UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Core Logic

2004/2005; 1st Semester
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Homework Set # 9 *Deadline:* November 17th, 2004

Exercise 26 (9 points total).

Let $\mathcal{L} = \{R\}$ be a language with one binary relation symbol. Consider the following seven $\mathcal{L}$-sentences:

$$\begin{aligned}
\varphi_{\text{(i)}} &:\simeq \forall x \neg Rxx \\
\varphi_{\text{(ii)}} &:\simeq \forall x \forall y (x \neq y \rightarrow (Rxy \vee Ryx)) \\
\varphi_{\text{(iii)}} &:\simeq \forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz) \\
\varphi_{\text{(iv)}} &:\simeq \forall x \exists y \exists z (Ryx \wedge Rxz) \\
\varphi_{\text{ME}} &:\simeq \exists x \forall y (Ryx \vee x = y) \\
\varphi_{\text{LEP}} &:\simeq \forall x \exists y \forall z (Rxz \rightarrow (Rzy \vee y = z))
\end{aligned}$$

Check whether the following sets of sentences are consistent. If they are, give a model. If they aren't, derive a contradiction (3 points each).

(1) $\{\varphi_{\text{(i)}}, \varphi_{\text{(iii)}}, \varphi_{\text{(iv)}}, \varphi_{\text{ME}}\}$,
(2) $\{\varphi_{\text{(i)}}, \varphi_{\text{(iii)}}, \varphi_{\text{LEP}}, \neg\varphi_{\text{ME}}\}$,
(3) $\{\varphi_{\text{(i)}}, \varphi_{\text{(ii)}}, \varphi_{\text{(iii)}}, \varphi_{\text{LEP}}, \neg\varphi_{\text{ME}}\}$,

Exercise 27 (7 points total).

Let $\mathcal{L} := \{+, \cdot, 0, 1, -\}$ be the language of Boolean algebras and Φ_{BA} be the axioms of Boolean algebras. Let

$$\begin{aligned}
\varphi &:\simeq \forall x \forall y \Big(\big((x \neq x \cdot y) \wedge (y \neq x \cdot y)\big) \rightarrow (x \cdot y = 0) \Big), \\
\psi &:\simeq \exists x \big((x \neq 0) \wedge (x \neq 1)\big).
\end{aligned}$$

Let Φ_0, Φ_1 and Φ_2 be the deductive closures of Φ_{BA}, $\Phi_{\text{BA}} \cup \{\varphi\}$ and $\Phi_{\text{BA}} \cup \{\varphi, \psi\}$, respectively. Investigate whether Φ_i is a complete theory. If it isn't, give a formula σ such that $\sigma \notin \Phi_i$ and $\neg\sigma \notin \Phi_i$. If it is complete, give a brief argument why.

Exercise 28 (6 points total).

Let PA be the first-order axiom system of Peano Arithmetic. Assume that PA is consistent.

(1) Show that there is a model $\mathfrak{M}$ of $\mathsf{PA} + \neg\mathrm{Cons}(\mathsf{PA})$ (1 point).
(2) Give an example of a sentence that is true in $\mathfrak{M}$ but not true in the metatheory (1 point).
(3) Consider the following symmetric version of Gödel's Second Incompleteness Theorem $\mathsf{SymG2}$:

   If T is a consistent recursively axiomatized theory such that $\mathsf{PA} \subseteq T$, then the theories $T + \mathrm{Cons}(T)$ and $T + \neg\mathrm{Cons}(T)$ are consistent as well.

   Give a counterexample to $\mathsf{SymG2}$ (4 points).

Exercise 29 (3 points total).

Give the names of the following logicians and mathematicians (1 point each):

- X was one of the students of David Hilbert who was a teacher at the *Gymnasium Arnoldinum* from 1929 to 1948.
- Y was an important figure in the history of the *Deutsche Mathematiker-Vereinigung*. He was married to the granddaughter of Hegel, and is popularly known for the "Y bottle", a two-dimensional manifold not embeddable into $\mathbb{R}^3$.
- Z received his PhD degree in 1924 at the UvA for a thesis entitled *Intuitionistische axiomatiek der projectieve meetkunde* and was the PhD supervisor of one of our guest speakers.

  (*One extra point:* What is the canonical webpage for finding information about supervisor-student relations in mathematics?)