UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Core Logic

2006/2007; 1st Semester
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Homework Set # 11 *Deadline:* December 6th, 2006

Exercise 36 (3 points).
Repeated from Homework Set # 10.

(1) Find wellorders $\mathbf{W}$ and $\mathbf{W}^*$ such that $\mathbf{W} \oplus \mathbf{W}^*$ is not isomorphic to $\mathbf{W}^* \oplus \mathbf{W}$ and explain why (1½ points).
(2) Similarly, find wellorders $\mathbf{W}$ and $\mathbf{W}^*$ such that $\mathbf{W} \otimes \mathbf{W}^*$ is not isomorphic to $\mathbf{W}^* \otimes \mathbf{W}$ and explain why (1½ points).

Exercise 37 (6 points).
Let PA be the first-order axiom system of Peano Arithmetic. Assume that PA is consistent.

(1) Show that there is a model $\mathfrak{M}$ of $\mathsf{PA} + \neg\mathrm{Cons}(\mathsf{PA})$ (1 point).
(2) Give an example of a sentence that is true in $\mathfrak{M}$ but not true in the metatheory (1 point).
(3) Consider the following symmetric version of Gödel's Second Incompleteness Theorem SymG2:

    If T is a consistent recursively axiomatized theory such that $\mathsf{PA} \subseteq T$, then the theories $T + \mathrm{Cons}(T)$ and $T + \neg\mathrm{Cons}(T)$ are consistent as well.

    Give a counterexample to SymG2 (4 points).

Exercise 38 (7 points).
Read the paper

> Richard **Zach**, The practice of finitism: Epsilon calculus and consistency proofs in Hilbert's Program, **Synthese** 137 (2003), p.211-259

and answer the following questions:

(1) Is the following statement true or false? "In his lecture course in *Sommersemester 1920*, Hilbert tried to axiomatize all of mathematics based on Frege's second-order logic from the *Grundlagen*." (1 point)
(2) According to Zach, in which semester did Hilbert use the ε-operator for the first time in his lecture course? (1 point)
(3) In the lectures from *Wintersemester 1921/22*, Hilbert uses operators τ and α. What is the relationship between these and the ε-operator? (2 points)
(4) As opposed to lectures of the earlier semesters, in his lectures of the *Wintersemester 1922/23*, Hilbert does not give axioms for addition and multiplication before the introduction of primitive recursive definitions. What (according to Zach) is the reason for this? (1 point)

(5) There is a major shift in the meaning of the ε-operator between Hilbert's 1923 paper and Ackermann's PhD dissertation (1924). What is it? (1 point)
(6) Who wrote to Hilbert in a letter in 1933 that one of the Hilbert-style consistency proofs "does not seem to harmonize with the work of Gödel"? (1 point)

Exercise 39 (6 points).
Let $2^{\mathbb{N}}$ be the set of all infinite 0-1 sequences. For $x \in 2^{\mathbb{N}}$, we define $\widehat{x}(n) := 1 - x(n)$. We call $\mathcal{C} \subseteq 2^{\mathbb{N}}$ a **symmetric class** if for every $x \in \mathcal{C}$, we also have $\widehat{x} \in \mathcal{C}$. A function $F : \mathbb{N} \to \mathcal{C}$ is called a $\mathcal{C}$-good parametrization if the sequence $\langle F(n)(n)\,;\, n \in \mathbb{N}\rangle$ is an element of $\mathcal{C}$ and F is a surjection.

(1) Show that no symmetric class $\mathcal{C}$ can have a $\mathcal{C}$-good parametrization (4 points).
(2) Derive Cantor's Theorem ("there is no bijection between $\mathbb{N}$ and $2^{\mathbb{N}}$") as a corollary (2 points).