# THE COLLAPSING GEOMETRY OF ALMOST RICCI-FLAT 4-MANIFOLDS

JOHN LOTT

Abstract. We consider Riemannian 4-manifolds that Gromov-Hausdorff converge to a lower dimensional limit space with the Ricci tensor going to zero. Among other things, we show that if the limit space is two dimensional then under some mild assumptions, the limiting four dimensional geometry away from the curvature blowup region is semiflat Kähler.

## 1. Introduction

When considering Einstein manifolds, or almost Einstein manifolds, the four dimensional case is especially interesting. This paper is about almost Ricci-flat 4-manifolds, meaning compact 4-manifolds $M$ that admit a sequence of Riemannian metrics $\{g_i\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} \| \mathrm{Ric}(M, g_i)\|_\infty \cdot \mathrm{diam}(M, g_i)^2 = 0$. Special cases come from Ricci-flat 4-manifolds. The known examples of the latter are finitely covered by a flat torus or by a Ricci-flat Kähler metric on a $K3$ manifold. There are almost Ricci-flat 4-manifolds that do not admit Ricci-flat metrics [2].

Fixing an upper diameter bound for $\{(M, g_i)\}_{i=1}^{\infty}$, one can divide the study of almost Ricci-flat 4-manifolds into the noncollapsed case, where there is a definite positive lower volume bound, and the collapsing case, where the volume goes to zero. In the noncollapsed case, a Gromov-Hausdorff limit (as the Ricci curvature goes to zero) is a four dimensional Ricci-flat orbifold with isolated orbifold points, as follows from work of Anderson [1], Bando-Kasue-Nakajima [5] and Tian [30]. The orbifold points are caused by noncompact Ricci-flat ALE manifolds (or orbifolds) that bubble off. There is a bubble tree description of the sequence [3, 4]. In [22], sufficient topological conditions were given for a noncollapsed almost Ricci-flat 4-manifold to admit a Ricci-flat metric. There are probably also noncollapsed almost Ricci-flat 4-manifolds that do not admit Ricci-flat metrics [6].

In the collapsing case, fundamental work was done by Cheeger and Tian [11]. Allowing the manifolds to vary, let $\{(M_i, g_i)\}_{i=1}^{\infty}$ be a sequence of compact connected orientable Riemannian 4-manifolds so that for some $C \in \mathbb{N}$ and $D < \infty$,

- $\chi(M_i) \le C$ for all $i$,
- $\mathrm{diam}(M_i, g_i) \le D$ for all $i$,
- $\lim_{i\to\infty} \| \mathrm{Ric}(M_i, g_i)\|_\infty = 0$ and
- $\lim_{i\to\infty} \mathrm{vol}(M_i, g_i) = 0$.

*Date*: March 19, 2018.

Research partially supported by NSF grant DMS-1510192.

After passing to a subsequence, we can assume that $\lim_{i\to\infty}(M_i, g_i) = (X, d_X)$ in the Gromov-Hausdorff topology, for some compact metric space $X$ whose Hausdorff dimension is less than four. As we will review in Subsection 3.1, Cheeger and Tian showed that for large $i$, each $(M_i, g_i)$ has a small "curvature blowup" region where the curvature concentrates in an $L^2$-sense, and a "regular" region with *a priori* curvature bounds. We can assume that the regular regions converge to a subset $X_{reg} \subset X$, whose complement in $X$ is a finite set. In particular, if $B$ is a connected component of $X_{reg}$ then taking the metric completion of $B$ amounts to adding a finite number of points.

We are interested in the four dimensional geometry of the regular regions. From work of Cheeger, Fukaya and Gromov, culminating in [8], collapsing regions with bounded curvature acquire continuous symmetries in the limit. (The results from [8] were localized in [11, Section 2].) A convenient language to formalize the collapsing limit, with its symmetries, is that of Riemannian groupoids, as described in [26, Section 5]. A brief introduction to the use of Riemannian groupoids in collapsing theory is in [27, Section 3]. Passing to a subsequence, we can assume that the regular regions, approaching $B$, also converge in the sense of Riemannian groupoids, to a four dimensional smooth Ricci-flat Riemannian groupoid $\mathcal{X}$ whose orbit space is $B$.

To state the main result of this paper, we recall that a (possibly incomplete) connected Riemannian manifold is *parabolic* if any nonnegative $C^2$-regular function $f$ that is bounded above, and satisfies $\triangle f \geq 0$, is constant. (If the manifold has boundary then we require $f$ to vanish on the boundary.) Some equivalent conditions for parabolicity are given in [16, Theorem 5.1]. There is a similar definition for Riemannian orbifolds.

For example, the complement of a finite number of points in a closed Riemannian manifold, of dimension greater than one, is parabolic [16, Corollary 5.4]. Whether or not a two dimensional Riemannian manifold is parabolic only depends on its underlying conformal structure.

Let $\sqrt{\det G}$ denote the relative volume function of the orbits of $\mathcal{X}$, a function on $B$. (For example, if $\mathcal{X}$ comes from a free torus action then $\sqrt{\det G}$ describes the relative volumes of the torus orbits.)

**Theorem 1.1.**
(1) *If* $\dim(B) = 3$ *then* $B$ *is an orbifold. If* $B$ *is parabolic and* $\sqrt{\det G}$ *is bounded above then* $B$ *is flat and the* 4*-dimensional geometry of* $\mathcal{X}$ *is flat.*
(2) *If* $\dim(B) = 2$ *then* $B$ *is an orbifold-with-boundary. If* $B$ *is parabolic and* $\sqrt{\det G}$ *is bounded above then* $B$ *is boundaryless with nonnegative scalar curvature, and the* 4*-dimensional geometry of* $\mathcal{X}$ *is semiflat Kähler.*
(3) *If* $\dim(B) = 1$ *then* $B$ *is a orbifold. The* 4*-dimensional geometry of* $\mathcal{X}$ *is flat, or a Riemannian Kasner geometry, or a Riemannian Taub Bianchi-II geometry.*
(4) *If* $\dim(B) = 0$ *then* $B$ *is a point and the* 4*-dimensional geometry of* $\mathcal{X}$ *is flat.*

The notion of a semiflat Kähler metric is given in [17, Section 3], [19, Section 3.2], [21] and [32, Section 3.1], among other places. The Riemannian Kasner geometry and the Riemannian Taub Bianchi-II geometry are defined in (3.22) and (3.23), respectively. Appropriate sequences of Ricci-flat $K3$ manifolds give examples of parts (1), (2) and (3) of Theorem 1.1; see Examples 3.7, 3.16 and 3.24. The constructions in [2] give further examples.

Theorem 1.1 can be viewed in two ways. On the one hand, it gives some explanation for the geometry of the regular regions seen in the known almost Ricci-flat examples, and indicates what other examples may exist. On the other hand, it shows what restrictions would have to be lifted in order to find exotic examples.

In the setting of Theorem 1.1, if $B$ is not a point then the local symmetry Lie algebra of $\mathcal{X}$ must be $\mathrm{nil}^3$ or $\mathbb{R}^N$, where $1 \leq N \leq 3$. The $\mathrm{nil}^3$ case can be handled separately, so the main task in proving Theorem 1.1 is to analyze the Ricci-flat equations on a manifold with a local $\mathbb{R}^N$-symmetry.

More generally, we look at Einstein manifolds of arbitrary dimension with a local $\mathbb{R}^N$-symmetry. In the case of a locally free action, the Riemannian metric gives a distribution that is transverse to the local orbits of the $\mathbb{R}^N$-action. One interesting feature is that when the quotient space is two dimensional, there are conserved quantities that, under suitable topological conditions, force the distribution to be integrable; see Corollary 2.16.

### 1.1. **Earlier work.**

Einstein manifolds with symmetries have been considered in many papers, including [7, 13, 33].

In [28], Naber and Tian looked at collapsing sequences of manifolds having bounded diameter and bounded curvature, with the Ricci tensor going to zero. (In the four dimensional case, this corresponds to not having any curvature blowup regions.) They showed that the Gromov-Hausdorff limit is a Ricci-flat orbifold. Instead of Riemannian groupoids, they used a notion of $N^*$-bundles. The argument used a formula for $\triangle \ln \det G$, along with the maximum principle; compare with (2.9).

In work in progress, Cheeger and Tian use the framework of [11] to study finite-volume complete Einstein 4-manifolds with negative Einstein constant and ends that are asymptotic to rays. They use the collapsing structure at infinity to identify the possible model geometries (real or complex hyperbolic cusps) and show that along an end, a model geometry is indeed asymptotically approached.

### 1.2. **Structure of the paper.**

In Section 2 we analyze Ricci-flatness for the total space of a (twisted) principal bundle with abelian structure group. The proof of Theorem 1.1 is given in Section 3. In fact, we prove the conclusions of Theorem 1.1 when the upper diameter bound is replaced by an upper volume bound, and the volume is only assumed to go to zero in the local sense of (3.2). With these more general assumptions, we have to introduce basepoints, which is why we only discuss the bounded diameter case in this introduction.

More detailed descriptions are given at the beginnings of the sections.

I thank Jeff Cheeger, Hans-Joachim Hein, Claude LeBrun, Gang Tian and Jeff Viaclovsky for helpful comments.

## 2. Isometric free local $\mathbb{R}^N$-actions

In this section we consider a (twisted) principal bundle with abelian structure group and an adapted Riemannian metric. In Subsection 2.1 we define the relevant bundles and metrics, and give the formula for the Ricci curvature of the total space. In Subsection 2.2 we show how the Ricci-flat condition simplifies when the base is parabolic and the fiberwise volume forms are relatively bounded. Subsection 2.3 gives the consequences when the fibers are one dimensional. In Subsection 2.4 we discuss the conserved quantities that arise when the base is two dimensional, and show what the results of Subsection 2.2 become in the two dimensional case. Finally, Subsection 2.5 is about a one dimensional base.

The results of this section extend directly to the case when the base is an orbifold. We use the Einstein summation convention freely.

2.1. **Ricci curvature equation.** Let $\mathcal{G}$ be a Lie group, with Lie algebra $\mathfrak{g}$. Let $\mathrm{Aut}(\mathcal{G})_\delta$ denote the automorphism group of $\mathcal{G}$, with the discrete topology, and let $\mathcal{G} \rtimes \mathrm{Aut}(\mathcal{G})_\delta$ denote the semidirect product. If $B$ is a connected smooth manifold, let $P \to B$ be a principal $\mathcal{G} \rtimes \mathrm{Aut}(\mathcal{G})_\delta$-bundle. From the homomorphism $\mathcal{G} \rtimes \mathrm{Aut}(\mathcal{G})_\delta \to \mathrm{Aut}(\mathcal{G})_\delta$, there is a corresponding principal bundle $\mathfrak{F}$ on $B$ with discrete structure group $\mathrm{Aut}(\mathcal{G})_\delta$. From the action of $\mathcal{G} \rtimes \mathrm{Aut}(\mathcal{G})_\delta$ on $\mathcal{G}$, there is also a fiber bundle $\mathfrak{E}$ on $B$ associated to $P \to B$, with fiber $\mathcal{G}$. One can think of $\mathfrak{E}$ as an $\mathfrak{F}$-twisted $\mathcal{G}$-principal bundle, in the sense that $\mathfrak{E}$ has free local $\mathcal{G}$-actions that are globally twisted by $\mathfrak{F}$. In addition, there is an flat vector bundle $e$ on $B$ associated to $\mathfrak{F}$, with fiber $\mathfrak{g}$.

In what follows, we will be interested in the case when $\mathcal{G}$ is an $N$-dimensional connected abelian Lie group with $N \geq 1$. (An example is when $\mathcal{G} = T^N$, $\mathfrak{F}$ is a trivial principal $\mathrm{GL}(N, \mathbb{Z})$-bundle and $\mathfrak{E}$ is a principal $T^N$-bundle on $B$.) Let $M$ be the total space of $\mathfrak{E}$. We write $\dim(B) = n$ and $\dim(M) = m = N + n$.

Let $\overline{g}$ be a Riemannian metric on $M$ with a free local isometric $\mathcal{G}$-action (globally twisted by $\mathfrak{F}$). In adapted local coordinates, we can write

$$\overline{g} \;=\; \sum_{I,J=1}^N G_{IJ}\,(dx^I + A^I)(dx^J + A^J) \;+\; \sum_{\alpha,\beta=1}^n g_{\alpha\beta}\,db^\alpha db^\beta. \tag{2.1}$$

Here the $x^I$'s are linear local coordinates on the fibers of $M \to B$, $(G_{IJ})$ is the local expression of a Euclidean inner product on $e$, $\sum_{\alpha,\beta=1}^n g_{\alpha\beta}\,db^\alpha db^\beta$ is the local expression of a metric $g_B$ on $B$ and $A^I = \sum_\alpha A^I_\alpha db^\alpha$ are the components of a local $e$-valued 1-form describing an connection $A$ on the twisted $\mathcal{G}$-bundle $M \to B$.

Put $F^I_{\alpha\beta} = \partial_\alpha A^I_\beta - \partial_\beta A^I_\alpha$. At a given point $b \in B$, we can assume that $A^I(b) = 0$. We write

$$G_{IJ;\alpha\beta} \;=\; G_{IJ;\alpha\beta} \;-\; \Gamma^\sigma{}_{\alpha\beta}\, G_{IJ,\sigma}, \tag{2.2}$$

where $\{\Gamma^\sigma{}_{\alpha\beta}\}$ are the Christoffel symbols for the metric $g_{\alpha\beta}$ on $B$.

From [27, Section 4.2], the Ricci tensor of $\overline{g}$ on $M$ is given in terms of the curvature tensor $R_{\alpha\beta\gamma\delta}$ of $B$, the 2-forms $F^I_{\alpha\beta}$ and the metrics $G_{IJ}$ by

$$\begin{aligned}(2.3)\qquad \overline{R}^{\overline{g}}_{IJ} &= -\frac{1}{2}\, g^{\alpha\beta}\, G_{IJ;\alpha\beta} - \frac{1}{4}\, g^{\alpha\beta}\, G^{KL}\, G_{KL,\alpha}\, G_{IJ,\beta} + \frac{1}{2}\, g^{\alpha\beta}\, G^{KL}\, G_{IK,\alpha}\, G_{LJ,\beta} + \\ &\quad \frac{1}{4}\, g^{\alpha\gamma}\, g^{\beta\delta}\, G_{IK}\, G_{JL}\, F^K_{\alpha\beta}\, F^L_{\gamma\delta} \\ \overline{R}^{\overline{g}}_{I\alpha} &= \frac{1}{2}\, g^{\gamma\delta}\, G_{IK}\, F^K_{\alpha\gamma;\delta} + \frac{1}{2}\, g^{\gamma\delta}\, G_{IK,\gamma}\, F^K_{\alpha\delta} + \frac{1}{4}\, g^{\gamma\delta}\, G_{Im}\, G^{KL}\, G_{KL,\gamma}\, F^m_{\alpha\delta} \\ \overline{R}^{\overline{g}}_{\alpha\beta} &= R^g_{\alpha\beta} - \frac{1}{2}\, G^{IJ}\, G_{IJ;\alpha\beta} + \frac{1}{4}\, G^{IJ}\, G_{JK,\alpha}\, G^{KL}\, G_{LI,\beta} - \frac{1}{2}\, g^{\gamma\delta}\, G_{IJ}\, F^I_{\alpha\gamma}\, F^J_{\beta\delta}.\end{aligned}$$

The scalar curvature is

$$\begin{aligned}(2.4)\qquad \overline{R}^{\overline{g}} &= R^g - g^{\alpha\beta} G^{IJ}\, G_{IJ;\alpha\beta} + \frac{3}{4}\, g^{\alpha\beta}\, G^{IJ}\, G_{JK,\alpha}\, G^{KL}\, G_{LI,\beta} \\ &\quad - \frac{1}{4}\, g^{\alpha\beta}\, G^{IJ}\, G_{IJ,\alpha}\, G^{KL}\, G_{KL,\beta} - \frac{1}{4}\, g^{\alpha\gamma}\, g^{\beta\delta}\, G_{IJ}\, F^I_{\alpha\beta}\, F^J_{\gamma\delta}.\end{aligned}$$

In the rest of this section we will assume that the flat vector bundle $e$ has holonomy in $\det^{-1}(\pm 1) \subset \mathrm{GL}(N, \mathbb{R})$, so that $\det G$ is globally defined on $B$.

2.2. **General characterization of Ricci-flat metrics.** Suppose that $\mathrm{Ric}(M, \overline{g}) = \lambda \overline{g}$ for some $\lambda \in \mathbb{R}$. Then the first equation in (2.3) gives

$$\begin{aligned}(2.5)\qquad N\lambda &= -\frac{1}{2}\, g^{\alpha\beta}\, G^{IJ}\, G_{IJ;\alpha\beta} - \frac{1}{4}\, g^{\alpha\beta}\, G^{KL}\, G_{KL,\alpha}\, G^{IJ}\, G_{IJ,\beta} + \\ &\quad \frac{1}{2}\, g^{\alpha\beta}\, G^{JI}\, G_{IK,\alpha}\, G^{KL}\, G_{LJ,\beta} + \frac{1}{4}\, |F|^2,\end{aligned}$$

where

$$(2.6)\qquad |F|^2 = g^{\alpha\gamma}\, g^{\beta\delta}\, G_{IJ}\, F^I_{\alpha\beta}\, F^J_{\gamma\delta}.$$

Now

$$(2.7)\qquad \nabla_\alpha (\det G)^{\frac{1}{2}} = \frac{1}{2} (\det G)^{\frac{1}{2}} G^{IJ} G_{IJ;\alpha}$$

and

$$\begin{aligned}(2.8)\qquad \triangle (\det G)^{\frac{1}{2}} &= (\det G)^{\frac{1}{2}} \cdot \\ &\left( \frac{1}{2} g^{\alpha\beta} G^{IJ} G_{IJ;\alpha\beta} - \frac{1}{2} g^{\alpha\beta} G^{JI} G_{IK;\alpha} G^{KL} G_{LJ;\beta} + \frac{1}{4} g^{\alpha\beta} G^{KL} G_{KL;\alpha} G^{IJ} G_{IJ;\beta} \right).\end{aligned}$$

Thus (2.5) becomes

$$(2.9)\qquad \triangle \sqrt{\det G} = \left( \frac{1}{4} |F|^2 - \lambda N \right) \sqrt{\det G}.$$

Recall the notion of a parabolic Riemannian manifold from the introduction. Information about parabolic Riemannian manifolds is in [16, Section 5].

**Proposition 2.10.** *If*

- *$B$ is parabolic,*
- *$\lambda \le 0$ and*
- *$\det G$ is bounded above*

*then*

(1) *$\det G$ is constant,*
(2) *$F = 0$,*
(3) *$\lambda = 0$ and*
(4) *$g^{\alpha\beta} \, G_{IJ;\alpha\beta} \, - \, g^{\alpha\beta} \, G^{KL} \, G_{IK,\alpha} \, G_{LJ,\beta} = 0$.*

*Proof.* From (2.9), $\sqrt{\det G}$ is subharmonic. Since $B$ is parabolic and $\sqrt{\det G}$ is bounded above, it must be constant. Then the right-hand side of (2.9) vanishes, which implies that $F = 0$ and $\lambda = 0$. Substituting this into the first equation of (2.3), whose left-hand side vanishes, proves the proposition. □

Normalizing $\det G$ to be one, the equation

$$g^{\alpha\beta} \, G_{IJ;\alpha\beta} \, - \, g^{\alpha\beta} \, G^{KL} \, G_{IK,\alpha} \, G_{LJ,\beta} = 0 \tag{2.11}$$

is the local expression for a harmonic map from $B$ to the symmetric space $\det^{-1}(\pm 1)/\operatorname{O}(N) \cong \operatorname{SL}(N,\mathbb{R})/\operatorname{SO}(N)$ [26, Proposition 4.17]. More globally, fixing a basepoint $b_0 \in B$, let $\rho : \pi_1(B, b_0) \to \det^{-1}(\pm 1)$ be the monodromy of the flat vector bundle $e$. Then (2.11) is the equation for a $\rho$-equivariant harmonic map $\widetilde{G} : \widetilde{B} \to \det^{-1}(\pm 1)/\operatorname{O}(N) \cong \operatorname{SL}(N,\mathbb{R})/\operatorname{SO}(N)$ on the universal cover $\widetilde{B}$.

2.3. **Codimension-one base.** Returning to the equations (2.3), suppose that $N = 1$. The matrix $(G_{IJ})$ just becomes a function $G$.

**Proposition 2.12.** *If*

- *$B$ is parabolic,*
- *$\lambda \le 0$ and*
- *$G$ is uniformly bounded above*

*then $G$ is a constant function, $F = 0$, $\lambda = 0$ and $B$ is Ricci-flat.*

*Proof.* From Proposition 2.10, $G$ is constant, $F = 0$ and $\lambda = 0$. The third equation in (2.3) implies that $B$ is Ricci-flat. □

2.4. **Two dimensional base.** Returning to the equations (2.3), suppose that $n = 2$ and $(M, \overline{g})$ is Einstein.

**Proposition 2.13.** *The second equation of (2.3) is equivalent to the statement that $\sqrt{\det G} G_{IJ} \frac{F^J}{\operatorname{dvol}_B}$ defines a flat section of $e^*$.*

*Proof.* Choose local isothermal coordinates on $B$ so that $g = e^{2\phi} \left( (dx^1)^2 + (dx^2)^2 \right)$. The nonzero Christoffel symbols, up to symmetries, are

$$\begin{aligned} \Gamma^1{}_{11} = \Gamma^2{}_{21} = -\Gamma^1{}_{22} = \partial_{x^1} \phi, \\ \Gamma^2{}_{22} = \Gamma^1{}_{12} = -\Gamma^2{}_{11} = \partial_{x^2} \phi. \end{aligned} \tag{2.14}$$

The second equation of (2.3) becomes equivalent to

$$\partial_\alpha \left( \sqrt{\det G}\, G_{IJ} e^{-2\phi} F^J_{12} \right) = 0. \tag{2.15}$$

As $\mathrm{dvol}_B = e^{2\phi} dx^1 \wedge dx^2$, we can express (2.15) in more invariant terms as saying that $\sqrt{\det G} G_{IJ} \frac{F^J}{\mathrm{dvol}_B}$ is locally constant. By assumption, $\det e$ is a trivial bundle, so $\sqrt{\det G} G_{IJ} \frac{F^J}{\mathrm{dvol}_B}$ is naturally a section of $e^*$. Equation (2.15) says that it is a locally constant section. □

**Corollary 2.16.** *If the monodromy representation $\rho : \pi_1(B, b_0) \to \det^{-1}(\pm 1) \subset \mathrm{GL}(N, \mathbb{R})$ of $e^*$ does not have a trivial one-dimensional subrepresentation then $F = 0$.*

*Example* 2.17. In the collapsing of a $K3$ manifold considered in [18], the base $B$ is $S^2$ minus 24 points; see Example 3.16 below. The vector bundle $e^*$ has unipotent holonomy when going around a small loop around any of the 24 punctures. However, the invariant subspaces do not line up globally and there is no nonzero flat section of $e^*$. Hence $F = 0$, as seen directly in [18].

*Remark* 2.18. In the Lorentzian setting, the expression $\sqrt{\det G}\, G_{IJ} \frac{F^J}{\mathrm{dvol}_B}$ coincides with the "twist constants" of general relativity, although in the relativity literature the relation to curvature seems to be missing, along with the topological meaning.

We note that parabolicity of a two dimensional Riemannian manifold just depends on the underlying conformal structure.

**Proposition 2.19.** *Suppose that $\dim(B) = 2$ and that the hypotheses of Proposition 2.10 are satisfied. Then $\widetilde{G}^* g_{\mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N)}$ is the pullback (from $B$ to $\widetilde{B}$) of a function times $g_B$.*

*Proof.* Using the conclusion of Proposition 2.10, the third equation of (2.3) becomes

$$G^{IJ} G_{JK,\alpha} G^{KL} G_{LI,\beta} = 4R_{\alpha\beta} = 2R g_{\alpha\beta}. \tag{2.20}$$

The left-hand side of (2.20) is the local expression for $G^* g_{\mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N)}$ [26, (4.16)]. The proposition follows. □

2.5. **One dimensional base.** Returning to the equations (2.3), suppose that $n = 1$. We only consider the case $\lambda = 0$. Automatically, $F = 0$. We give $B$ a unit speed parametrization $s$.

**Proposition 2.21.** *Either $G$ is constant or, up to a change of basis of $e$ and an isometric reparametrization of $s$, we have $G(s) = s^A$ where $A$ is a symmetric $N \times N$ matrix satisfying $\mathrm{Tr}(A) = 2$ and $\mathrm{Tr}(A^2) = 4$.*

*Proof.* From (2.9), we have $\frac{d^2}{ds^2} \sqrt{\det G} = 0$. Then either $\det G$ is constant or, after an isometric reparametrization of $s$, we can write $\det G = as^2$ for some $a > 0$, with $s \in (c_1, c_2) \subset (0, \infty)$.

If $\det G$ is constant then the first equation of (2.3) gives the matrix equation

$$G_{ss} - G_s G^{-1} G_s = 0 \tag{2.22}$$

and hence

$$\mathrm{Tr}(G^{-1}G_{ss}) - \mathrm{Tr}(G^{-1}G_sG^{-1}G_s) = 0, \tag{2.23}$$

while the third equation of (2.3) gives

$$\mathrm{Tr}(G^{-1}G_{ss}) - \frac{1}{2}\,\mathrm{Tr}(G^{-1}G_sG^{-1}G_s) = 0. \tag{2.24}$$

Hence $\mathrm{Tr}(G^{-1}G_sG^{-1}G_s) = 0$, or $\mathrm{Tr}\left(\left(G^{-\frac{1}{2}}G_sG^{-\frac{1}{2}}\right)^2\right) = 0$. This implies that $G_s = 0$, so $G$ is constant.

If $\det G = as^2$ then the first equation of (2.3) becomes

$$G_{ss} + \frac{1}{s}G_s - G_sG^{-1}G_s = 0, \tag{2.25}$$

or $\partial_s(sG^{-1}G_s) = 0$. Thus $G^{-1}G_s = \frac{A}{s}$ for some matrix $A$. By a linear change of basis of $\{x^I\}_{I=1}^N$, we can assume that $G(1) = \mathrm{Id}$. Then $G(s) = s^A$. As $G(s)$ is symmetric, the matrix $A$ must also be symmetric. As $\det G = as^2$, we must have $a = 1$ and $\mathrm{Tr}(A) = 2$. The third equation in (2.3) again becomes (2.24), which now implies that $\mathrm{Tr}(A^2 - A) - \frac{1}{2}\,\mathrm{Tr}(A^2) = 0$. Hence $\mathrm{Tr}(A^2) = 2\,\mathrm{Tr}(A) = 4$. □

## 3. Collapsing of almost Ricci-flat 4-manifolds

In this section we prove Theorem 1.1 in the setting of 4-manifolds whose Ricci curvature goes to zero relative to the volume, and that are locally volume collapsed. Subsection 3.1 has a review of some of the results of [11] and their consequences. Subsections 3.2, 3.3, 3.4 and 3.5 give the proof of Theorem 1.1 when the limit space has dimension three, two, one and zero, respectively.

3.1. **General convergence arguments.** We consider four dimensional compact connected orientable Riemannian manifolds that have Ricci curvature going to zero, relative to the volume. This is more general than the setup of the introduction. In order to prove properties of such manifolds by contradiction, one considers sequences $\{(M_i, \overline{g}_i)\}_{i=1}^\infty$ where each $M_i$ is a compact connected orientable four dimensional manifold and $\lim_{i\to\infty} \| \mathrm{Ric}(M_i, \overline{g}_i)\|_\infty \cdot \mathrm{vol}(M_i, \overline{g}_i)^{\frac{1}{2}} = 0$.

*Example* 3.1. If $M$ is the underlying 4-manifold of a complex elliptic surface then LeBrun showed that $M$ admits a sequence of Riemannian metrics $\{\overline{g}_i\}_{i=1}^\infty$ with $\lim_{i\to\infty} \| \mathrm{Ric}(M, \overline{g}_i)\|_\infty \cdot \mathrm{vol}(M, \overline{g}_i)^{\frac{1}{2}} = 0$ if and only if $M$ is relatively minimal, i.e. has no smooth rational $(-1)$-curves in the fibers [23, Theorem 4].

After rescaling, we can assume that $\lim_{i\to\infty} \| \mathrm{Ric}(M_i, \overline{g}_i)\|_\infty = 0$ and $\mathrm{vol}(M_i, \overline{g}_i) \le V$ for all $i$, for some $V < \infty$. We first address the noncollapsing case. Suppose that for some $s, v > 0$, after passing to a subsequence there are points $m_i \in M_i$ so that $\mathrm{vol}(B_s(m_i)) \ge vs^4$ for all $i$. After passing to a further subsequence, there is a pointed Gromov-Hausdorff limit $\lim_{i\to\infty}(M_i, \overline{g}_i, m_i) = (X, d_X, x_\infty)$, where $X$ is a complete locally compact metric space whose Hausdorff dimension is four.

If there is a uniform upper bound on the $L^2$-norms of the curvatures of $\{(M_i, \overline{g}_i)\}_{i=1}^\infty$ then $X$ is a four dimensional Ricci-flat orbifold [1, 5, 30]. Such an $L^2$-curvature bound is guaranteed if there is a uniform upper bound on the Euler characteristics of the $M_i$'s [11, Remark 1.4]; this in turn is guaranteed if there is a uniform upper diameter bound [10].

The subject of this paper is rather the collapsing case when

$$\lim_{i\to\infty} s^{-4} \sup_{m\in M_i} \operatorname{vol}(B_s(m)) = 0 \tag{3.2}$$

for each $s > 0$. To apply the results of [11], we need a uniform upper bound on the $L^2$-norms of the curvatures of the $\{(M_i, \overline{g}_i)\}_{i=1}^\infty$. Again, it suffices to have a uniform upper bound on the Euler characteristics of the $M_i$'s.

From (3.2), for any sequence $\{s_i\}_{i=1}^\infty$ of positive numbers converging to zero, after passing to a further subsequence of $\{(M_i, \overline{g}_i)\}_{i=1}^\infty$ we can assume that $s_i^{-4} \operatorname{vol}(B_{s_i}(m)) \le \frac{1}{i}$ for all $m \in M_i$. From [11, Theorem 0.1 and Remark 5.11], there is some positive integer $\mathcal{N}$ so that for all large $i$, there are points $\{p_{i,j}\}_{j=1}^{\mathcal{B}_i}$ in $M_i$, with $\mathcal{B}_i \le \mathcal{N}$, such that

$$\int_{M_i - \bigcup_{j=1}^{\mathcal{B}_i} B_{p_{i,j}}(s_i)} |\operatorname{Rm}|^2 \, \operatorname{dvol}_{M_i} \le \text{const.}\, i^{-1}. \tag{3.3}$$

Choose basepoints $m_i \in M_i$. After passing to a further subsequence, we can assume that $\lim_{i\to\infty}(M_i, \overline{g}_i, m_i) = (X, d_X, x_\infty)$ in the pointed Gromov-Hausdorff topology, where $X$ is a complete locally compact metric space whose Hausdorff dimension is less than four. We can also assume that the $\mathcal{B}_i$'s are all the same number, say $\mathcal{B}$. Then we can assume that for each $j \in \{1, \ldots, \mathcal{B}\}$, either $\lim_{i\to\infty} p_{i,j} = x_j$ for some $x_j \in X$ or $\lim_{i\to\infty} d_{M_i}(m_i, p_{i,j}) = \infty$. After removing repetitions, let $\{x_j\}_{j=1}^C \subset X$ be the limits.

From [11, Theorem 0.8 and Remark 8.22], for any compact subset $K$ of $X - \{x_1, \ldots, x_C\}$, there is some $\epsilon_K > 0$ so that for any $q \in [1, \infty)$ and for large $i$, on the subset of $(M_i, \overline{g}_i)$ that is $\epsilon_K$-close to $K$ we have uniformly bounded $W^{2,q}$-covering geometry; see also [24, Theorem 1.1]. Hence we can apply techniques from bounded curvature collapse [11, Remark 2.7]. We will use convergence of Riemannian groupoids, as in [26, Sections 5.1-5.4]. Choose $x' \in X - \{x_1, \ldots, x_C\}$. (The choice of $x'$ will be irrelevant.) Choose a sequence $m'_i \in M_i$ with $\lim_{i\to\infty} m'_i = x'$. Using an exhaustion of $X - \{x_1, \ldots, x_C\}$ by precompact open sets containing $x'$, after passing to a subsequence, we can assume that $\lim_{i\to\infty} \left( M_i - \bigcup_{j=1}^{\mathcal{B}_i} B_{p_{i,j}}(s_i), \overline{g}_i, m'_i \right) = (\mathcal{X}, g_{\mathcal{X}}, x')$, where $(\mathcal{X}, g_{\mathcal{X}}, x')$ is a four dimensional closed Hausdorff pointed Riemannian groupoid whose orbit space $\mathcal{O}$ is $X - \{x_1, \ldots, x_C\}$, and we can think of $x'$ as an orbit. Taking the metric completion of $\mathcal{O}$ amounts to adding a finite number of points. The unit space of $\mathcal{X}$ carries a structure sheaf $\underline{\mathfrak{g}}$ of nilpotent Lie algebras, which acts on the unit space by local Killing vector fields. The local Killing vector fields do not simultaneously vanish at any point in the unit space, unless $\mathfrak{g} = 0$.

The metric convergence to $(\mathcal{X}, g_{\mathcal{X}})$ is in the weak $W^{2,q}$-topology, for any $1 \le q < \infty$. Hence the metric $g_{\mathcal{X}}$ on the unit space of $\mathcal{X}$ is Ricci-flat. As the limit is constructed using harmonic coordinates, it follows that the metric on the unit space is smooth. (If each $M_i$ is Ricci-flat then the convergence is $C^\infty_{loc}$.)

If $\mathcal{X}$ has trivial isotropy groups then the orbit space $\mathcal{O}$ is smooth. Then for any precompact open subset $U \subset \mathcal{O}$ and for large $i$, there is a subset of the "regular" region $M_i - \bigcup_{j=1}^{\mathcal{B}_i} B_{p_{i,j}}(s_i)$ that is the total space of a fiber bundle over $U$, with infranil fibers. We are interested in the abelian case with $N$-torus fibers. In this case, the underlying groupoid of the limit $\mathcal{X}$ can be described in terms of the following transition maps. Let $\{U_a\}_{a \in \mathcal{A}}$ be a covering of $\mathcal{O}$ by contractible open sets. For $a, b \in \mathcal{A}$, the transition map $\phi_{ab}$ is a smooth map from $U_a \cap U_b$ to $\frac{\mathbb{R}^N}{\mathbb{R}^N_\delta} \rtimes \mathrm{GL}(N, \mathbb{R})_\delta$, where the $\delta$-subscript denotes the discrete topology [27, (5.3)]. (Compare with the transition maps for a principal $T^N$-bundle, which take value in $\frac{\mathbb{R}^N}{\mathbb{Z}^N}$, and the transition maps for a twisted principal $T^N$-bundle, which take value in $\frac{\mathbb{R}^N}{\mathbb{Z}^N} \rtimes \mathrm{GL}(N, \mathbb{Z})$.) That is, $\phi_{ab}$ is represented by a pair $(f_{ab}, \gamma_{ab})$ where $f_{ab} : U_a \cap U_b \to \mathbb{R}^N$ is a smooth map and $\gamma_{ab} \in \mathrm{GL}(N, \mathbb{R})$. Two pairs $(f_{ab}, \gamma_{ab})$ and $(f'_{ab}, \gamma'_{ab})$ are equivalent if $\gamma_{ab} = \gamma'_{ab}$ and $f_{ab} = f'_{ab} + v_{ab}$ for some constant vector $v_{ab} \in \mathbb{R}^N$. The $\{\phi_{ab}\}_{a,b \in \mathcal{A}}$ have to satisfy the cocycle condition. Up to equivalence, such structures on $\mathcal{O}$ are classified by a set $\mathrm{H}^1(\mathcal{O}, \underline{\mathcal{E}})$ that fits into an exact sequence of pointed sets [27, (5.7)]

$$\mathrm{GL}(N, \mathbb{R}) \longrightarrow \mathrm{H}^2(\mathcal{O}; \mathbb{R}^N) \longrightarrow \mathrm{H}^1(\mathcal{O}, \underline{\mathcal{E}}) \longrightarrow \mathrm{H}^1(\mathcal{O}, \mathrm{GL}(N, \mathbb{R})_\delta). \tag{3.4}$$

Here $\mathrm{GL}(N, \mathbb{R})$ acts on $\mathrm{H}^2(\mathcal{O}; \mathbb{R}^N)$. The set $\mathrm{H}^1(\mathcal{O}, \mathrm{GL}(N, \mathbb{R})_\delta)$ classifies the flat $\mathbb{R}^N$-vector bundles on $\mathcal{O}$.

A Riemannian metric on the groupoid has the following description. For each $a \in \mathcal{A}$, let $G_a$ be a smooth map from $U_a$ to the positive-definite symmetric $(N \times N)$-matrices, let $A_a$ be an $\mathbb{R}^N$-valued 1-form on $U_a$ and let $g_a$ be a Riemannian metric on $U_a$. Then the triples $\{(G_a, A_a, g_a)\}_{a \in \mathcal{A}}$ define a Riemannian metric on the groupoid if for each $a, b \in \mathcal{A}$, on $U_a \cap U_b$ we have $(G_b, A_b, g_b) = (\gamma_{ab} G_a \gamma_{ab}^T, \gamma_{ab}(A_a + df_{ab}), f_{ba}^* g_a)$. (Compare with (2.1).)

We claim that in the collapsing case, the flat $\mathbb{R}^N$-vector bundle on $\mathcal{O}$ corresponding to $\{\gamma_{ab}\}_{a,b \in \mathcal{A}}$ has holonomy in $\det^{-1}(\pm 1) \subset \mathrm{GL}(N, \mathbb{R})$. The reason is that we are considering the case when for any precompact open set $U \subset \mathcal{O}$ and for large $i$, part of the regular region of $M_i$ is the total space of a $T^N$-bundle over $U$. Given such an $i$, there is a flat $\mathbb{R}^N$-bundle on $U$, whose fiber over a point in $U$ is the first cohomology of the torus fiber over the point. As $\mathrm{H}^N(T^N; \mathbb{Z}) \cong \mathbb{Z}$, the holonomy of this flat $\mathbb{R}^N$-bundle lies in $\det^{-1}(\pm 1) \subset \mathrm{GL}(N, \mathbb{R})$. Passing to the limit as $i \to \infty$, the holonomy will still lie in $\det^{-1}(\pm 1) \subset \mathrm{GL}(N, \mathbb{R})$. This is true for any such $U$.

Thus we can apply the results of Section 2 about Ricci-flatness to $\mathcal{X}$. If $\mathcal{X}$ has finite isotropy groups then there is a similar statement, with $\mathcal{O}$ becoming an orbifold.

Let $B$ be a connected component of $X - \{x_1, \ldots, x_C\}$. We replace $\mathcal{X}$ by its subgroupoid consisting of the orbits corresponding to points in $B$. In the next four subsections we prove the properties of $\mathcal{X}$ asserted in Theorem 1.1.

Theorem 1.1 is stated in the introduction for a sequence with bounded diameter, Ricci curvature going to zero and volume going to zero. In this case there is a uniform upper volume bound and (3.2) holds. Hence the discussion in this subsection applies.

3.2. **Three dimensional limit space.** Suppose that $\dim(B) = 3$. Then it is a three-dimensional Riemannian orbifold, as the groupoid $\mathcal{X}$ must have finite isotropy groups in this case.

*Remark* 3.5. The appearance of possible orbifold points in $B$ has nothing to do with the points in $X$ that could arise as limits of the curvature blowup regions in the $(M_i, \overline{g}_i)$'s. These were already removed in forming $B$. There could be orbifold points in $B$ even if the manifolds $(M_i, \overline{g}_i)$ are flat.

The matrix $(G_{IJ})$ is just a function $G$ on $B$.

**Proposition 3.6.** *If $B$ is parabolic and $G$ is uniformly bounded above then $G$ is a constant function, $F = 0$ and $B$ is flat.*

*Proof.* The first two statements follow from Proposition 2.10, which also says that $B$ is Ricci-flat. Since it is three dimensional, it is flat. □

*Example* 3.7. Examples of the hypotheses of Proposition 3.6 come from the construction of collapsing Ricci-flat metrics on $K3$ in [15]. There is an $\mathbb{Z}_2$-action on $S^1$ by complex conjugation, and hence on $T^3$, with eight fixed points. The paper [15] constructs sequences of Ricci-flat metrics on $K3$ that converge to $X = T^3/\mathbb{Z}_2$ in the Gromov-Hausdorff topology. The subset $B$ is $X$ minus the eight fixed points and a certain number of other points, where the number can be chosen between 0 and 16. We note that $B$ is parabolic.

During the collapse, ALF gravitational instantons of dihedral type bubble off from the eight fixed points.

3.3. **Two dimensional limit space.** Suppose that $\dim(B) = 2$.

3.3.1. *Finite isotropy groups.* Consider first the case when the groupoid $\mathcal{X}$ has trivial isotropy groups. Then $B$ is a smooth surface. For large $i$, the corresponding subset of $M_i$ is the total space of a fiber bundle over $B$. As $M_i$ is orientable, the fibers must be 2-tori. We identify $\mathrm{SL}(2, \mathbb{R}) / \mathrm{SO}(2)$ with the hyperbolic plane $H^2$, the latter having a fixed orientation.

**Proposition 3.8.** *If $\det(G)$ is bounded above and $B$ is parabolic then*

- *$\det(G)$ is constant,*
- *$F = 0$ and*
- *With the right choice of orientation on $\widetilde{B}$, the $\rho$-equivariant map $\widetilde{G} : \widetilde{B} \to H^2$ is holomorphic.*

*Proof.* The first two statements follow from Proposition 2.10. From Proposition 2.19, the map $\widetilde{G}$ is conformal. Then with the right choice of orientation on $\widetilde{B}$, it is holomorphic. □

*Remark* 3.9. Examples where $\det(G)$ is bounded above come from collapsing Kähler manifolds $(M_i, \overline{g}_i)$ that admit holomorphic fiberings over $X$ with the generic fiber being a torus. Then for each $i$, all of the regular fibers have the same volume.

The notion of a semiflat Kähler metric is given in [17, Section 3], [19, Section 3.2], [21] and [32, Section 3.1], among other places. It is usually considered for torus fibrations, but also makes sense in our context.

**Corollary 3.10.** *Under the hypotheses of Proposition 3.8, the metric $g_{\mathcal{X}}$ on the unit space $M$ of $\mathcal{X}$ is a semiflat Kähler metric.*

From (2.20), the pullback of the Ricci tensor of $B$, to $\widetilde{B}$, is $\frac{1}{4}\widetilde{G}^* g_{H^2}$.

To be more explicit about the semiflat Kähler metric, let $\tau = \tau(z)$ be a holomorphic map from an open set $U$ in $B$ to the upper half plane. A corresponding map $(G_{IJ})$ from $U$ to symmetric matrices is

$$G = \frac{1}{\Im(\tau)} \begin{pmatrix} 1 & \Re(\tau) \\ \Re(\tau) & |\tau|^2 \end{pmatrix}. \tag{3.11}$$

The complex structure on $M$ can be defined by local holomorphic coordinates. One local coordinate is the pullback of a local holomorphic coordinate on $B$. The other one is $w = x^1 + \tau x^2$. The Kähler form corresponding to $\overline{g}$ is

$$\overline{\omega} = \omega_B + \frac{i}{2\Im(\tau)} \left( dw - \frac{w - \overline{w}}{\tau - \overline{\tau}} d\tau \right) \wedge \left( d\overline{w} - \frac{w - \overline{w}}{\tau - \overline{\tau}} d\overline{\tau} \right), \tag{3.12}$$

where $\omega_B$ is the Kähler form for $g$. If $\phi_B$ is a local Kähler potential for $\omega_B$, meaning that $\omega_B = i\partial\overline{\partial}\phi_B$, then a local Kähler potential for $\overline{\omega}$ is

$$\overline{\phi} = \phi_B - \frac{1}{4\Im(\tau)} (w - \overline{w})^2. \tag{3.13}$$

Equation (2.20) becomes the statement that the Ricci form on $B$ is

$$\mathrm{Ric}_B = \frac{i}{4\Im(\tau)^2} d\tau \wedge d\overline{\tau} = \frac{i|\tau'(z)|^2}{4\Im(\tau)^2} dz \wedge d\overline{z}. \tag{3.14}$$

*Remark* 3.15. Although the metric completion of $B$ amounts to adding a finite number of points, it does not follow from this that $B$ is parabolic. For example, the domain $U = \{z \in \mathbb{C} : 1 < |z| < 2\}$ is nonparabolic. One can construct a metric $e^{2\phi} g_{Eucl}$ on $U$ so that the metric completion consists of adding two points.

*Example* 3.16. The paper [18] considers a collapsing sequence of Ricci-flat metrics on $K3$, for which $B$ is $S^2$ minus 24 points. In this case, $B$ is parabolic. The semiflat metric is described in [18, Example 2.2].

To briefly summarize the geometry of the collapse, as taken from [18], during the collapse there are 24 Taub-NUT gravitational instantons that bubble off. A Taub-NUT gravitational instanton is of ALF type, i.e. has cubic volume growth. It may not be evident how truncated ALF instantons get attached in the collapsing limit to the semiflat metric on a $T^2$-fibration over the complement of 24 balls in $S^2$, since there seems to be a discrepancy in the limiting dimensions (3 vs. 2). This is the role of the (incomplete) Ricci-flat Ooguri-Vafa metric, which provides the approximate geometry over a ball around any of the 24 points in $S^2$, for the collapsing $K3$ manifold. The Ooguri-Vafa manifold contains an approximate (truncated and rescaled) Taub-NUT metric. In effect, the Ooguri-Vafa manifold gives a Ricci-flat transition region as a cobordism between the boundary 3-sphere of the truncated Taub-NUT metric, with the Hopf $S^1$-action, and the Nil-manifold that lives over the boundary of the ball and has a twisted $T^2$-action. This transition region carries a mixed $F$-structure in the sense of [9].

Now consider the case when isotropy groups are finite. Then the orbit space $B$ is an orbifold. The results of this subsubsection extend to the orbifold setting.

3.3.2. *Infinite isotropy groups.* Suppose now that some points in the unit space of $\mathcal{X}$ have isotropy group isomorphic to $\mathrm{SO}(2)$, and the other isotropy groups are trivial. Then $B$ is a surface with boundary. Consider the interior of $B$, i.e. the subset of $B$ corresponding to points in the unit space with trivial isotropy group.

**Proposition 3.17.** $\det(G)$ *extends continuously to be zero on* $\partial B$.

*Proof.* Let $m$ be a point in the unit space of $\mathcal{X}$ with isotropy group $\mathrm{SO}(2)$. From the slice theorem, a neighborhood of $m$ in the unit space is equivariantly diffeomorphic to a neighborhood of the origin in $\mathbb{R} \times (\mathbb{R} \times \mathbb{C})$. Here the group action is on the $\mathbb{R} \times \mathbb{C}$ factor, with translations of the $\mathbb{R}$-term and rotations of the $\mathbb{C}$-term. The points in $\mathbb{R} \times (\mathbb{R} \times \mathbb{C})$ with trivial isotropy are $\mathbb{R} \times (\mathbb{R} \times \mathbb{C}^*)$. Their quotient by the group action is $\mathbb{R} \times (0, \infty)$, from which one obtains local coordinates for the part of the interior of $B$ approaching a boundary point.

The metric $\overline{g}$ is smooth in the coordinates given by $\mathbb{R} \times (\mathbb{R} \times \mathbb{C})$. To describe $\overline{g}$ in terms of the setup of Subsection 2.1, we use the local parametrization of $\mathbb{R} \times (\mathbb{R} \times \mathbb{C}^*)$ by $(b^1, b^2, x^1, x^2) \to (b^1, x^1, b^2 \cos(x^2), b^2 \sin(x^2))$, where $b^2 > 0$. Then for $b^2$ small, the matrix $G$ is asymptotic to $\begin{pmatrix} 1 & 0 \\ 0 & (b^2)^2 \end{pmatrix}$. In particular, $\lim_{b^2 \to 0} \det(G) = 0$, showing that $\det(G)$ vanishes on $\partial B$. □

Let $\Sigma$ be a two dimensional Riemannian manifold with nonempty boundary. We say that $\Sigma$ is parabolic if any nonnegative $C^2$-regular subharmonic function on $\Sigma$ that vanishes on $\partial \Sigma$, and is bounded above, must be zero. (The reference [16] instead imposes Neumann boundary conditions.)

**Proposition 3.18.** *Suppose that* $\det(G)$ *is bounded above. Suppose that the isotropy groups of points in the unit space of* $\mathcal{X}$ *are trivial or isomorphic to* $\mathrm{SO}(2)$*, with not all of them being trivial. Then* $B$ *cannot be parabolic.*

*Proof.* By assumption, $B$ is a smooth surface with nonempty boundary. If it is parabolic then as in Proposition 3.8, the function $\det(G)$ is a nonzero constant. This contradicts Proposition 3.17. □

Finally, suppose just that not all points in the unit space of $\mathcal{X}$ have a finite isotropy group. Then $B$ is a two dimensional orbifold-with-boundary, with a nonempty boundary. It makes sense to talk about $B$ being parabolic. Proposition 3.18 has the following extension.

**Proposition 3.19.** *Suppose that* $\det(G)$ *is bounded above. Suppose that not all points in the unit space of* $\mathcal{X}$ *have a finite isotropy group. Then* $B$ *cannot be parabolic.*

*Example* 3.20. Suppose that $\det(G)$ is bounded above and $B$ is compact. Then it is a compact orbifold-with-boundary and hence is parabolic. From Propositions 3.8 and 3.19, and (3.14), $B$ is a boundaryless orbifold with nonnegative scalar curvature. Hence the orbifold universal cover $\widetilde{B}$ is either isometric to $\mathbb{R}^2$ or has underlying topological space $S^2$ with zero, one or two orbifold singular points. In either case, the holomorphic map $\widetilde{G} : \widetilde{B} \to H^2$ must be a constant map. Then (3.14) implies that $B$ is flat, so $\widetilde{B} = \mathbb{R}^2$. In conclusion, the Riemannian groupoid $\mathcal{X}$ is flat.

3.4. **One dimensional limit space.** Suppose that $\dim(B) = 1$. Then $B$ is a connected one-dimensional orbifold. After passing to a cover, we can assume that $B$ is a circle or an open interval.

The sheaf $\underline{\mathfrak{g}}$ is a sheaf of three dimensional nilpotent Lie algebras. For large $i$, the corresponding subset of $M_i$ is a fiber bundle over $B$, whose fiber is a three dimensional infranilmanifold. By [25, p. 291], after passing to a finite cover we can assume that the fiber is a nilmanifold. (This is obvious when $B$ is an open interval.)

Suppose first that the Lie algebra $\mathfrak{g}$ is abelian. Then the fiber is a 3-torus. If $B$ is a circle then from Proposition 2.21, $(G_{IJ})$ is locally constant and the monodromy of $e$ around $B$ is orthogonal. In particular, $\overline{g}$ is flat. If $B$ is an interval then from Proposition 2.21, the metric on the unit space of $\mathcal{X}$ is locally isometric to

$$ds^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{3.21}$$

or

$$ds^2 + s^{2p_1}(dx^1)^2 + s^{2p_2}(dx^2)^2 + s^{2p_3}(dx^3)^2, \tag{3.22}$$

where $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$. The metric (3.22) is a Riemannian version of the Ricci-flat Kasner metric from general relativity [14, Section 9.1.1].

Going back to the possibility that $B$ is an orbifold, if there is a singular orbifold point then the pullback geometry has a $\mathbb{Z}_2$-symmetry and hence must be flat.

The other possibility for $\mathfrak{g}$ is the three dimensional Heisenberg Lie algebra. In this case, the ODE arguments for Ricci-flat Lorentzian metrics with a spatially homogeneous cross-section of nil$^3$-type [14, Section 6.3.2] extend to the Riemannian case, to show that the metric on the unit space of $\mathcal{X}$ is locally isometric to a metric of the form

$$A^2 ds^2 + s^{2p_1}A^{-2}(dx^1 + 4p_1 b x^3 dx^2)^2 + s^{2p_2}A^2(dx^2)^2 + s^{2p_3}A^2(dx^3)^2. \tag{3.23}$$

Here $b > 0$, $A^2 = 1 + b^2 s^{4p_1}$ and $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$. The metric (3.23) is a Riemannian version of the Taub Bianchi-II metric of general relativity [14, Section 9.2.1].

*Example* 3.24. There are Ricci-flat $K3$ manifolds that collapse to a closed interval, for which the regular regions approach a flat four dimensional geometry over an open interval [12, Section 5]. During the collapse, an ALH gravitational instanton bubbles off from each end of the interval.

There are also Ricci-flat $K3$ manifolds that collapse to a closed interval, with the subset $B$ being two open intervals, so that over each of these intervals, the regular regions approach a Riemannian Taub Bianchi-II geometry [20]. The construction uses the fact that in the four dimensional case, the Ricci-flat metrics constructed in [31, Theorem 4.1] have an asymptotic geometry of Riemannian Taub Bianchi-II type.

We do not know if there are almost Ricci-flat metrics that collapse to an interval, for which the regular regions approach Riemannian Kasner geometries over open intervals.

3.5. **Zero dimensional limit space.** If $\mathcal{O}$ is a point then the unit space of $\mathcal{X}$ is locally homogeneous with a Ricci-flat metric. Hence it must be flat [29].

## References


[1] M. Anderson, "Convergence and rigidity of manifolds under Ricci curvature bounds", Invent. Math. 102, p. 429-445 (1990)

[2] M. Anderson, "Hausdorff perturbations of Ricci-flat manifolds and the splitting theorem", Duke Math. J. 68, p. 67-82 (1992)

[3] M. Anderson and J. Cheeger, "Diffeomorphism finiteness for manifolds with Ricci curvature and $L^{n/2}$-norm of curvature bounded", Geom. Funct. Anal. 1, p. 231252 (1991)

[4] S. Bando, "Bubbling out of Einstein manifolds", Tohoku Math. J. 42, p. 205216 (1990)

[5] S. Bando, A. Kasue and H. Nakajima, "On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth", Invent. Math. 97 (1989), p. 313349 (1989)

[6] S. Brendle and N. Kapouleas, "Gluing Eguchi-Hanson metrics and a question of Page", Comm. Pure Appl. Math 70, p. 1366-1401 (2017)

[7] D. Calderbank and H. Pedersen, "Selfdual Einstein metrics with torus symmetry", J. Diff. Geom. 3, p. 485-521 (2002)

[8] J. Cheeger, K. Fukaya and M. Gromov, "Nilpotent structures and invariant metrics on collapsed manifolds", J. of the Amer. Math. Soc. 5, p. 327-372 (1992)

[9] J. Cheeger and M. Gromov, "Collapsing Riemannian manifolds while keeping their curvature bounded I", J. Diff. Geom. 23, p. 309-346 (1986)

[10] J. Cheeger and A. Naber, "Regularity of Einstein manifolds and the codimension 4 conjecture" Ann. Math. 182, p. 1093-1165 (2015)

[11] J. Cheeger and G. Tian, "Curvature and injectivity radius estimates for Einstein 4-manifolds", J. Amer. Math. Soc. 19, p. 487-525 (2006)

[12] G. Chen and X. Chen, "Gravitational instantons with faster than quadratic curvature decay (III)", preprint, https://arxiv.org/abs/1603.08465 (2016)

[13] B. Dammerman, "Metrics of special curvature with symmetry", preprint, https://arxiv.org/pdf/math/0610738.pdf (2004)

[14] G. Ellis and J. Wainwright, eds, Dynamical systems in cosmology, Cambridge University Press, Cambridge (1997)

[15] L. Foscolo, "ALF gravitational instantons and collapsing Ricci-flat metrics on the K3 surface", preprint, https://arxiv.org/abs/1603.06315 (2016)

[16] A. Grigoryan, "Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds", Bull. Amer. Math. Soc. 36, p. 135-249 (1999)

[17] M. Gross. V. Tosatti and Y. Zhang, "Collapsing of abelian fibered Calabi-Yau manifolds", Duke Math J. 162, p. 517-551 (2013)

[18] M. Gross and P. Wilson, "Large complex structure limits of $K3$ surfaces", J. Diff. Geom. 55, p. 475-546 (2000)

[19] H.-J. Hein, "Gravitational instantons from rational elliptic surfaces", J. Amer. Math. Soc. 25, p. 355-393 (2012)
[20] H.-J. Hein, Song Sun, Jeff Viaclovsky and Ruobing Zhang, in preparation
[21] H.-J. Hein and V. Tosatti, "Remarks on the collapsing of torus fibered CalabiYau manifolds", Bull. London Math. Soc. 47, p. 1021-1027 (2015)
[22] V. Kapovitch and J. Lott, "On noncollapsed almost Ricci-flat 4-manifolds", to appear, Amer. J. Math.
[23] C. LeBrun, "Kodaira dimension and the Yamabe problem", Comm. Anal. Geom. 7, p. 133-156 (1999)
[24] Y. Li, "Smoothing Riemannian metrics with bounded Ricci curvatures in dimension four", Adv. Math. 223, p. 1924-1957 (2010)
[25] J. Lott, "Collapsing and the differential form Laplacian : the case of a smooth limit space", Duke Math. J. 114, p. 267-306 (2002)
[26] J. Lott, "On the long-time behavior of type-III Ricci flow solutions", Math. Ann. 339, p. 627-666 (2007)
[27] J. Lott, "Dimensional reduction and the long-time behavior of Ricci flow", Comm. Math. Helv. 85, p. 485-534 (2010)
[28] A. Naber and G. Tian, "Geometric structures of collapsing Riemannian manifolds II", to appear, J. für die Reine und Angew. Math. (Crelle's journal)
[29] A. Spiro, "A remark on locally homogeneous Riemannian spaces", Results Math. 24, p. 318-325 (1993)
[30] G. Tian, "On Calabi's conjecture for complex surfaces with positive first Chern class", Invent. Math. 101, p. 101-172 (1990)
[31] G. Tian and S.-T. Yau, "Complete Kähler manifolds with zero Ricci curvature I", J. Amer. Math. Soc. 3, p. 579-609 (1990)
[32] V. Tosatti and Y. Zhang, "Collapsing hyperKähler manifolds", preprint, https://arxiv.org/abs/1705.03299 (2017)
[33] M. Wang, "Einstein metrics from symmetry and bundle constructions: a sequel", in Differential geometry: under the influence of S.-S. Chern, Advanced Lectures in Math. 252, Higher Education Press/International Press, p. 253-309 (2012)

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840, USA
*E-mail address*: lott@berkeley.edu