the base space B.

Proof of Theorem 2

For simplicity, we omit reference to p. Let g_0^{TM} denote the Riemannian metric on M. From [17] or Lemma 3, if a Riemannian metric g_1^{TM} on M is ϵ-close to g_0^{TM}, then the spectrum of $\triangle^M$, computed with g_1^{TM}, is $J\epsilon$-close to the spectrum computed with

g_0^{TM}. We use the geometric results of [12] to find a metric g_2^{TM} on M which is close to g_0^{TM} and to which we can apply Theorem 1.

First, as in [12, (2.4.1)], by the smoothing results of U. Abresch and others (see [12, Theorem 1.12]), we can find metrics on M and B which are ϵ-close to the original metrics such that the new metrics satisfy $\|\nabla^i R\|_\infty \leq A_i(n,\epsilon)$ for some appropriate sequence $\{A_i(n,\epsilon)\}_{i=0}^\infty$. By rescaling, we may assume that $\|R^M\|_\infty \leq 1$, $\|R^B\|_\infty \leq 1$, and $\mathrm{inj}(B) \geq 1$. Let g_1^{TM} denote the new metric on M. We now apply [12, Theorem 2.6], with B fixed. It implies that there are positive constants $\lambda(n)$ and $c(n,\epsilon)$, so that if $d_{GH}(M,B) \leq \lambda(n)$, then there is a fibration $f : M \to B$ such that

(1) $\mathrm{diam}(f^{-1}(b)) \leq c(n,\epsilon)\, d_{GH}(M,B)$,

(2) f is a $c(n,\epsilon)$-almost Riemannian submersion,

(3) $\|\Pi_{f^{-1}(b)}\|_\infty \leq c(n,\epsilon)$.

As in [20], the Gauss-Codazzi equation, the curvature bound on M, and the second fundamental form bound on $f^{-1}(b)$ imply a uniform bound on $\{\|R^{f^{-1}(b)}\|_\infty\}_{b\in B}$. Along with the diameter bound on $f^{-1}(b)$, this implies that if $d_{GH}(M,B)$ is sufficiently small, then $f^{-1}(b)$ is almost flat.

From [12, Propositions 3.6 and 4.9], we can find another metric g_2^{TM} on M which is ϵ-close to g_1^{TM} so that the fibration $f : M \to B$ gives M the structure of a Riemannian affine fiber bundle. Furthermore, by [12, Proposition 4.9], there is a sequence $\{A_i'(n,\epsilon)\}_{i=0}^\infty$, so that we may assume that g_1^{TM} and g_2^{TM} are close in the sense that

$$\|\nabla^i(g_1^{TM} - g_2^{TM})\|_\infty \leq A_i'(n,\epsilon)\, d_{GH}(M,B), \tag{6.1}$$

where the covariant derivative in (6.1) is that of the Levi-Civita connection of g_2^{TM} (see also [30, Theorem 1.1] for an explicit statement). In particular, there is an upper bound on $\|R^M(g_2^{TM})\|_\infty$ in terms of B, n, ϵ, and K.

We now apply Theorem 1 to the Riemannian affine fiber bundle with metric g_2^{TM}. It remains to estimate the geometric terms appearing in (1.3). We have an estimate on $\|\Pi\|_\infty$ as above. Applying O'Neill's formula (see [6, (9.29c)]) to the Riemannian affine fiber bundle, we can estimate $\|T\|_\infty^2$ in terms of $\|R^M\|_\infty$ and $\|R^B\|_\infty$. Putting this together, the theorem follows. □

The vector bundles E and Euclidean inner products h^E which appear in Theorem 2 are not completely arbitrary. For example, E^0 is the trivial $\mathbb{R}$-bundle on B. More substantially, if E is a real $\mathbb{Z}$-graded topological vector bundle on B, let $\mathscr{C}_E$ be the space of grading-preserving connections on E, let $\mathscr{G}_E$ be the group of smooth grading-preserving $\mathrm{GL}(E)$ gauge transformations on E, and let $\mathscr{H}_E$ be the space of graded Euclidean inner products on E. We equip $\mathscr{C}_E$ and $\mathscr{H}_E$ with the C^∞-topology. Give $(\mathscr{C}_E \times \mathscr{H}_E)/\mathscr{G}_E$ the quotient topology. Let ∇^E denote the connection part $A'_{[1]}$ of A'.

PROPOSITION 3

In Theorem 2 we may assume that E is one of a finite number of isomorphism classes of real $\mathbb{Z}$-graded topological vector bundles $\{E_i\}$ on B. Furthermore, there are compact subsets $C_{E_i} \subset (\mathscr{C}_{E_i} \times \mathscr{H}_{E_i})/\mathscr{G}_{E_i}$ depending on n, ϵ, and K, such that we may assume that the gauge-equivalence class of the pair (∇^E, h^E) lies in C_E.

Proof

As the infinitesimal holonomy of the connection $T^H M$ lies in $\mathrm{aff}(Z) = \mathfrak{n}_R^F$, its action on $\mathfrak{n}$, which is through the adjoint representation, is nilpotent. Hence its action on $\Lambda^*(\mathfrak{n}^*)^F$ is also nilpotent. Given $b \in B$, it follows that the local holonomy group of $T^H M$ at b acts unipotently on E_b. Then the structure group of E can be topologically reduced to a discrete group, and so E admits a flat connection. The rank of E is at most $2^{\dim(M)-\dim(B)}$. By an argument of G. Lusztig, only a finite number of isomorphism classes of real topological vector bundles over B of a given rank admit a flat connection (see [22, p. 22]). This proves the first part of the proposition.

To prove the second part of the proposition, we first reduce to the case $F = \{e\}$. Recall that $\widehat{Z}$ is a nilmanifold that covers Z, with covering group F. Given $g \in \mathrm{Aff}(Z)$, we can lift it to some $\widehat{g} \in \mathrm{Aff}(\widehat{Z})$. There is an automorphism $\alpha_{\widehat{g}} \in \mathrm{Aut}(F)$ such that for all $f \in F$ and $\widehat{z} \in \widehat{Z}$,

$$\alpha_{\widehat{g}}(f) \cdot \widehat{z} = (\widehat{g} f \widehat{g}^{-1})(\widehat{z}). \tag{6.2}$$

Considering the different possible liftings of g, we obtain a well-defined homomorphism $\mathrm{Aff}(Z) \to \mathrm{Out}(F)$. Then there is an exact sequence

$$1 \longrightarrow \mathrm{Aff}(\widehat{Z})^F \longrightarrow \mathrm{Aff}(Z) \longrightarrow \mathrm{Out}(F). \tag{6.3}$$

Let P be the principal $\mathrm{Aff}(Z)$-bundle such that $M = P \times_{\mathrm{Aff}(Z)} Z$. Put $\widehat{M} = P \times_{\mathrm{Aff}(\widehat{Z})^F} \widehat{Z}$. Then $\widehat{M}$ is an affine fiber bundle that regularly covers M, with the order of the covering group bounded in terms of $|F|$. Again, there is a uniform upper bound on $|F|$ in terms of $\dim(Z)$ (see [9]). Instead of considering M, it suffices to consider $\widehat{M}$ and work equivariantly with respect to the covering group. Thus we assume that Z is a nilmanifold with $\Gamma \subset N_L$ and $F = \{e\}$.

As the fiber of E^j is $\Lambda^j(\mathfrak{n}^*)$, it suffices to prove the second part of the proposition for E^1 with fiber $\mathfrak{n}^*$. Let us consider instead for a moment $(E^1)^*$ with fiber $\mathfrak{n}$. With respect to the lower central series (1.8) of $\mathfrak{n}$, let $(E^1)^*_{[k]}$ be the vector bundle associated to P with fiber $\mathfrak{n}'_{[k]}$. Then there is a filtration

$$(E^1)^* = (E^1)^*_{[0]} \supset (E^1)^*_{[1]} \supset \cdots \supset (E^1)^*_{[S]} \supset 0. \tag{6.4}$$

Let Spl be the set of splittings of the short exact sequences

$$0 \longrightarrow (E^1)^*_{[k+1]} \longrightarrow (E^1)^*_{[k]} \longrightarrow (E^1)^*_{[k]}/(E^1)^*_{[k+1]} \longrightarrow 0. \tag{6.5}$$

Put $\mathscr{V}_{[k]} = (E^1)^*_{[k]}/(E^1)^*_{[k+1]}$ and

$$\mathscr{V} = \bigoplus_{k=0}^{S} \mathscr{V}_{[k]}. \tag{6.6}$$

Let $\mathscr{H}_{\mathscr{V}}$ be the set of graded Euclidean inner products on the $\mathbb{Z}$-graded real vector bundle $\mathscr{V}$. A Euclidean inner product $h^{(E^1)^*}$ determines splittings $\{s_k\}_{k=0}^{S-1}$ of (6.5) and a Euclidean inner product $h^{\mathscr{V}} \in \mathscr{H}_{\mathscr{V}}$. Conversely, one recovers $h^{(E^1)^*}$ from the splittings $\{s_k\}_{k=0}^{S-1}$ and $h^{\mathscr{V}}$. Thus there is an isomorphism $\mathscr{H}_{(E^1)^*} \cong \mathrm{Spl} \times \mathscr{H}_{\mathscr{V}}$.

Let $\mathscr{C}_{\mathrm{fil}}$ denote the set of connections on $(E^1)^*$ which preserve the filtration (6.4). Let $\mathscr{C}_{\mathscr{V}}$ be the set of connections on $\mathscr{V}$ which are grading-preserving with respect to (6.6). Let $\mathrm{End}^<(\mathscr{V})$ be the set of endomorphisms of $\mathscr{V}$ which are strictly lower triangular with respect to (6.6). Given $(\nabla^{(E^1)^*}, h^{(E^1)^*}) \in \mathscr{C}_{\mathrm{fil}} \times \mathscr{H}_{(E^1)^*}$, let $i : (E^1)^* \to \mathscr{V}$ be the isomorphism induced by $h^{(E^1)^*}$. Then $i \circ \nabla^{(E^1)^*} \circ i^{-1} \in \mathscr{C}_{\mathscr{V}} \times \Omega^1(B; \mathrm{End}^<(\mathscr{V}))$. In this way there is an isomorphism

$$\mathscr{C}_{\mathrm{fil}} \times \mathscr{H}_{(E^1)^*} \cong \mathscr{C}_{\mathscr{V}} \times \Omega^1\left(B; \mathrm{End}^<(\mathscr{V})\right) \times \mathrm{Spl} \times \mathscr{H}_{\mathscr{V}}. \tag{6.7}$$

Let $\mathscr{G}_{\mathrm{fil}}$ be the set of filtration-preserving gauge transformations of $(E^1)^*$, and let $\mathscr{G}_{\mathscr{V}}$ be the set of grading-preserving gauge transformations of $\mathscr{V}$. Note that the set of splittings of (6.5) is acted upon freely and transitively by the gauge transformations of $(E^1)^*_{[k]}$ which preserve $(E^1)^*_{[k+1]}$ and act as the identity on $(E^1)^*_{[k]}/(E^1)^*_{[k+1]}$. It follows that $\mathrm{Ker}(\mathscr{G}_{\mathrm{fil}} \to \mathscr{G}_{\mathscr{V}})$ acts freely and transitively on Spl. Then

$$\left(\mathscr{C}_{\mathrm{fil}} \times \mathscr{H}_{(E^1)^*}\right) / \mathrm{Ker}\left(\mathscr{G}_{\mathrm{fil}} \to \mathscr{G}_{\mathscr{V}}\right) \cong \mathscr{C}_{\mathscr{V}} \times \Omega^1\left(B; \mathrm{End}^<(\mathscr{V})\right) \times \mathscr{H}_{\mathscr{V}}, \tag{6.8}$$

and so

$$\left(\mathscr{C}_{\mathrm{fil}} \times \mathscr{H}_{(E^1)^*}\right) / \mathscr{G}_{\mathrm{fil}} \cong \left(\mathscr{C}_{\mathscr{V}} \times \Omega^1(B; \mathrm{End}^<(\mathscr{V})) \times \mathscr{H}_{\mathscr{V}}\right) / \mathscr{G}_{\mathscr{V}}. \tag{6.9}$$

There is an obvious continuous map $(\mathscr{C}_{\mathrm{fil}} \times \mathscr{H}_{(E^1)^*})/\mathscr{G}_{\mathrm{fil}} \to (\mathscr{C}_{(E^1)^*} \times \mathscr{H}_{(E^1)^*})/\mathscr{G}_{(E^1)^*}$.

As $\mathrm{Aff}(Z)$ preserves the lower-central-series filtration of $\mathfrak{n}$, in our case the dual connection to ∇^{E^1} lies in $\mathscr{C}_{\mathrm{fil}}$. Then considering dual spaces in (6.9), it is enough for us to show that there is a compact subset of

$$\left(\mathscr{C}_{\mathscr{V}^*} \times \Omega^1(B; \mathrm{End}^>(\mathscr{V}^*)) \times \mathscr{H}_{\mathscr{V}^*}\right) / \mathscr{G}_{\mathscr{V}^*} \tag{6.10}$$

in which we may assume that the gauge equivalence of the pair (∇^{E^1}, h^{E^1}) lies. We can then map the compact subset into $(\mathscr{C}_{E^1} \times \mathscr{H}_{E^1})/\mathscr{G}_{E^1}$.

As the local holonomy of E^1 comes from an N_R-action, it factors through the coadjoint action of N on $\mathfrak{n}^*$. Letting $\nabla^{\mathscr{V}^*} = \bigoplus_{k=1}^{S} \nabla^{\mathscr{V}^*_{[k]}}$ be the component of ∇^{E^1} in $\mathscr{C}_{\mathscr{V}^*}$, it follows that the local holonomy of $\nabla^{\mathscr{V}^*_{[k]}}$ is trivial and so $\nabla^{\mathscr{V}^*_{[k]}}$ is flat. We first

claim that there is a compact subset $C_{\mathcal{V}^*_{[k]}} \subset (\mathscr{C}_{\mathcal{V}^*_{[k]}} \times \mathscr{H}_{\mathcal{V}^*_{[k]}})/\mathscr{G}_{\mathcal{V}^*_{[k]}}$ such that we may assume that the gauge equivalence class of the pair $(\nabla^{\mathcal{V}^*_{[k]}}, h^{\mathcal{V}^*_{[k]}})$ lies in $C_{\mathcal{V}^*_{[k]}}$.

For simplicity of notation, fix $k \in [0, S]$ and let $\mathscr{E}$ denote $\mathcal{V}^*_{[k]}$. Let $\mathscr{F}_{\mathscr{E}}$ be the space of flat connections on $\mathscr{E}$, with the subspace topology from $\mathscr{C}_{\mathscr{E}}$. We show that there is a compact subset of $(\mathscr{F}_{\mathscr{E}} \times \mathscr{H}_{\mathscr{E}})/\mathscr{G}_{\mathscr{E}}$ in which we may assume that the gauge equivalence class of the pair $(\nabla^{\mathscr{E}}, h^{\mathscr{E}})$ lies. Then the claim follows from mapping the compact subset into $(\mathscr{C}_{\mathscr{E}} \times \mathscr{H}_{\mathscr{E}})/\mathscr{G}_{\mathscr{E}}$.

Let $\widetilde{\mathscr{E}}$ be the lift of $\mathscr{E}$ to the universal cover $\widetilde{B}$ of B. Fix a basepoint $\widetilde{b}_0 \in \widetilde{B}$ with projection $b_0 \in B$, and let $\widetilde{\mathscr{E}}_{\widetilde{b}_0}$ be the fiber of $\widetilde{\mathscr{E}}$ over $\widetilde{b}_0$. Then a flat connection $\nabla^{\mathscr{E}}$ gives a trivialization $\widetilde{\mathscr{E}} = \widetilde{B} \times \widetilde{\mathscr{E}}_{\widetilde{b}_0}$. Let $\rho : \pi_1(B, b_0) \to \mathrm{Aut}(\widetilde{\mathscr{E}}_{\widetilde{b}_0})$ be the holonomy of $\nabla^{\mathscr{E}}$. Then a Euclidean inner product $h^{\mathscr{E}}$ on $\mathscr{E}$ can be identified with a Euclidean inner product $h^{\widetilde{\mathscr{E}}}$ on $\widetilde{\mathscr{E}}$ which satisfies

$$h^{\widetilde{\mathscr{E}}}(\gamma^{-1}\widetilde{b}) = \rho(\gamma)^T h^{\widetilde{\mathscr{E}}}(\widetilde{b})\rho(\gamma) \tag{6.11}$$

for all $\gamma \in \pi_1(B, b_0)$ and $\widetilde{b} \in \widetilde{B}$. In short, we can identify $(\mathscr{F}_{\mathscr{E}} \times \mathscr{H}_{\mathscr{E}})/\mathscr{G}_{\mathscr{E}}$ with the pairs $(\rho, h^{\widetilde{\mathscr{E}}})$ satisfying (6.11), modulo $\mathrm{Aut}(\widetilde{\mathscr{E}}_{\widetilde{b}_0})$. We can use the $\mathrm{Aut}(\widetilde{\mathscr{E}}_{\widetilde{b}_0})$-action to identify $\widetilde{\mathscr{E}}_{\widetilde{b}_0}$ with $\mathbb{R}^N$, with the standard inner product $h^{\mathbb{R}^N}$. If we put

$$\begin{aligned} X_{\mathscr{E}} = \big\{(\rho, h^{\widetilde{\mathscr{E}}}) \in \mathrm{Hom}\big(\pi_1(B, b_0), \mathrm{GL}(N, \mathbb{R})\big) \times \mathscr{H}_{\widetilde{B}\times\mathbb{R}^N} : h^{\widetilde{\mathscr{E}}}(\widetilde{b}_0) = h^{\mathbb{R}^N} \\ \text{and for all } \gamma \in \pi_1(B, b_0) \text{ and } \widetilde{b} \in \widetilde{B}, h^{\widetilde{\mathscr{E}}}(\gamma^{-1}\widetilde{b}) = \rho(\gamma)^T h^{\widetilde{\mathscr{E}}}(\widetilde{b})\rho(\gamma)\big\}, \end{aligned} \tag{6.12}$$

then we have identified $(\mathscr{F}_{\mathscr{E}} \times \mathscr{H}_{\mathscr{E}})/\mathscr{G}_{\mathscr{E}}$ with $X_{\mathscr{E}}/O(N)$. Let $\{\gamma_j\}$ be a finite generating set of $\pi_1(B, b_0)$. The topology on $X_{\mathscr{E}}$ comes from the fiber bundle structure

$$X_{\mathscr{E}} \to \mathrm{Hom}\big(\pi_1(B, b_0), \mathrm{GL}(N, \mathbb{R})\big), \tag{6.13}$$

whose fiber over $\rho \in \mathrm{Hom}(\pi_1(B, b_0), \mathrm{GL}(N, \mathbb{R}))$ is

$$\begin{aligned} \big\{h^{\widetilde{\mathscr{E}}} \in \mathscr{H}_{\widetilde{B}\times\mathbb{R}^N} : h^{\widetilde{\mathscr{E}}}(\widetilde{b}_0) = h^{\mathbb{R}^N} \text{ and for all } \gamma \in \pi_1(B, b_0) \text{ and } \widetilde{b} \in \widetilde{B}, \\ h^{\widetilde{\mathscr{E}}}(\gamma^{-1}\widetilde{b}) = \rho(\gamma)^T h^{\widetilde{\mathscr{E}}}(\widetilde{b})\rho(\gamma)\big\}. \end{aligned} \tag{6.14}$$

Here $\mathrm{Hom}(\pi_1(B, b_0), \mathrm{GL}(N, \mathbb{R}))$ has a topology as a subspace of $\mathrm{GL}(N, \mathbb{R})^{\{\gamma_j\}}$, and the fiber (6.14) has the C^∞-topology. Thus it suffices to show that $(\rho, h^{\widetilde{\mathscr{E}}})$ lies in a predetermined compact subset $C_{\mathscr{E}}$ of $X_{\mathscr{E}}$.

By [20, (1)–(7)], we may assume that we have uniform bounds on the second fundamental form Π of the Riemannian affine fiber bundle M, along with its covariant derivatives. As Π determines how the Riemannian metrics on nearby fibers vary (with respect to $T^H M$) and h^{E^1} comes from the inner product on the

parallel differential forms on the fibers $\{Z_b\}_{b\in B}$, we obtain uniform bounds on $(h^{E_1})^{-1}(\nabla^{E_1}h^{E_1})$ and its covariant derivatives. In particular, we also have a uniform bound on $(h^{\mathscr{E}})^{-1}(\nabla^{\mathscr{E}}h^{\mathscr{E}})$ and hence on $(h^{\widetilde{\mathscr{E}}})^{-1}(dh^{\widetilde{\mathscr{E}}})$. For the finite generating set $\{\gamma_j\}$, using the fact that $h^{\widetilde{\mathscr{E}}}(\widetilde{b}_0) = h^{\mathbb{R}^N}$, we obtain in this way uniform bounds on $\{h^{\widetilde{\mathscr{E}}}(\gamma_j^{-1}\widetilde{b}_0)\}$. The equivariance (6.11) then gives uniform bounds on $\{\rho(\gamma_j)^T\rho(\gamma_j)\}$ and hence on $\{\rho(\gamma_j)\}$. Thus ρ lies in a predetermined compact subset of the representation space $\mathrm{Hom}(\pi_1(B,b_0),\mathrm{GL}(N,\mathbb{R}))$. Given ρ, the uniform bounds on the covariant derivatives of $h^{\widetilde{\mathscr{E}}}$ over a fundamental domain in $\widetilde{B}$ show that we have compactness in the fiber (6.14). As these bounds can be made continuous in ρ, the claim follows.

Fix a Euclidean inner product $h_0^{\mathscr{V}^*}$ on $\mathscr{V}^*$. Given a pair $(\nabla^{\mathscr{V}^*}, h^{\mathscr{V}^*}) \in \mathscr{C}_{\mathscr{V}^*} \times \mathscr{H}_{\mathscr{V}^*}$, we can always perform a gauge transformation to transform the Euclidean inner product to $h_0^{\mathscr{V}^*}$. Let $\mathscr{O}_{\mathscr{V}^*}$ be the orthogonal gauge transformations with respect to $h_0^{\mathscr{V}^*}$. Then we can identify $(\mathscr{C}_{\mathscr{V}^*} \times \mathscr{H}_{\mathscr{V}^*})/\mathscr{G}_{\mathscr{V}^*}$ with $\mathscr{C}_{\mathscr{V}^*}/\mathscr{O}_{\mathscr{V}^*}$. Similarly,

$$\left(\mathscr{C}_{\mathscr{V}^*} \times \Omega^1(B;\mathrm{End}^{>}(\mathscr{V}^*)) \times \mathscr{H}_{\mathscr{V}^*}\right)/\mathscr{G}_{\mathscr{V}^*} \cong \left(\mathscr{C}_{\mathscr{V}^*} \times \Omega^1(B;\mathrm{End}^{>}(\mathscr{V}^*))\right)/\mathscr{O}_{\mathscr{V}^*}. \tag{6.15}$$

There is a singular fibration $p : \left(\mathscr{C}_{\mathscr{V}^*} \times \Omega^1(B;\mathrm{End}^{>}(\mathscr{V}^*))\right)/\mathscr{O}_{\mathscr{V}^*} \to \mathscr{C}_{\mathscr{V}^*}/\mathscr{O}_{\mathscr{V}^*}$. The fiber over a gauge equivalence class $[\nabla^{\mathscr{V}^*}]$ is $\Omega^1(B;\mathrm{End}^{>}(\mathscr{V}^*))/G$, where G is the group of orthogonal gauge transformations that are parallel with respect to $\nabla^{\mathscr{V}^*}$. In particular, upon choosing a basepoint $b_0 \in B$, we can view G as contained in the finite-dimensional orthogonal group $O(\mathscr{V}^*_{b_0})$.

From what we have already shown, we know that we are restricted to a compact subset of the base $(\mathscr{C}_{\mathscr{V}^*} \times \mathscr{H}_{\mathscr{V}^*})/\mathscr{G}_{\mathscr{V}^*} \cong \mathscr{C}_{\mathscr{V}^*}/\mathscr{O}_{\mathscr{V}^*}$ of the singular fibration p. Let $(\nabla^{E^1})^T$ be the adjoint connection to ∇^{E^1} with respect to h^{E^1}. The uniform bounds on $(h^{E^1})^{-1}(\nabla^{E^1}h^{E^1})$ and its derivatives give uniform C^∞-bounds on the part of ∇^{E^1} which does not preserve the metric h^{E^1}, that is, on $\nabla^{E^1} - (\nabla^{E^1})^T \in \Omega^1(B;\mathrm{End}(E^1))$. In particular, using the upper triangularity of ∇^{E^1}, we obtain uniform C^∞-bounds on the part of ∇^{E^1} in $\Omega^1(B;\mathrm{End}^{>}(\mathscr{V}^*))$. As the bounds can be made continous with respect to $[\nabla^{\mathscr{V}^*}] \in \mathscr{C}_{\mathscr{V}^*}/\mathscr{O}_{\mathscr{V}^*}$, we have shown that there is a fixed compact subset of $\left(\mathscr{C}_{\mathscr{V}^*} \times \Omega^1(B;\mathrm{End}^{>}(\mathscr{V}^*)) \times \mathscr{H}_{\mathscr{V}^*}\right)/\mathscr{G}_{\mathscr{V}^*}$ in which we may assume that the pair (∇^{E^1}, h^{E^1}) lies.

To summarize, we have shown that the topological vector bundle E^1 has a flat structure $\mathscr{V}^*$, with flat connection $\nabla^{\mathscr{V}^*}$. We showed that there are bounds on the holonomy of $\nabla^{\mathscr{V}^*}$ which are uniform in n, ϵ, and K. We then showed that h^{E^1} and $\nabla^{E^1} - \nabla^{\mathscr{V}^*}$ are C^∞-bounded in terms of n, ϵ, and K. (More precisely, we showed that these statements are true after an appropriate gauge transformation is made.) The proposition follows. □

Let $\mathscr{S}_E$ be the space of degree-1 superconnections on E, with the C^∞-topology.

PROPOSITION 4
With reference to Proposition 3, there are compact subsets $D_{E_i} \subset (\mathscr{S}_{E_i} \times \mathscr{H}_{E_i})/\mathscr{G}_{E_i}$ depending on n, ϵ, and K such that we may assume that the gauge-equivalence class of the pair $\left(A', h^E\right)$ lies in D_E.

Proof
Let E be as in Proposition 3. As in the proof of Proposition 3, upon choosing h_0^E, we have identifications $(\mathscr{C}_E \times \mathscr{H}_E)/\mathscr{G}_E \cong \mathscr{C}_E/\mathscr{O}_E$ and $(\mathscr{S}_E \times \mathscr{H}_E)/\mathscr{G}_E \cong \mathscr{S}_E/\mathscr{O}_E$. There is a singular fibration $p : \mathscr{S}_E/\mathscr{O}_E \to \mathscr{C}_E/\mathscr{O}_E$ coming from the projection $A' \to A'_{[1]}$. The fiber of p over a gauge equivalence class $[\nabla^E]$ is $\bigoplus_{a+b=1} \Omega^a(B; \mathrm{End}(E^*, E^{*+b}))/G$, where G is the group of orthogonal gauge transformations that are parallel with respect to ∇^E.

From Proposition 3 we know that we are restricted to a compact subset of the base $(\mathscr{C}_E \times \mathscr{H}_E)/\mathscr{G}_E \cong \mathscr{C}_E/\mathscr{O}_E$. The superconnection on E has the form (5.22). We measure norms on $\Omega(B; \mathrm{End}(E))$ using h_0^E. The differential $d^{\mathfrak{n}}$ comes from exterior differentiation on the parallel forms on the fibers of the Riemannian affine fiber bundle. Note that as A' is flat, $d^{\mathfrak{n}}$ is parallel with respect to ∇^E. As we have a uniform n-, ϵ- and K-dependent bound on the curvatures of the fibers Z, Lemma 1 gives a uniform bound on the structure constants $\{c^i{}_{jk}\}$ and hence a uniform bound on $\|d^{\mathfrak{n}}\|_\infty$. The operator i_T is also parallel with respect to ∇^E. From O'Neill's formula (see [6, (9.29)]), we obtain a uniform bound on $\|i_T\|_\infty$. Thus we have uniform C^∞-bounds on $A' - \nabla^E \in \Omega(B; \mathrm{End}(E))$, and so we have compactness in the fibers of p. As the bounds can be made continuous with respect to $[\nabla^E] \in \mathscr{C}_E/\mathscr{O}_E$, the proposition follows. □

Proof of Theorem 3
Propositions 3 and 4 prove the theorem. □

We need certain eigenvalue statements. Let $\mathscr{E}$ be a $\mathbb{Z}_2$-graded real topological vector bundle on a smooth closed manifold B. Let $\mathscr{S}_\mathscr{E}$ be the space of superconnections on $\mathscr{E}$, let $\mathscr{G}_\mathscr{E}$ be the $\mathrm{GL}(\mathscr{E})$ gauge group of $\mathscr{E}$, and let $\mathscr{H}_\mathscr{E}$ be the space of Euclidean metrics on $\mathscr{E}$. Fix a Euclidean metric $h_0^\mathscr{E} \in \mathscr{H}_\mathscr{E}$. Given a pair $(A', h^\mathscr{E}) \in \mathscr{S}_\mathscr{E} \times \mathscr{H}_\mathscr{E}$, we can always perform a gauge transformation to transform the Euclidean metric to $h_0^\mathscr{E}$. Let $\mathscr{O}_\mathscr{E}$ be the group of orthogonal gauge transformations of $\mathscr{E}$ with respect to $h_0^\mathscr{E}$. Then we can identify $(\mathscr{S}_\mathscr{E} \times \mathscr{H}_\mathscr{E})/\mathscr{G}_\mathscr{E}$ with $\mathscr{S}_\mathscr{E}/\mathscr{O}_\mathscr{E}$. Given $A' \in \mathscr{S}_\mathscr{E}$, let $(A')^*$ be its adjoint with respect to $h_0^\mathscr{E}$, and put $\triangle_{A'} = A'(A')^* + (A')^* A'$, acting on $\Omega(B; \mathscr{E})$. For $j \in \mathbb{Z}^+$, let $\mu_j(A')$ be the jth eigenvalue of $\triangle_{A'}$, counted with multiplicity. It is $\mathscr{O}_\mathscr{E}$-invariant. Equivalently, μ_j is $\mathscr{G}_\mathscr{E}$-invariant as a function of the pair (A', h^E).

Proof of Theorem 4
As E^p admits a flat connection, there is some $r \in \mathbb{N}$ such that for all p, $E^p \otimes \mathbb{R}^r$ is topologically isomorphic to the trivial vector bundle $B \times \mathbb{R}^{r \cdot rk(E^p)}$. Hence $E \otimes \mathbb{R}^r$ is topologically isomorphic to the $\mathbb{Z}$-graded trivial vector bundle $\mathscr{E} = B \times \mathbb{R}^{r \cdot rk(E)}$.

For simplicity, we omit reference to p. In view of Theorem 2, it suffices to show that there is a positive constant $D(n, \epsilon, K)$ such that $|\lambda_j(B; E)^{1/2} - \lambda'_j(B)^{1/2}| \leq D(\epsilon, n, K)$.

The operator $\triangle^E \otimes \mathrm{Id}$ on $\Omega(B; E) \otimes \mathbb{R}^r$ has a spectrum that is the same as that of $\triangle^E$, but with multiplicities multiplied by r. Hence it is enough to compare the spectrum of $\triangle^E \otimes \mathrm{Id}$, acting on $\Omega(B; E \otimes \mathbb{R}^r)$, with that of the standard Laplacian on $\Omega(B; \mathscr{E})$.

From Theorem 3, we may assume that the gauge equivalence class of the pair (A', h^E) lies in a predetermined compact subset $D \subset (\mathscr{S}_E \times \mathscr{H}_E)/\mathscr{G}_E$. Put $A'_1 = A' \otimes \mathrm{Id}$, acting on $\Omega(B; E \otimes \mathbb{R}^r)$, and put $h_1 = h^E \otimes h^{\mathbb{R}^r}$ on $E \otimes \mathbb{R}^r$. Using the isomorphism $\mathscr{E} \cong E \otimes \mathbb{R}^r$, we may assume that the gauge equivalence class of the pair (A'_1, h_1) lies in a predetermined compact subset $D_1 \subset (\mathscr{S}_\mathscr{E} \times \mathscr{H}_\mathscr{E})/\mathscr{G}_\mathscr{E}$.

Let A'_2 be the trivial flat connection on $\mathscr{E}$, and let h_2 be the product Euclidean inner product on $\mathscr{E}$. With an appropriate gauge transformation $g \in \mathscr{G}_\mathscr{E}$, we can transform (A'_1, h_1) to $(g \cdot A'_1, h_2)$ without changing the eigenvalues. Under the identification $(\mathscr{S}_\mathscr{E} \times \mathscr{H}_\mathscr{E})/\mathscr{G}_\mathscr{E} = \mathscr{S}_\mathscr{E}/\mathscr{O}_\mathscr{E}$, we can assume that the equivalence class of $g \cdot A'_1$ in $\mathscr{S}_\mathscr{E}/\mathscr{O}_\mathscr{E}$ lies in a predetermined compact subset $D_2 \subset \mathscr{S}_\mathscr{E}/\mathscr{O}_\mathscr{E}$.

The eigenvalues of the Laplacian associated to the superconnection A'_1 and the Euclidean inner product h_2 are unchanged when the group of orthogonal gauge transformations $\mathscr{O}_\mathscr{E}$ acts on A'_1. Consider the function $l : \mathscr{S}_\mathscr{E} \times \mathscr{S}_\mathscr{E} \to \mathbb{R}$ given by

$$l(A'_1, A'_2) = \inf_{g' \in \mathscr{O}_\mathscr{E}} \|g' \cdot A'_1 - A'_2\|. \tag{6.16}$$

An elementary argument shows that l is continuous. Hence it descends to a continuous function on $(\mathscr{S}_\mathscr{E}/\mathscr{O}_\mathscr{E}) \times (\mathscr{S}_\mathscr{E}/\mathscr{O}_\mathscr{E})$. When Lemma 4 is applied to $g' \cdot (g \cdot A'_1)$ and A'_2, the compactness of D_2 and the finiteness statement in Theorem 3 give the desired eigenvalue estimate. The theorem follows. □

7. Small positive eigenvalues
In this section we characterize the manifolds M for which the p-form Laplacian has small positive eigenvalues. We first describe a spectral sequence that computes the cohomology of a flat degree-1 superconnection A'. We use the compactness result of Theorem 3 to show that if M has j small eigenvalues of the p-form Laplacian with $j > \mathrm{b}_p(M)$, and M collapses to a smooth manifold B, then there is an associated flat degree-1 superconnection A'_∞ on B with $\dim(\mathrm{H}^p(A'_\infty)) \geq j$. We then use the spectral sequence of A'_∞ to characterize when this can happen. In Corollary 1 we give

a bound on the number of small eigenvalues of the 1-form Laplacian. In Corollary 2 we give a bound on the number of small eigenvalues of the p-form Laplacian when one is sufficiently close to a smooth limit space of dimension $\dim(M) - 1$. Theorem 5 describes when a collapsing sequence can have small positive eigenvalues of the p-form Laplacian, in terms of the topology of the affine fiber bundle $M \to B$. Corollary 3 gives a precise description of when there are small positive eigenvalues of the 1-form Laplacian in a collapsing sequence. In Corollaries 4 and 5 we look at collapsing sequences with limit spaces of dimension 1 or $\dim(M)-1$, respectively. Finally, given an affine fiber bundle, in Theorem 6 we give a collapsing construction that produces small eigenvalues of the p-form Laplacian.

In the collapsing arguments in this section, when the limit space is a smooth manifold, we can always assume that its Riemannian metric is smooth. At first sight the smoothness assumption on the metric may seem strange as the limit space of a bounded sectional curvature collapse, when a smooth manifold, generally only has a $C^{1,\alpha}$-metric. The point is that we are interested in the case when an eigenvalue goes to zero, which gives a zero eigenvalue of $\triangle^E$ in the limit. The property of having a zero eigenvalue is essentially topological in nature and so is also true for a smoothed metric. For this reason, we can apply smoothing results to the metrics and so ensure that the limit metric is smooth.

Let B be a smooth connected closed manifold. Let $E = \bigoplus_{j=0}^{m} E^j$ be a $\mathbb{Z}$-graded real vector bundle on B, and let $A' = \sum_{i\geq 0} A'_{[i]}$ be a flat degree-1 superconnection on E. Let $\mathrm{H}^p(A')$ denote the degree-p cohomology of the differential A' on $\Omega(B;\, E)$, where the latter has the total grading. Given $a, b \in \mathbb{N}$, we write $\omega^{a,b}$ for an element of $\Omega^a(B;\, E^b)$.

In order to compute $\mathrm{H}^p(A')$, let us first consider the equation $A'\omega = 0$. Putting

$$\omega = \omega^{p,0} + \omega^{p-1,1} + \omega^{p-2,2} + \cdots, \tag{7.1}$$

we obtain

$$(A'_{[0]} + A'_{[1]} + A'_{[2]} + \cdots)(\omega^{p,0} + \omega^{p-1,1} + \omega^{p-2,2} + \cdots) = 0 \tag{7.2}$$

or

$$\begin{aligned} A'_{[0]}\omega^{p,0} &= 0,\\ A'_{[0]}\omega^{p-1,1} + A'_{[1]}\omega^{p,0} &= 0,\\ A'_{[0]}\omega^{p-2,2} + A'_{[1]}\omega^{p-1,1} + A'_{[2]}\omega^{p,0} &= 0,\\ &\vdots \end{aligned} \tag{7.3}$$

We can try to solve these equations iteratively.

Formalizing this procedure, we obtain a spectral sequence to compute $\mathrm{H}^p(A')$. Put $E_0^{a,b} = \Omega^a(B; E^b)$, and define $d_0 : E_0^{a,b} \to E_0^{a,b+1}$ by $d_0\omega^{a,b} = A_{[0]}\omega^{a,b}$. For $r \geq 1$, put

$$E_r^{a,b} = \frac{\{\{\omega^{a+s,b-s}\}_{s=0}^{r-1} : \text{ for } 0 \leq s \leq r-1, \ \sum_{t=0}^{s} A'_{[s-t]}\omega^{a+t,b-t} = 0\}}{\{\{\omega^{a+s,b-s}\}_{s=0}^{r-1} : \omega^{a+s,b-s} = \sum_{t=0}^{s} A'_{[s-t]}\widehat{\omega}^{a+t,b-t-1} \text{ for some } \{\widehat{\omega}^{a+s,b-s-1}\}_{s=0}^{r-1}\}}. \tag{7.4}$$

Define a differential $d_r : E_r^{a,b} \to E_r^{a+r,b-r+1}$ by

$$d_r\{\omega^{a+s,b-s}\}_{s=0}^{r-1} = \Big\{ \sum_{t=0}^{r-1} A'_{[r+s-t]}\omega^{a+t,b-t} \Big\}_{s=0}^{r-1}. \tag{7.5}$$

Then $E_{r+1} \cong \mathrm{Ker}(d_r)/\mathrm{Im}(d_r)$. The spectral sequence $\{E_r^{*,*}\}_{r=0}^{\infty}$ has a limit $E_\infty^{*,*}$ with

$$\mathrm{H}^p(A') \cong \bigoplus_{a+b=p} E_\infty^{a,b}. \tag{7.6}$$

From [8, Proposition 2.5], for each $b \in \mathbb{N}$, $\mathrm{H}^b(A'_{[0]})$ is a flat vector bundle on B. Then

$$\begin{aligned} E_0^{a,b} &= \Omega^a(B; E^b), \\ E_1^{a,b} &= \Omega^a\big(B; \mathrm{H}^b(A'_{[0]})\big), \\ E_2^{a,b} &= \mathrm{H}^a\big(B; \mathrm{H}^b(A'_{[0]})\big). \end{aligned} \tag{7.7}$$

Example 4
If $M \to B$ is a fiber bundle, E is the infinite-dimensional vector bundle W of vertical differential forms (see [8, Section III(a)]), and A' is the superconnection arising from exterior differentiation on M, then we recover the Leray spectral sequence to compute $\mathrm{H}^*(M; \mathbb{R})$.

Example 5
If $M \to B$ is an affine fiber bundle, E is the vector bundle of parallel differential forms on the fibers, and A' is as in (5.22), then it follows from [28, Corollary 7.28] that $E_r^{*,*}$ is the same as the corresponding term in the Leray spectral sequence for $\mathrm{H}^*(M; \mathbb{R})$ if $r \geq 1$.

Suppose that M is a connected closed manifold with at least j small eigenvalues of $\triangle_p$ for some $j > \mathrm{b}_p(M)$. Consider a sequence of Riemannian metrics $\{g_i\}_{i=1}^{\infty}$ in $\mathscr{M}(M, K)$ with $\lim_{i\to\infty} \lambda_{p,j}(M, g_i) = 0$. As in the proof of Theorem 2, for any $\epsilon > 0$ there is a sequence $\{A_k(n, \epsilon)\}_{k=0}^{\infty}$, so that for all i we can find a new metric g_i' on M which is ϵ-close to g_i, with $\|\nabla^k R^M(g_i')\|_\infty \leq A_k(n, \epsilon)$. Fix ϵ to be, say, $1/2$. From [17] or Lemma 3, we have that $\lambda_{p,j}(M, g_i')$ is $J\epsilon$-close to $\lambda_{p,j}(M, g_i)$ for some

fixed integer J. Thus, without loss of generality, we may replace g_i by g_i'. We relabel g_i' as g_i.

As $j > \mathrm{b}_p(M)$, there must be a subsequence of $\{(M, g_i)\}_{i=1}^{\infty}$ which Gromov-Hausdorff converges to a lower-dimensional limit space X. That is, we are in the collapsing situation. Suppose that the limit space is a smooth manifold B. From [12, Section 5], the regularity of the metrics on M implies that B has a smooth Riemannian metric g^{TB}. (We are in the situation in which the limit space $\check{X}$ of the frame bundles, a smooth Riemannian manifold, has an $O(n)$-action with a single orbit type.) From Theorem 2, for large i there are vector bundles E_i on B, flat degree-1 superconnections A_i' on E_i, and Euclidean inner products h^{E_i} on E_i such that $\lambda_{p,j}(M, g_i)$ is ϵ-close to $\lambda_{p,j}(B; E_i)$. From Theorem 3, after taking a subsequence, we may assume that all of the E_i's are topologically equivalent to a single vector bundle E on B and that the pairs (A_i', h^{E_i}) converge after gauge transformation to a pair $(A_\infty', h^{E_\infty})$. Then from Lemmas 3 and 4, the Laplacian associated to $(A_\infty', h^{E_\infty})$ satisfies $\dim \mathrm{Ker}(\triangle_p^E) \geq j$. Applying standard Hodge theory to the superconnection Laplacian $\triangle^E$, we obtain $\dim(\mathrm{H}^p(A_\infty')) \geq j$. On the other hand, looking at the E_2-term of the spectral sequence gives $\dim(\mathrm{H}^p(A_\infty')) \leq \sum_{a+b=p} \dim\big(\mathrm{H}^a(B; \mathrm{H}^b(A'_{\infty,[0]}))\big)$. Thus

$$j \leq \sum_{a+b=p} \dim\big(\mathrm{H}^a(B; \mathrm{H}^b(A'_{\infty,[0]}))\big). \tag{7.8}$$

Proof of Corollary 1
In the case $p = 1$, we obtain

$$j \leq \dim\big(\mathrm{H}^1(B; \mathrm{H}^0(A'_{\infty,[0]}))\big) + \dim\big(\mathrm{H}^0(B; \mathrm{H}^1(A'_{\infty,[0]}))\big). \tag{7.9}$$

As $\mathrm{H}^0(A'_{\infty,[0]})$ is the trivial $\mathbb{R}$-bundle on B, $\dim\big(\mathrm{H}^1(B; \mathrm{H}^0(A'_{\infty,[0]}))\big) = \mathrm{b}_1(B)$. As $A'_{\infty,[0]}$ acts by zero on E^0, there is an injection $\mathrm{H}^1(A'_{\infty,[0]}) \to E^1$. Then

$$\dim\big(\mathrm{H}^0(B; \mathrm{H}^1(A'_{\infty,[0]}))\big) \leq \dim\big(\mathrm{H}^1(A'_{\infty,[0]})\big) \leq \dim(E^1) \leq \dim(M) - \dim(B). \tag{7.10}$$

Thus $j \leq \mathrm{b}_1(B) + \dim(M) - \dim(B)$. On the other hand, the spectral sequence for $\mathrm{H}^*(M; \mathbb{R})$ gives

$$\mathrm{H}^1(M; \mathbb{R}) = \mathrm{H}^1(B; \mathbb{R}) \oplus \mathrm{Ker}\big(\mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R})) \to \mathrm{H}^2(B; \mathbb{R})\big). \tag{7.11}$$

In particular, $\mathrm{b}_1(B) \leq \mathrm{b}_1(M)$. The corollary follows. □

Remark. Using heat equation methods (see [3]), one can show that there is an increasing function f such that if $\mathrm{Ric}(M) \geq -(n-1)\lambda^2$ and $\mathrm{diam}(M) \leq D$, then the number of small eigenvalues of the 1-form Laplacian is bounded above by $f(\lambda D)$.

This result is weaker than Corollary 1 when applied to manifolds with sectional curvature bounds, but it is more general in that it applies to manifolds with just a lower Ricci curvature bound.

Proof of Corollary 2
From Fukaya's fibration theorem, if a manifold M^n with $\|R^M\|_\infty \leq K$ is sufficiently Gromov-Hausdorff close to B, then M is the total space of a circle bundle over B. Suppose that the claim of the corollary is not true. Then there is a sequence of connected closed n-dimensional Riemannian manifolds $\{(M_i, g_i)\}_{i=1}^\infty$ with $\|R^{M_i}(g_i)\|_\infty \leq K$ and $\lim_{i\to\infty} M_i = B$ which provides a counterexample. As there is a finite number of isomorphism classes of flat real line bundles on B, after passing to a subsequence, we may assume that each M_i is a circle bundle over B with a fixed orientation bundle $\mathscr{O}$ and that $\lim_{i\to\infty} \lambda_{p,j}(M_i, g_i) = 0$ for $j = \mathrm{b}_p(B) + \mathrm{b}_{p-1}(B; \mathscr{O}) + 1$. Following the argument before the proof of Corollary 1, we obtain $E = E^0 \oplus E^1$ on B, with E^0 a trivial $\mathbb{R}$-bundle and $E^1 = \mathscr{O}$, and a limit superconnection A'_∞ on E with $A'_{\infty,[0]} = 0$ and $A'_{\infty,[1]} = \nabla^E$, the canonical flat connection. Then, as in (7.8), we obtain

$$j \leq \mathrm{b}_p(B) + \mathrm{b}_{p-1}(B; \mathscr{O}), \tag{7.12}$$

which is a contradiction. □

Proof of Theorem 5
As in the proof of Theorem 2, without loss of generality we may assume that each (M, g_i) is a Riemannian affine fiber bundle structure on the affine fiber bundle $M \to B$. Suppose that for each $q \in [0, p]$, $b_q(Z) = \dim(\Lambda^q(\mathfrak{n}^*)^F)$ and the holonomy representation of the flat vector bundle $\mathrm{H}^q(Z; \mathbb{R})$ on B is semisimple. Let $E \to B$ be the real vector bundle associated to the affine fiber bundle $M \to B$ as in Section 5. Then $E \cong \mathrm{H}^*(Z; \mathbb{R})$. The superconnection A'_E on E, from Section 5, has $A'_{E,[0]} = 0$ and $A'_{E,[1]} = \nabla^E$, the canonical flat connection on $E \cong \mathrm{H}^*(Z; \mathbb{R})$. As the affine fiber bundle is fixed, each E_i equals E and each A'_i equals A'_E. However, the Euclidean metrics $\{h_i^E\}_{i=1}^\infty$ on E vary. There is a sequence of gauge transformations $\{g_i\}_{i=1}^\infty$, so that after passing to a subsequence, $\lim_{i\to\infty} g_i \cdot (A'_i, h_i^E) = (A'_\infty, h_\infty^E)$ for some pair (A'_∞, h_∞^E). Clearly, $A'_{\infty,[0]} = 0$ and $A'_{\infty,[1]} = \lim_{i\to\infty} g_i \cdot \nabla^E$. As the holonomy representation of $\mathrm{H}^q(Z; \mathbb{R})$ is semisimple for $q \in [0, p]$, the connection $A'_{\infty,[1]}\big|_{E^q}$ is gauge equivalent to ∇^{E^q}. That is, the connection does not degenerate. (In the complex case this follows from [25, Theorem 1.27], and the real case follows from [29, Theorem 11.4].)

Equation (7.8) now implies

$$j \leq \sum_{a+b=p} \dim\big(\mathrm{H}^a(B; \mathrm{H}^b(Z; \mathbb{R}))\big). \tag{7.13}$$

If the Leray spectral sequence to compute $H^*(M;\mathbb{R})$ degenerates at the E_2-term, then

$$b_p(M) = \sum_{a+b=p} \dim\big(H^a(B;H^b(Z;\mathbb{R}))\big), \tag{7.14}$$

which contradicts the assumption that $j > b_p(M)$. □

Example 6
Let Z be an almost flat manifold as in Example 1. Put $M = Z \times B$. If there is a sequence of affine parallel metrics on Z which give it r_p small eigenvalues of the p-form Laplacian, then M has $\sum_{a+b=p} r_a \cdot b_b(B)$ small eigenvalues of the p-form Laplacian. This gives an example of Theorem 5.1.

Example 7 ([24])
Let N be the Heisenberg group of upper-diagonal unipotent (3×3)-matrices, and let Γ be the integer lattice in G. Put $M = \Gamma\backslash N$. Then M fibers over S^1, the fiber being T^2 and the monodromy being given by the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. One has $b_1(M) = 2$, but for any $K > 0$, $a_{1,3,K} = 0$. That is, one can collapse M to a circle by a sequence of affine parallel metrics, while producing 3 small eigenvalues of the 1-form Laplacian. This gives an example of Theorem 5.2.

Example 8
Consider M as in Example 2. If the Leray spectral sequence to compute $H^p(M;\mathbb{R})$ does not degenerate at the E_2-term, then there are small positive eigenvalues of the p-form Laplacian on M. This gives an example of Theorem 5.3.

Proof of Corollary 3
The affine fiber bundle $M \to B$ induces a vector bundle $E \to B$ and a flat degree-1 superconnection A'_E, as in Section 5. As in Example 5, the spectral sequence associated to A'_E is the same as the Leray spectral sequence for computing $H^*(M;\mathbb{R})$. Let A'_∞ denote the limit superconnection arising as in the proof of Theorem 5. The spectral sequence for $H^*(A'_\infty)$ gives

$$H^1(A'_\infty) = H^1(B;\mathbb{R}) \oplus \mathrm{Ker}\big(H^0(B;H^1(A'_{\infty,[0]})) \to H^2(B;\mathbb{R})\big). \tag{7.15}$$

In particular,

$$\dim\big(H^1(A'_\infty)\big) = b_1(B) + \dim\big(\mathrm{Ker}(H^0(B;H^1(A'_{\infty,[0]})) \to H^2(B;\mathbb{R}))\big). \tag{7.16}$$

We wish to compare this with the corresponding spectral sequence for $H^*(A'_E)$, that is, (7.11).

Suppose that the differential $d_2 : \mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R})) \to \mathrm{H}^2(B; \mathbb{R})$ vanishes. Then from (7.11),

$$\mathrm{b}_1(M) = \mathrm{b}_1(B) + \dim\big(\mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R}))\big). \tag{7.17}$$

By assumption, $\dim(\mathrm{H}^1(A'_\infty)) = j > \mathrm{b}_1(M)$. This implies that

$$\dim\big(\mathrm{H}^0(B; \mathrm{H}^1(A'_{\infty,[0]}))\big) > \dim\big(\mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R}))\big). \tag{7.18}$$

In terms of the original superconnection A'_E, we have that $\mathrm{H}^1(Z; \mathbb{R})$ is a flat subbundle of $\mathrm{H}^1(A'_{E,[0]})$. After taking limits, we obtain a flat subbundle $\mathrm{H}^1(Z; \mathbb{R})_\infty$ of $\mathrm{H}^1(A'_{\infty,[0]})$. Here the fibers of $\mathrm{H}^1(Z; \mathbb{R})_\infty$ are again isomorphic to the first real cohomology group of Z, but the flat structure could be different from that of the bundle which we denoted by $\mathrm{H}^1(Z; \mathbb{R})$. In particular,

$$\dim\big(\mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R})_\infty)\big) \geq \dim\big(\mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R}))\big). \tag{7.19}$$

Clearly,

$$\dim\big(\mathrm{H}^0(B; \mathrm{H}^1(A'_{\infty,[0]}))\big) \geq \dim\big(\mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R})_\infty)\big). \tag{7.20}$$

Then from (7.18) we must have

$$\dim\big(\mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R})_\infty)\big) > \dim\big(\mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R}))\big) \tag{7.21}$$

or

$$\dim\big(\mathrm{H}^0(B; \mathrm{H}^1(A'_{\infty,[0]}))\big) > \dim\big(\mathrm{H}^0(B; \mathrm{H}^1(Z; \mathbb{R})_\infty)\big). \tag{7.22}$$

If (7.21) holds, then the holonomy representation of the flat vector bundle $\mathrm{H}^1(Z; \mathbb{R})$ must have a nontrivial unipotent subrepresentation (see [29, Theorem 11.4 and Proposition 11.14]). If (7.22) holds, then there is a nonzero covariantly constant section of the vector bundle $(H^1(A'_{\infty,[0]}))/(H^1(Z; \mathbb{R})_\infty)$ on B, where the flat connection on $(H^1(A'_{\infty,[0]}))/(H^1(Z; \mathbb{R})_\infty)$ is induced from the flat connection on $\mathrm{H}^1(A'_{\infty,[0]})$. This proves the corollary. □

Proof of Corollary 4

Suppose that for $q \in \{p-1, p\}$, $b_q(Z) = \dim(\Lambda^q(\mathfrak{n})^F)$. From the Leray spectral sequence, $\mathrm{H}^p(M) \cong \mathrm{Ker}(\Phi^p - I) \oplus \mathrm{Coker}(\Phi^{p-1} - I)$. Let $\mathrm{H}^*(Z; \mathbb{R})_\infty$ denote the limiting flat vector bundle on S^1, as in the proof of Corollary 3, with holonomy $\Phi^p_\infty \in \mathrm{Aut}(\mathrm{H}^p(Z; \mathbb{R}))$. The spectral sequence for $\mathrm{H}^*(A'_\infty)$ gives $\mathrm{H}^p(A'_\infty) \cong \mathrm{Ker}(\Phi^p_\infty - I) \oplus \mathrm{Coker}(\Phi^{p-1}_\infty - I)$. We have $\dim(\mathrm{Ker}(\Phi^p_\infty - I)) \geq \dim(\mathrm{Ker}(\Phi^p - I))$ and $\dim(\mathrm{Coker}(\Phi^p_\infty - I)) \geq \dim(\mathrm{Coker}(\Phi^p - I))$. By assumption, $j = \dim(\mathrm{H}^p(A'_\infty)) > \dim(\mathrm{H}^p(M; \mathbb{R})) = \mathrm{b}_p(M)$. If $\dim(\mathrm{Ker}(\Phi^p_\infty - I)) > \dim(\mathrm{Ker}(\Phi^p - I))$, then Φ^p must have a nontrivial unipotent subfactor. Similarly, if $\dim(\mathrm{Coker}(\Phi^p_\infty - I)) > \dim(\mathrm{Coker}(\Phi^p - I))$, then Φ^{p-1} must have a nontrivial unipotent subfactor. □

Example 9
Suppose that the affine fiber bundle $M \to S^1$ has fiber $Z = T^2$. If M has a Sol geometry or an $\mathbb{R}^3$-geometry, then Corollary 4 implies that there are no small positive eigenvalues in a collapsing sequence associated to $M \to S^1$. On the other hand, if M has a Nil geometry, then Example 7 shows that there are small positive eigenvalues of the 1-form Laplacian. (See [24] for further examples of homogeneous collapsings.)

Proof of Corollary 5
The E_2-term of the spectral sequence for computing $\mathrm{H}^*(M;\mathbb{R})$ consists of $E_2^{p,0} = \mathrm{H}^p(X;\mathbb{R})$ and $E_2^{p,1} = \mathrm{H}^p(X;\mathscr{O})$. The differential is $\mathscr{M}_\chi$. The corollary now follows from Theorem 5. □

Proof of Theorem 6
As in [20, §6], we can reduce the structure group of the fiber bundle $P \to B$ so that the local holonomy lies in a maximal connected compact subgroup of $\mathrm{Aff}(Z)$, a torus group. Choose a horizontal distribution $T^H M$ on M whose local holonomy lies in this torus group. Add vertical Riemannian metrics g^{TZ}, parallel along the fibers, and a Riemannian metric g^{TB} on B to give $M \to B$ the structure of a Riemannian affine fiber bundle. We use Theorem 1 to make statements about the eigenvalues of the differential form Laplacian on M.

There is a vector space isomorphism $\mathfrak{n}^* \cong \bigoplus_{k=0}^{S} \mathfrak{r}_{[k]}^*$. Define a number operator on $\mathfrak{n}^*$ to be multiplication by 3^k on $\mathfrak{r}_{[k]}^*$. Extend this to a number operator on $\Lambda^*(\mathfrak{n}^*)^F$ and to a number operator N on the vector bundle E^* over B.

For $\epsilon > 0$, rescale g^{TZ} to a new metric g_ϵ^{TZ} by multiplying it by ϵ^{3^k} on $\mathfrak{r}_{[k]} \subset \mathfrak{n}$. Let g_ϵ^{TM} be the corresponding Riemannian metric on M. The rescaling does not affect d^M. The adjoint of d^M with respect to the new metric is $(d^M)_\epsilon^* = \epsilon^N (d^M)^* \epsilon^{-N}$. Putting

$$\begin{aligned} C'_\epsilon &= \epsilon^{-N/2}\, d^M \epsilon^{N/2}, \\ C''_\epsilon &= \epsilon^{N/2} (d^M)^* \epsilon^{-N/2}, \end{aligned} \tag{7.23}$$

we have that C'_ϵ is a flat degree-1 superconnection, with C''_ϵ being its adjoint with respect to g^{TZ}. The Laplacian $\triangle^M$ coming from g_ϵ^{TM} is conjugate to $C'_\epsilon C''_\epsilon + C''_\epsilon C'_\epsilon$.

By [20, §6], $\lim_{\epsilon\to 0}(M, g_\epsilon^{TM}) = B$ with bounded sectional curvature in the limit. (The proof in [20, §6] uses a scaling by ϵ^{2^k}, but the proof goes through for a scaling by ϵ^{3^k}. The phrase "The element Y_i of $\mathfrak{g}$, through the right action of $G, \ldots$" in [20, p. 349, line b9] should read "... the left action of $G, \ldots$".) Let A'_ϵ denote the superconnection on E constructed by restricting C'_ϵ to the fiberwise-parallel forms. We show that $\lim_{\epsilon\to 0} A'_\epsilon = \nabla^G$, the flat connection on G. The theorem then follows from Theorem 1 and Lemma 4.

Consider $A'_{\epsilon,[0]}$. It acts on a fiber of E by $\epsilon^{-N/2} d^{\mathfrak{n}} \epsilon^{N/2}$. Consider first its action on a fiber $(\mathfrak{n}^*)^F \cong \bigoplus_{k=0}^S (\mathfrak{r}^*_{[k]})^F$ of E^1. As $d^{\mathfrak{n}}$ acts on $\Lambda^1(\mathfrak{n}^*)$ by the dual of the Lie bracket and $[\mathfrak{r}_{[k]}, \mathfrak{r}_{[l]}] \subset \bigoplus_{m > \max(k,l)} \mathfrak{r}_{[m]}$, we have $d^{\mathfrak{n}} \mathfrak{r}^*_{[m]} \subset \bigoplus_{k,l<m} \mathfrak{r}^*_{[k]} \wedge \mathfrak{r}^*_{[l]}$. It follows that

$$\epsilon^{-N/2} d^{\mathfrak{n}} \epsilon^{N/2} : (\mathfrak{r}^*_{[m]})^F \to (\mathfrak{r}^*_{[k]} \wedge \mathfrak{r}^*_{[l]})^F \tag{7.24}$$

is $O(\epsilon^{(3^m - 3^k - 3^l)/2})$. We obtain that the action of $A'_{\epsilon,[0]}$ on E^1 is $O(\epsilon^{1/2})$ as $\epsilon \to 0$. A similar argument shows that the action of $A'_{\epsilon,[0]}$ on E^* is $O(\epsilon^{1/2})$.

Now consider $A'_{\epsilon,[1]} = \epsilon^{-N/2} \nabla^E \epsilon^{N/2}$. Put $F^*_k = P \times_{\mathrm{Aff}(Z)} (\mathfrak{r}^*_{[k]})^F$, so that $E^1 \cong \bigoplus_{k=0}^S F^*_k$. (Here the $*$ in F^*_k denotes an adjoint, not a $\mathbb{Z}$-grading.) Consider first the action of $C'_{\epsilon,[1]}$ on $C^\infty(B; E^1)$. As the holonomy of ∇^E comes from an $\mathrm{Aff}(Z)$ action, we have $\nabla^E : C^\infty(B; F^*_k) \to \bigoplus_{l \leq k} C^\infty(B; F^*_l)$ (see the proof of Proposition 3). If $l < k$, then the component

$$\epsilon^{-N/2} \nabla^E \epsilon^{N/2} : C^\infty(B; F^*_k) \to C^\infty(B; F^*_l) \tag{7.25}$$

of $\epsilon^{-N/2} \nabla^E \epsilon^{N/2}$ is $O(\epsilon^{1/2})$. On the other hand, the component $\nabla^E : C^\infty(B; F^*_k) \to C^\infty(B; F^*_k)$ is the restriction of the flat connection ∇^G from G^1 to F^*_k. A similar argument applies to all of E to show that as $\epsilon \to 0$, $A'_{\epsilon,[1]} = \nabla^G + O(\epsilon^{1/2})$.

Finally, consider $A'_{\epsilon,[2]} = \epsilon^{-N/2} i_T \epsilon^{N/2}$. The curvature T of the fiber bundle $M \to B$ is independent of ϵ. As T acts by interior multiplication on the fibers of E, the action of $\epsilon^{-N/2} i_T \epsilon^{N/2}$ on $(\mathfrak{r}^*_{[k]})^F \subset E^1$ is $O(\epsilon^{3^k/2})$. A similar argument applies to all of E to show that as $\epsilon \to 0$, $A'_{\epsilon,[2]} = O(\epsilon^{1/2})$.

The theorem follows. □

Note. After this paper was finished, we learned of the preprint version of [15] which, among other things, contains proofs of Corollaries 2 and 5 in the case when M and B are oriented. Paper [24] is also related to the present paper.

Acknowledgments. I thank Bruno Colbois, Gilles Courtois, and Pierre Jammes for corrections to an earlier version of this paper. I thank the referee for a very careful reading of the manuscript and many useful remarks, among these suggesting a simplification of the proof of Proposition 2.

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