Geometries (1).

- We call a structure $\langle P, L, I\rangle$ a plane geometry if $I \subseteq P \times L$ is a relation.
- We call the elements of P “points”, the elements of L “lines” and we read $pI\ell$ as “p lies on ℓ”.
- If ℓ and ℓ^* are lines, we say that ℓ and ℓ^* are parallel if there is no point p such that $pI\ell$ and $pI\ell^*$.
- **Example.** If $P = \mathbb{R}^2$, then we call $\ell \subseteq P$ a line if

$$\ell = \{\langle x, y\rangle \,;\, y = a \cdot x + b\}$$

for some $a, b \in \mathbb{R}$. Let $\mathcal{L}$ be the set of lines. We write $pI\ell$ if $p \in \ell$. Then $\langle P, \mathcal{L}, I\rangle$ is a plane geometry.

Geometries (2).

- (A1) For every $p \neq q \in P$ there is exactly one $\ell \in L$ such that $pI\ell$ and $qI\ell$.
- (A2) For every $\ell \neq \ell^* \in L$, either ℓ and ℓ^* are parallel, or there is exactly one $p \in P$ such that $pI\ell$ and $pI\ell^*$.
- (N) For every $p \in P$ there is an $\ell \in L$ such that p doesn't lie on ℓ and for every $\ell \in L$ there is an $p \in P$ such that p doesn't lie on ℓ.
- (P2) For every $\ell \neq \ell^* \in L$, there is exactly one $p \in P$ such that $pI\ell$ and $pI\ell^*$.

A plane geometry that satisfies (A1), (A2) and (N) is called a plane. A plane geometry that satisfies (A1), (P2) and (N) is called a projective plane.

Geometries (3).

- (A1) For every $p \neq q \in P$ there is exactly one $\ell \in L$ such that $pI\ell$ and $qI\ell$.
- (A2) For every $\ell \neq \ell^* \in L$, either ℓ and ℓ^* are parallel, or there is exactly one $p \in P$ such that $pI\ell$ and $pI\ell^*$.
- (N) For every $p \in P$ there is an $\ell \in L$ such that p doesn't lie on ℓ and for every $\ell \in L$ there is an $p \in P$ such that p doesn't lie on ℓ.

Let $\mathbf{P} := \langle \mathbb{R}^2, \mathcal{L}, \in \rangle$. Then $\mathbf{P}$ is a plane.

- (WE) ("the weak Euclidean postulate") For every $\ell \in L$ and every $p \in P$ such that p doesn't lie on ℓ, there is an $\ell^* \in L$ such that $pI\ell^*$ and ℓ and ℓ^* are parallel.
- (SE) ("the strong Euclidean postulate") For every $\ell \in L$ and every $p \in P$ such that p doesn't lie on ℓ, there is exactly one $\ell^* \in L$ such that $pI\ell^*$ and ℓ and ℓ^* are parallel.

$\mathbf{P}$ is a strongly Euclidean plane.

Geometries (4).

Question. Do (A1), (A2), (N), and (WE) imply (SE)?

It is easy to see what a positive solution would be, but a negative solution would require reasoning over all possible proofs.

Semantic version of the question. Is every weakly Euclidean plane strongly Euclidean?

Syntactic versus semantic.

	Does Φ imply ψ?	**Does every Φ-structure satisfy ψ?**
Positive	Give a proof	Check all structures
	$\exists$	$\forall$
Negative	Check all proofs	Give a counterexample
	$\forall$	$\exists$

Euclid's Fifth Postulate.

"the scandal of elementary geometry" (D'Alembert 1767)

"In the theory of parallels we are even now not further than Euclid. This is a shameful part of mathematics..." (Gauss 1817)

Johann Carl Friedrich Gauss	Nikolai Ivanovich Lobachevsky	János Bolyai
(1777-1855)	(1792-1856)	(1802-1860)
		
1817	1829	1823

A non-Euclidean geometry.

Take the usual geometry $\mathbf{P} = \langle \mathbb{R}^2, \mathcal{L}, \in \rangle$ on the Euclidean plane.

Consider $\mathbb{U} := \{x \in \mathbb{R}^2\,;\ \|x\| < 1\}$. We define the restriction of $\mathcal{L}$ to $\mathbb{U}$ by $\mathcal{L}^{\mathbb{U}} := \{\ell \cap \mathbb{U}\,;\ \ell \in \mathcal{L}\}$.

$\mathbf{U} := \langle \mathbb{U}, \mathcal{L}^{\mathbb{U}}, \in \rangle$.

Theorem. $\mathbf{U}$ is a weakly Euclidean plane which is not strongly Euclidean.

Cantor (1).

Georg Cantor
(1845-1918)
studied in Zürich, Berlin, Göttingen
Professor in Halle

- Work in analysis leads to the notion of cardinality (1874): most real numbers are transcendental.
- Correspondence with Dedekind (1831-1916): bijection between the line and the plane.
- Perfect sets and iterations of operations lead to a notion of ordinal number (1880).

Cantor (2).

Georg Cantor (1845-1918)

- 1877. Leopold Kronecker (1823-1891) tried to prevent publication of Cantor’s work.
- Cantor is supported by Dedekind and Felix Klein.
- 1884: Cantor suffers from a severe depression.
- 1888-1891: Cantor is the leading force in the foundation of the *Deutsche Mathematiker-Vereinigung*.
- Development of the foundations of set theory: 1895-1899.

Cardinality (1).

$$\begin{array}{lcccccccccc} \text{The natural numbers} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ \\ \text{The even numbers} & 0 & & 2 & & 4 & & 6 & & 8 & \cdots \end{array}$$

- There is a 1-1 correspondence (bijection) between $\mathbb{N}$ and the even numbers.
- There is a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.
- There is a bijection between $\mathbb{Q}$ and $\mathbb{N}$.
- There is no bijection between the set of infinite 0-1 sequences and $\mathbb{N}$.
- There is no bijection between $\mathbb{R}$ and $\mathbb{N}$.

Cardinality (2).

Theorem (Cantor). There is no bijection between the set of infinite 0-1 sequences and $\mathbb{N}$.

Theorem (Cantor). There is a bijection between the real line and the real plane.

Proof. Let's just do it for the set of infinite 0-1 sequences and the set of pairs of infinite 0-1 sequences:
If x is an infinite 0-1 sequence, then let

$$x_0(n) := x(2n), \text{ and}$$

$$x_1(n) := x(2n+1).$$

Let $F(x) := \langle x_0, x_1 \rangle$. F is a bijection. q.e.d.

Cantor to Dedekind (1877): "*Ich sehe es, aber ich glaube es nicht!*"

Transfiniteness (1).

If $X \subseteq \mathbb{R}$ is a set of reals, we call $x \in X$ isolated in X if no sequence of elements of X converges to x.

Cantor's goal: Given any set X, give a construction of a nonempty subset that doesn't contain any isolated points.

Idea: Let X^{isol} be the set of all points isolated in X, and define $X' := X \backslash X^{\mathrm{isol}}$.

Problem: It could happen that $x \in X'$ was the limit of a sequence of points isolated in X. So it wasn't isolated in X, but is now isolated in X'.

Solution: Iterate the procedure: $X_0 := X$ and $X_{n+1} := (X_n)'$.

Transfiniteness (2).

$X' := X \backslash X^{\text{isol}}$; $X_0 := X$ and $X_{n+1} := (X_n)'$.

Question: Is $\bigcap_{n\in\mathbb{N}} X_n$ a set without isolated points?

Answer: In general, no!

So, you could set $X_\infty := \bigcap_{n\in\mathbb{N}} X_n$, and then $X_{\infty+1} := (X_\infty)'$; in general, $X_{\infty+n+1} := (X_{\infty+n})'$.

The indices used in transfinite iterations like this are called ordinals.

Sets (1).

The notion of cardinality needs a general notion of function as a special relation between sets. In order to make the notion of an ordinal precise, we also need sets.

What is a set?

> *Eine Menge ist eine Zusammenfassung bestimmter, wohlunterschiedener Dinge unserer Anschauung oder unseres Denkens zu einem Ganzen.* (Cantor 1895)

Sets (2).

Eine Menge ist eine Zusammenfassung bestimmter, wohlunterschiedener Dinge unserer Anschauung oder unseres Denkens zu einem Ganzen. (Cantor 1895)

Example. Call a linear ordering $\leq$ on a set X a wellorder if any nonempty set $A \subseteq X$ has a $\leq$-least element.

Question. Can we define a wellorder on the set $\mathbb{R}$ of real numbers?

Answer (Zermelo 1908). Yes! The proof uses the following statement about sets: "Whenever I is an index set and for each $i \in I$, the set X_i is nonempty, then the set C of functions $f : I \to \bigcup X_i$ such that for all i, we have $f(i) \in X_i$ is nonempty as well."

$\leadsto$ Problems in the Foundations of Mathematics (next week)

Syllogistics versus Propositional Logic.

Deficiencies of Syllogistics:

Not expressible:

Every X is a Y and a Z. *Ergo...* Every X is a Y.

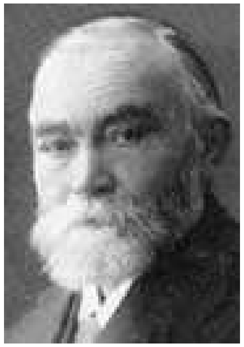
Deficiencies of Propositional Logic:

- $X\mathrm{a}Y$ can be represented as $Y \rightarrow X$.
- $X\mathrm{e}Y$ can be represented as $Y \rightarrow \neg X$.

Not expressible:

$X\mathrm{i}Y$ and $X\mathrm{o}Y$.

Frege.



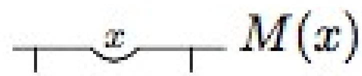
Gottlob Frege
1848 - 1925

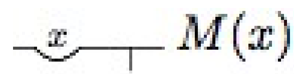
- Studied in Jena and Göttingen.
- Professor in Jena.
- *Begriffsschrift* (1879).
- *Grundgesetze der Arithmetik* (1893/1903).

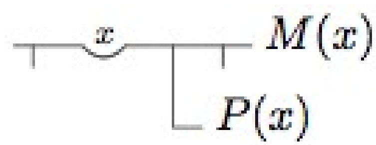
“Every good mathematician is at least half a philosopher, and every good philosopher is at least half a mathematician. (G. Frege)”

Frege's logical framework.

"Everything is M"  $\forall x\, M(x)$

"Something is M"  $\exists x\, M(x) \equiv \neg\forall x\, \neg M(x)$

"Nothing is M"  $\forall x\, \neg M(x)$

"Some P is an M"  $\exists x\, (P(x) \wedge M(x))$

$\equiv \neg\forall x\, (P(x) \rightarrow \neg M(x))$

Second order logic allowing for quantification over properties.

Frege's importance.

- Notion of a formal system.
- Formal notion of proof in a formal system.
- Analysis of number-theoretic properties in terms of second-order properties.
  $\rightsquigarrow$ Russell's Paradox
  (*Grundlagekrise der Mathematik*)

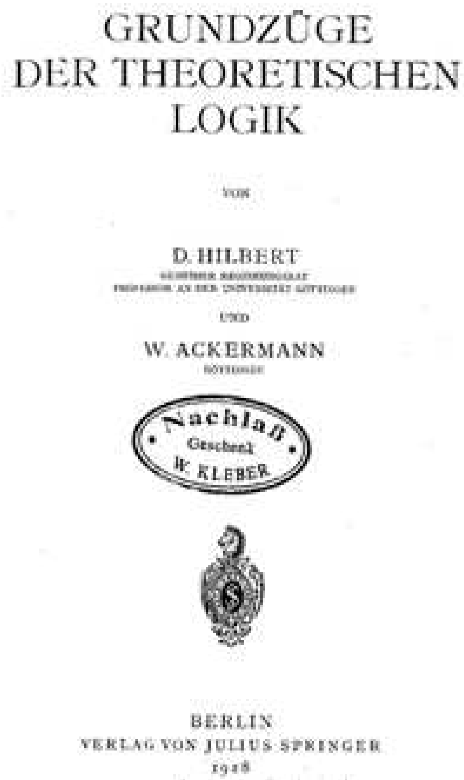
Hilbert (1).

David Hilbert (1862-1943)
Student of Lindemann
1886-1895 Königsberg
1895-1930 Göttingen

1899: *Grundlagen der Geometrie*

"*Man muss jederzeit an Stelle von 'Punkten', 'Geraden', 'Ebenen' 'Tische', 'Stühle', 'Bierseidel' sagen können.*"
"It has to be possible to say 'tables', 'chairs' and 'beer mugs' instead of 'points', 'lines' and 'planes' at any time."

Hilbert (2).



GRUNDZÜGE
DER THEORETISCHEN
LOGIK

VON

D. HILBERT

UND

W. ACKERMANN

Nachlaß
Geschenk
W. KLEBER

BERLIN
VERLAG VON JULIUS SPRINGER
1928

1928: **Hilbert-Ackermann**
Grundzüge der Theoretischen Logik

Wilhelm Ackermann (1896-1962)

First order logic (1).

A first-order language $\mathcal{L}$ is a set $\{\dot{\mathrm{f}}_i\,;\, i \in I\} \cup \{\dot{\mathrm{R}}_j\,;\, j \in J\}$ of function symbols and relation symbols together with a signature $\sigma : I \cup J \to \mathbb{N}$.

- $\sigma(\dot{\mathrm{f}}_i) = n$ is interpreted as "$\dot{\mathrm{f}}_i$ represents an n-ary function".
- $\sigma(\dot{\mathrm{R}}_i) = n$ is interpreted as "$\dot{\mathrm{R}}_i$ represents an n-ary relation".

In addition to the symbols from $\mathcal{L}$, we shall be using the logical symbols $\forall$, $\exists$, $\wedge$, $\vee$, $\to$, $\neg$, $\leftrightarrow$, equality $=$, and a set of variables Var.

First order logic (2).

We fix a first-order language $\mathcal{L} = \{\dot{\mathrm{f}}_i \,;\, i \in I\} \cup \{\dot{\mathrm{R}}_j \,;\, j \in J\}$ and a signature $\sigma : I \cup J \to \mathbb{N}$.

Definition of an $\mathcal{L}$-term.

- Every variable is an $\mathcal{L}$-term.
- If $\sigma(\dot{\mathrm{f}}_i) = n$, and $t_1, ..., t_n$ are $\mathcal{L}$-terms, then $\dot{\mathrm{f}}_i(t_1, ..., t_n)$ is an $\mathcal{L}$-term.
- Nothing else is an $\mathcal{L}$-term.

Example. Let $\mathcal{L} = \{\dot{\times}\}$ be a first order language with a binary function symbol.

- $\dot{\times}(x, x)$ is an $\mathcal{L}$-term (normally written as $x \dot{\times} x$, or x^2).
- $\dot{\times}(\dot{\times}(x, x), x)$ is an $\mathcal{L}$-term (normally written as $(x \dot{\times} x) \dot{\times} x$, or x^3).

First order logic (3).

Definition of an $\mathcal{L}$-formula.

- If t and t^* are $\mathcal{L}$-terms, then $t = t^*$ is an $\mathcal{L}$-formula.
- If $\sigma(\dot{\mathrm{R}}_i) = n$, and $t_1, ..., t_n$ are $\mathcal{L}$-terms, then $\dot{\mathrm{R}}_i(t_1, ..., t_n)$ is an $\mathcal{L}$-formula.
- If φ and ψ are $\mathcal{L}$-formulae and x is a variable, then $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$, $\forall x\,(\varphi)$ and $\exists x\,(\varphi)$ are $\mathcal{L}$-formulae.
- Nothing else is an $\mathcal{L}$-formula.

An $\mathcal{L}$-formula without free variables is called an $\mathcal{L}$-sentence.