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Fixed points of commutative Lüders operations

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Abstract
This paper verifies a conjecture posed in a pair of papers on the fixed point sets for a class of quantum operations. Specifically, it is proved that if a quantum operation has mutually commuting operation elements that are effects forming a resolution of the identity, then the fixed point set of the quantum operation is exactly the commutant of the operation elements.

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1. Introduction

Let H be a complex Hilbert space, $\mathcal{B}(H)$ be the bounded linear operator set on H. If $A \in \mathcal{B}(H)$ and $0 \leqslant A \leqslant I$, then A is called a *quantum effect* on H. Each quantum effect can be used to represent a yes–no measurement that may be unsharp [1–6]. The set of all quantum effects on H is denoted by $\mathcal{E}(H)$; the set of all orthogonal projection operators on H is denoted by $\mathcal{P}(H)$. Each element P of $\mathcal{P}(H)$ can be used to represent a yes–no measurement that is sharp [1–6]. Let $\mathcal{T}(H)$ be the set of all trace class operators on H and $\mathcal{D}(H)$ be the set of all density operators on H, i.e. $\mathcal{D}(H) = \{\rho : \rho \in \mathcal{T}(H), \rho \geqslant 0, \mathrm{tr}(S) = 1\}$. Each element ρ of $\mathcal{D}(H)$ represents a state of the quantum system H.

Let $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ be the quantum measurement, that is $\sum_{i=1}^n E_i^2 = I$ in the strong operator topology, where $1 \leqslant n \leqslant \infty$, then the probability of outcome E_i measured in the state ρ is given by $\mathrm{tr}(\rho E_i)$, and the new quantum state after the measurement $\mathcal{A}$ is performed is defined by

$$\Phi(\rho) = \sum_{i=1}^n E_i \rho E_i.$$

Note that $\Phi : \rho \to \sum_{i=1}^n E_i \rho E_i$ defined a transformation on the state set $\mathcal{D}(H)$; we call it the *Lüders transformation* [6, 7]. In physics, the question whether a state ρ is not disturbed

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by the measurement $\mathcal{A} = \{E_i\}_{i=1}^n$ becomes equivalent to the fact that ρ is a solution of the equation

$$\Phi(\rho) = \sum_{i=1}^n E_i \rho E_i = \rho.$$

It was showed in [8] that the measurement $\mathcal{A} = \{E_i\}_{i=1}^2$ does not disturb ρ if and only if ρ commutes with each E_i, $i = 1, 2$.

Moreover, if we define the *Lüders quantum operation* $\Phi_{\mathcal{A}}$ on $\mathcal{B}(H)$ as

$$\Phi_{\mathcal{A}} : \mathcal{B}(H) \to \mathcal{B}(H), \qquad B \to \Phi_{\mathcal{A}}(B) = \sum_{i=1}^n E_i B E_i,$$

then an interesting problem is that if $B \in \mathcal{B}(H)$ is a fixed point of $\Phi_{\mathcal{A}}$, that is, $\Phi_{\mathcal{A}}(B) = \sum_{i=1}^n E_i B E_i = B$, then B commutes with each E_i ? $i = 1, 2, \ldots, n$.

In [9, 10], we knew the conclusion is true if H is a finite-dimensional complex Hilbert space. In [9–11], it was showed that the conclusion is not true when $n = 5$ or $n = 3$ for infinite-dimensional complex Hilbert space. Thus, the general conclusion for infinite-dimensional cases is false. On the other hand, Busch and Singh in [8] showed that for $n = 2$ the conclusion is true for all complex Hilbert spaces. Note that in this case, $E_1^2 + E_2^2 = I$, so $E_1E_2 = E_2E_1$, that is, $\mathcal{A} = \{E_1, E_2\}$ is commutative. This motivated Arias, Gheonda, Gudder and Nagy to conjecture when $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ is commutative, then the conclusion is true, that is, the fixed point set of $\Phi_{\mathcal{A}}$ is exactly the commutant $\mathcal{A}'$ of the operation elements $\mathcal{A} = \{E_i\}_{i=1}^n$. Moreover, Nagy in [12] showed that if the conjecture is true, then

$$\Phi_{\mathcal{A}}(E) = \sum_{i=1}^n E_i E E_i = I - E$$

has the unique solution $\frac{1}{2}I$ in $\mathcal{E}(H)$; in physics, it showed that if the measurement $\mathcal{A}$ disturbs the quantum effect E completely into its supplement $I - E$, then E has to be $\frac{1}{2}I$.

As showed in [13–16], the structures of fixed point sets of quantum operations have important applications in quantum information theory; in particular, in [15, theorem 3], the fixed point set is a matrix algebra which shares an elegant structure, played a central role in identifying the protected structures.

In this paper, by using the spectral theory of self-adjoint operators, we prove the conjecture affirmatively. Moreover, when $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ is commutative and $F = \sum_{i=1}^n E_i^2 < I$, we also obtain a nice conclusion. Note that the von Neumann algebra $\mathcal{N}$ generated by $\{E_i\}_{i=1,\ldots,n}$ is Abelian which can be embed into a maximal Abelian von Neumann algebra. Since a maximal Abelian von Neumann algebra $\mathcal{M}$ on a separable Hilbert space is always a direct sum of $\mathcal{M}_1$ and $\mathcal{M}_2$. Here $\mathcal{M}_1$ is isometric to $\bigoplus_{i=1}^\infty C_i$ and $\mathcal{M}_2$ is isometric to $L_\infty(B)$, where B is a compact subset of the real number set R. Thus, $\mathcal{A}'$ has the form $\bigoplus_{i=1}^\infty M_k \otimes 1_{n_k} \bigoplus L_\infty(C)$, where C is a subset of B and M_k is a matrix algebra whose dimension is k and n_k ranges from 0 to ∞ [17]. So our conclusions are analogous with the finite-dimensional cases' concise shape in theorem 3 in [15].

2. Element lemmas and proofs

Let $1 \leqslant n < \infty$ and $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ be commutative. Firstly, for each E_i, $1 \leqslant i \leqslant n$, we have the spectral representation theorem

$$E_i = \int_0^1 \lambda \, \mathrm{d}F_\lambda^{(i)},$$

where $\{F_\lambda^{(i)}\}_{\lambda\in\mathbb{R}}$ is the identity resolution of E_i satisfying that $\{F_\lambda^{(i)}\}_{\lambda\in\mathbb{R}}$ is right continuous in the strong operator topology and $F_\lambda^{(i)} = 0$ if $\lambda < 0$ and $F_\lambda^{(i)} = I$ if $\lambda \geqslant \|E_i\|$, moreover, for each $\lambda \in \mathbb{R}$, $F_\lambda^{(i)} = P^{E_i}(-\infty, \lambda]$, where P^{E_i} is the spectral measure of E_i [17]. Now, for the fixed integers $m, k_1, k_2, \ldots, k_n$, we denote

$$F^m_{k_1,\ldots,k_n} = P^{E_1}\left(\frac{k_1}{m}, \frac{k_1+1}{m}\right] \ldots P^{E_n}\left(\frac{k_n}{m}, \frac{k_n+1}{m}\right].$$

Since E_i and E_j are commutative for any $i, j = 1, 2, \ldots, n$, so $F^m_{k_1,\ldots,k_n}$ is a well-defined orthogonal projection operator.

Lemma 2.1. *Let $1 \leqslant n < \infty$, $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ be commutative and $B \in \mathcal{B}(H)$. If for any integers m and $k_1, k_2, \ldots, k_n$, B commutes with $F^m_{k_1,\ldots,k_n}$, then B is commutative with each E_i in $\mathcal{A} = \{E_i\}_{i=1}^n$.*

Proof. For each rational number $q = \frac{p}{l}$, where p, l are integers. If $\frac{p}{l} < 0$, then $F^{(i)}_{\frac{p}{l}} = 0$, and if $\frac{p}{l} \geqslant 1$, then $F^{(i)}_{\frac{p}{l}} = I$. Let $l > p \geqslant 0$, so $0 \leqslant \frac{p}{l} < 1$. Then $F^{(i)}_{\frac{p}{l}} = P^{E_i}\left(\frac{-1}{l}, 0\right] + P^{E_i}\left(0, \frac{1}{l}\right] + \cdots + P^{E_i}\left(\frac{p-1}{l}, \frac{p}{l}\right]$; thus, we can prove easily that

$$F^{(i)}_{\frac{p}{l}} = \sum_{k_i<p}\left(\sum_{k_1,\ldots,k_{i-1},k_i,k_{i+1},\ldots,k_n} F^l_{k_1,\ldots,k_n}\right).$$

So, for each rational number $q = \frac{p}{l}$, $F^{(i)}_{\frac{p}{l}}$ commutes with B; note that $\{F^{(i)}\}_{\lambda\in\mathbb{R}}$ is right continuous in the strong operator topology, so B commutes with each E_i, $i = 1, 2, \ldots, n$. □

Lemma 2.2. *Let $1 \leqslant n < \infty$, $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ be commutative and $B \in \mathcal{B}(H)$. If B does not commute with some E_{i_0} in $\mathcal{A}$, then there are integers m, $k_1, k_2, \ldots, k_n$ and $k_1', k_2', \ldots, k_n'$, such that $k_i \neq k_i'$ for at least one i and $F^m_{k_1,k_2,\ldots,k_n} B F^m_{k_1',k_2',\ldots,k_n'} \neq 0$.*

Proof. Without loss of generality, we suppose that B does not commute with E_1. By lemma 2.1, there are integers m and $k_1, k_2, \ldots, k_n$ such that $F^m_{k_1,k_2,\ldots,k_n} B \neq F^m_{k_1,k_2,\ldots,k_n} B F^m_{k_1,k_2,\ldots,k_n}$ or $B F^m_{k_1,k_2,\ldots,k_n} \neq F^m_{k_1,k_2,\ldots,k_n} B F^m_{k_1,k_2,\ldots,k_n}$. If $F^m_{k_1,k_2,\ldots,k_n} B \neq F^m_{k_1,k_2,\ldots,k_n} B F^m_{k_1,k_2,\ldots,k_n}$, then there exist integers $k_1', k_2', \ldots, k_n'$, $k_i \neq k_i'$ for at least one i such that $F^m_{k_1,k_2,\ldots,k_n} B F^m_{k_1',k_2',\ldots,k_n'} \neq 0$. In fact, if not, we will get that

$$F^m_{k_1,k_2,\ldots,k_n} B = \sum_{k_1',k_2',\ldots,k_n'} F^m_{k_1,k_2,\ldots,k_n} B F^m_{k_1',k_2',\ldots,k_n'} = F^m_{k_1,k_2,\ldots,k_n} B F^m_{k_1,k_2,\ldots,k_n}.$$

This is a contradiction. Similarly, if $B F^m_{k_1,k_2,\ldots,k_n} \neq F^m_{k_1,k_2,\ldots,k_n} B F^m_{k_1,k_2,\ldots,k_n}$, we will also get the same conclusion. The lemma is proven. □

Moreover, we have a stronger conclusion in the following.

Lemma 2.3. *Let $A \in \mathcal{E}(H)$ and $B \in \mathcal{B}(H)$. If B does not commute with A, then there exist integers m, k and j with $|k - j| \geqslant 2$ such that*

$$P^A\left(\frac{k}{m}, \frac{k+1}{m}\right] B P^A\left(\frac{j}{m}, \frac{j+1}{m}\right] \neq 0.$$

Proof. By lemma 2.2, we can find $k_1 \neq j_1$ such that $C = P^A\left(\frac{k_1}{m}, \frac{k_1+1}{m}\right] B P^A\left(\frac{j_1}{m}, \frac{j_1+1}{m}\right] \neq 0$. If $|k_1 - j_1| \geqslant 2$, then we get the m, k, j satisfy the lemma. If $j_1 = k_1 + 1$, we replace m by

$2m$ and let $k_2 = 2k_1$, $j_2 = 2j_1$. Then

$$P^A\left(\frac{k_1}{m}, \frac{k_1+1}{m}\right] = P^A\left(\frac{k_2}{2m}, \frac{k_2+1}{2m}\right] + P^A\left(\frac{k_2+1}{2m}, \frac{k_2+2}{2m}\right],$$
$$P^A\left(\frac{j_1}{m}, \frac{j_1+1}{m}\right] = P^A\left(\frac{j_2}{2m}, \frac{j_2+1}{2m}\right] + P^A\left(\frac{j_2+1}{2m}, \frac{j_2+2}{2m}\right].$$

Now we consider k_2, k_2+1 and j_2, j_2+1, if we still cannot take $|k-j| \geqslant 2$ satisfy the conclusion, then

$$P^A\left(\frac{k_2}{2m}, \frac{k_2+1}{2m}\right] BP^A\left(\frac{j_2}{2m}, \frac{j_2+1}{2m}\right] = 0,$$
$$P^A\left(\frac{k_2}{2m}, \frac{k_2+1}{2m}\right] BP^A\left(\frac{j_2+1}{2m}, \frac{j_2+2}{2m}\right] = 0,$$
$$P^A\left(\frac{k_2+1}{2m}, \frac{k_2+2}{2m}\right] BP^A\left(\frac{j_2+1}{2m}, \frac{j_2+2}{2m}\right] = 0.$$

So we have $C = P^A\left(\frac{k_2+1}{2m}, \frac{k_2+2}{2m}\right] BP^A\left(\frac{j_2}{2m}, \frac{j_2+1}{2m}\right]$.

Following this, we find the integers k, j which satisfy the conclusion or we get a sequence $\{p_i, p_i+1, 2^{i-1}m\}_{i=1}^{\infty}$ such that $p_i + 1 = 2^{i-1}j_1$ and $C = P^A\left(\frac{p_i}{2^{i-1}m}, \frac{p_i+1}{2^{i-1}m}\right] BP^A\left(\frac{p_i+1}{2^{i-1}m}, \frac{p_i+2}{2^{i-1}m}\right]$. If the first case occurs, then we proved the lemma. If the second case occurs, note that

$$\bigcap_{i=1}^{\infty}\left(\frac{p_i+1}{2^{i-1}m}, \frac{p_i+2}{2^{i-1}m}\right] = \emptyset,$$

and

$$\bigcap_{i=1}^{\infty}\left(\frac{p_i}{2^{i-1}m}, \frac{p_i+1}{2^{i-1}m}\right] = \left\{\frac{j_1}{m}\right\},$$

so $\lim_{i\to\infty} P^A\left(\frac{p_i}{2^{i-1}m}, \frac{p_i+1}{2^{i-1}m}\right] = P^A\left\{\frac{j_1}{m}\right\}$ and $\lim_{i\to\infty} P^A\left(\frac{p_i+1}{2^{i-1}m}, \frac{p_i+2}{2^{i-1}m}\right] = 0$ in strong operator topology; thus,

$$\lim_{i\to\infty} P^A\left(\frac{p_i}{2^{i-1}m}, \frac{p_i+1}{2^{i-1}m}\right] BP^A\left(\frac{p_i+1}{2^{i-1}m}, \frac{p_i+2}{2^{i-1}m}\right] = 0$$

in strong operator topology [17]. But for each positive integer i,

$$C = P^A\left(\frac{p_i}{2^{i-1}m}, \frac{p_i+1}{2^{i-1}m}\right] BP^A\left(\frac{p_i+1}{2^{i-1}m}, \frac{p_i+2}{2^{i-1}m}\right],$$

so we get $C = 0$; this is a contradiction, and the lemma is proved in this case.

If $k_1 + 1 = j_1$, we just need to take all the above calculations in adjoint and interchange the indices j and k. The proof is similar; thus, we proved the lemma. □

Lemma 2.4. *Let $1 \leqslant n < \infty$, $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ be commutative and $\sum_{i=1}^n E_i^2 \leqslant I$. If $X \in B(H)$ is not commutative with E_1, then there exists a positive integer m such that for each positive integer p, there exist projection operators $P, Q \in \mathcal{A}'$, $PQ = 0$, $Y = PXQ \neq 0$, and*

$$\frac{\|Y\| - \|\Phi_{\mathcal{A}}(Y)\|}{\|Y\|} \geqslant \frac{p^2 - 4\sqrt{n}mp - 2n}{2(pm)^2}.$$

Proof. Since X does not commute with E_1, it follows from lemma 2.3 that there exist integers $m, k,$ and j such that $|k-j| \geqslant 2$ and $P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] XP^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] \neq 0$. Note that

$$P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] XP^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] = \sum_{k_2,\ldots,k_n} \sum_{k_2',\ldots,k_n'} F^m_{k,k_2,\ldots,k_n} X F^m_{j,k_2',\ldots,k_n'},$$

so there exist $k, k_2, \ldots, k_n$ and $j, k'_2, \ldots, k'_n$ such that $|k-j| \geqslant 2$ and

$$F^m_{k,k_2,\ldots,k_n} X F^m_{j,k'_2,\ldots,k'_n} \neq 0.$$

Let $P_0 = F^m_{k,k_2,\ldots,k_n}$, $Q_0 = F^m_{j,k'_2,\ldots,k'_n}$, $Y_0 = P_0 X Q_0$. Then P_0 and Q_0 are projection operators and $P_0, Q_0 \in \mathcal{A}'$, $P_0 Q_0 = 0$, $Y_0 = P_0 X Q_0 \neq 0$. Moreover, for each $i = 1, 2, \ldots, n$, if we denote $k_1 = k$, $k'_1 = j$, then

$$\begin{aligned}
\|E_i Y_0 E_i\| &= \left\| E_i P^{E_i}\left(\frac{k_i}{m}, \frac{k_i+1}{m}\right] Y_0 P^{E_i}\left(\frac{k'_i}{m}, \frac{k'_i+1}{m}\right] E_i \right\| \\
&\leqslant \left\| E_i P^{E_i}\left(\frac{k_i}{m}, \frac{k_i+1}{m}\right]\right\| \|Y_0\| \left\| P^{E_i}\left(\frac{k'_i}{m}, \frac{k'_i+1}{m}\right] E_i \right\| \\
&\leqslant \frac{k_i+1}{m} \|Y_0\| \frac{k'_i+1}{m} \\
&= \frac{k_i+1}{m}\frac{k'_i+1}{m} \|Y_0\|. \qquad (1)
\end{aligned}$$

Thus, we have

$$\left\| \sum_{i=1}^n E_i Y_0 E_i \right\| \leqslant \sum_{i=1}^n \|E_i Y_0 E_i\| \leqslant \left(\sum_{i=1}^n \frac{k_i k'_i}{m^2} + \sum_{i=1}^n \frac{k_i + k'_i}{m^2} + \frac{n}{m^2} \right) \|Y_0\|. \qquad (2)$$

Since $\sum_{i=1}^n E_i^2 \leqslant I$ and

$$\begin{aligned}
F^m_{k,k_2,\ldots,k_n}\left(I - \sum_{i=1}^n E_i^2\right) &= F^m_{k,k_2,\ldots,k_n} - F^m_{k,k_2,\ldots,k_n} \sum_{i=1}^n E_i^2 \\
&\leqslant F^m_{k,k_2,\ldots,k_n} - \sum_{i=1}^n \frac{k_i^2}{m^2} F^m_{k,k_2,\ldots,k_n} \\
&= \left(1 - \sum_{i=1}^n \frac{k_i^2}{m^2}\right) F^m_{k,k_2,\ldots,k_n}, \qquad (3)
\end{aligned}$$

so, we have $\sum_{i=1}^n k_i^2 \leqslant m^2$. Similarly, we have also $\sum_{i=1}^n k_i'^2 \leqslant m^2$. Moreover, note that

$$\begin{aligned}
2m^2\left(1 - \sum_{i=1}^n \frac{k_i k'_i}{m^2} - \sum_{i=1}^n \frac{k_i + k'_i}{m^2} - \frac{n}{m^2}\right) &= m^2 + m^2 - 2\sum_{i=1}^n k_i k'_i - 2\sum_{i=1}^n (k_i + k'_i) - 2n \\
&\geqslant \sum_{i=1}^n k_i^2 + \sum_{i=1}^n k_i'^2 - 2\sum_{i=1}^n k_i k'_i - 2\sum_{i=1}^n (k_i + k'_i) - 2n \\
&= \sum_{i=1}^n (k_i - k'_i)^2 - 2\sum_{i=1}^n (k_i + k'_i) - 2n \\
&\geqslant (k_1 - k'_1)^2 - 2\sum_{i=1}^n (k_i + k'_i) - 2n, \qquad (4)
\end{aligned}$$

and $\left(\sum_{i=1}^n k_i\right)^2 \leqslant n\left(\sum_{i=1}^n k_i^2\right) \leqslant nm^2$, $\left(\sum_{i=1}^n k'_i\right)^2 \leqslant n\left(\sum_{i=1}^n k_i'^2\right) \leqslant nm^2$, we have

$$2m^2\left(1 - \sum_{i=1}^n \frac{k_i k'_i}{m^2} - \sum_{i=1}^n \frac{k_i + k'_i}{m^2} - \frac{n}{m^2}\right) \geqslant (j-k)^2 - 4\sqrt{n}m - 2n. \qquad (5)$$

On the other hand, it follows from

$$\|Y_0\| - \left\|\sum_{i=1}^{n} E_i Y_0 E_i\right\| \geqslant \|Y_0\| - \sum_{i=1}^{n} \|E_i Y_0 E_i\|$$
$$\geqslant \left[1 - \left(\sum_{i=1}^{n} \frac{k_i k_i'}{m^2} + \sum_{i=1}^{n} \frac{k_i + k_i'}{m^2} + \frac{n}{m^2}\right)\right]\|Y_0\|$$

and (5) that

$$\frac{\|Y_0\| - \|\Phi_{\mathcal{A}}(Y_0)\|}{\|Y_0\|} \geqslant \frac{(j-k)^2 - 4\sqrt{n}m - 2n}{2m^2}.$$

For each positive integer p, we replace m with pm. Note that

$$Y_0 = \sum_{s_1,s_2,\ldots,s_n} \sum_{s_1',s_2',\ldots,s_n'} F^{pm}_{s_1,s_2,\ldots,s_n} Y_0 F^{mp}_{s_1',s_2'\ldots,s_n'} \neq 0,$$

so there exist $s_1, s_2, \ldots, s_n$ and $s_1', s_2', \ldots, s_n'$ such that

$$Y = F^{pm}_{s_1,\ldots,s_n} Y_0 F^{pm}_{s_1',\ldots,s_n'} \neq 0.$$

Thus, it is easily to prove that $\frac{k_i}{m} \leqslant \frac{s_i}{pm} \leqslant \frac{k_i+1}{m}$ and $\frac{k_i'}{m} \leqslant \frac{s_i'}{pm} \leqslant \frac{k_i'+1}{m}$. Note that $k_1 = k, k_1' = j$ and $\left|\frac{j-k}{m}\right| \geqslant \frac{2}{m}$, we have

$$\left\|\frac{s_1 - s_1'}{pm}\right\| \geqslant \left\|\frac{k_1 + 1 - k_1'}{m}\right\| \geqslant 1/m;$$

thus

$$\|s_1 - s_1'\| \geqslant p.$$

By the similar analysis methods as (5), we get

$$2(pm)^2\left(1 - \sum_{i=1}^{n} \frac{s_i s_i'}{(pm)^2} - \sum_{i=1}^{n} \frac{s_i + s_i'}{(pm)^2} - \frac{n}{(pm)^2}\right) \geqslant p^2 - 4\sqrt{n}mp - 2n. \tag{6}$$

On the other hand, we also have

$$\|Y\| - \left\|\sum_{i=1}^{n} E_i Y E_i\right\| \geqslant \|Y\| - \sum_{i=1}^{n} \|E_i Y E_i\|$$
$$\geqslant \left[1 - \left(\sum_{i=1}^{n} \frac{k_i k_i'}{m^2} + \sum_{i=1}^{n} \frac{k_i + k_i'}{(pm)^2} + \frac{n}{(pm)^2}\right)\right]\|Y\|.$$

Let $P = F^{pm}_{s_1,s_2,\ldots,s_n} P_0$ and $Q = Q_0 F^{pm}_{s_1',s_2',\ldots,s_n'}$. Then it is clear that $P, Q \in \mathcal{A}'$, $PQ = 0$, $Y = PXQ \neq 0$, and

$$\frac{\|Y\| - \|\Phi_{\mathcal{A}}(Y)\|}{\|Y\|} \geqslant \frac{p^2 - 4\sqrt{n}m - 2n}{2(pm)^2}.$$

The lemma is proved. □

It follows from the proof of lemma 2.4 that we have the following important conclusion:

Corollary 2.1. *Let $1 \leqslant n < \infty$, $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ be commutative and $\sum_{i=1}^n E_i^2 \leqslant I$. If $X \in B(H)$ and there exist integers m, k, and j with $|k - j| \geqslant 2$ such that*

$$P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] X P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] \neq 0,$$

then for each positive integer p, there exist projection operators $P, Q \in \mathcal{A}'$, $PQ = 0$, $Y = PXQ \neq 0$, and

$$\frac{\|Y\| - \|\Phi_{\mathcal{A}}(Y)\|}{\|Y\|} \geqslant \frac{p^2 - 4\sqrt{n}mp - 2n}{2(pm)^2}.$$

3. Main results and proofs

Let $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ and $\Phi_{\mathcal{A}}$ be the Lüders quantum operation which is decided by $\mathcal{A}$. It is easy to prove that $\|\Phi_{\mathcal{A}}\| = \left\|\sum_{i=1}^n E_i^2\right\|$ [9]. Now, we denote $B(H)^{\Phi_{\mathcal{A}}}$ to be the fixed point set of $\Phi_{\mathcal{A}}$ and $\mathcal{A}'$ to be the commutant of $\mathcal{A}$, that is, $B(H)^{\Phi_{\mathcal{A}}} = \{B \in B(H) \mid \Phi_{\mathcal{A}}(B) = B\}$, $\mathcal{A}' = \{B \in \mathcal{B}(H) \mid BE_i = E_iB, 1 \leqslant i \leqslant n\}$. It is clear that if $\sum_{i=1}^n E_i^2 = I$ in strong operator topology, then $\mathcal{A}' \subseteq B(H)^{\Phi_{\mathcal{A}}}$.

Theorem 3.1. *Let $1 \leqslant n \leqslant \infty$, $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ be commutative and $\sum_{i=1}^n E_i^2 = I$ in strong operator topology. Then*

$$\mathcal{B}(H)^{\Phi_{\mathcal{A}}} = \left\{B \in \mathcal{B}(H) | \Phi_{\mathcal{A}}(B) = \sum_{i=1}^n E_iBE_i = B\right\} = \mathcal{A}'.$$

Proof. Since $\mathcal{A}' \subseteq \mathcal{B}(H)^{\Phi_{\mathcal{A}}}$, in order to prove the converse containing relation, we suppose that $B \in \mathcal{B}(H)^{\Phi_{\mathcal{A}}}\backslash\mathcal{A}'$. Without loss of generality, we can suppose that B is not commutative with E_1. By lemma 2.3, there is a triple integer set $\{m, j, k\}$ such that $|k - j| \geqslant 2$ and $P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right]BP^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] \neq 0$.

For each positive integer $q \leqslant n$, let $F_q = \sum_{i=1}^q E_i^2$ and $\Phi_q : \mathcal{B}(H) \to \mathcal{B}(H)$ be defined by $\Phi_q(A) = \sum_{i=1}^q E_iAE_i$. Then $F_q \to I$ in strong operator topology and Φ_q is a completely positive map. If $P_q = P^{F_q}\left(\left(1 - \frac{1}{4m^2}, 1\right]\right)$, then $P_q \to I$ in strong operator topology (see [[18], P_{248}]). Now we show that $P_qP^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right]BP^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right]P_q = 0$. In fact, if not, note that

$$\begin{aligned} &P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right]P_qP^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right]BP^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right]P_qP^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] \\ &\quad = P_qP^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right]BP^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right]P_q \neq 0, \end{aligned}$$

so, by corollary 2.1, for each positive integer p, there exist projection operators P and Q, $P, Q \in \mathcal{A}'$, $PQ = 0$, such that

$$\begin{aligned} Y &= PP_qP^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right]BP^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right]P_qQ \\ &= P_qPP^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right]BP^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right]QP_q \neq 0 \end{aligned}$$

and

$$\frac{\|Y\| - \|\Phi_q(Y)\|}{\|Y\|} \geqslant \frac{p^2 - 4\sqrt{q}mp - 2q}{2(pm)^2}.$$

Since

$$\frac{p^2 - 4\sqrt{q}mp - 2q}{2(pm)^2} \to \frac{1}{2m^2}$$

as $p \to \infty$. So we can choose Y such that

$$\frac{\|Y\| - \|\Phi_q(Y)\|}{\|Y\|} \geqslant \frac{3}{8m^2}.$$

Note that $P_q E_i = E_i P_q$ and $P_q Y = Y P_q$ for each $1 \leqslant i \leqslant n$, $\mathcal{A}_1 = \{P_q E_i\}_{i=q+1}^n$ decides a *Lüders* operation $\Phi_{\mathcal{A}_1}$, and

$$\|\Phi_{\mathcal{A}_1}\| = \left\| \sum_{i=q+1}^n P_q E_i^2 P_q \right\| = \left\| P_q \left(\sum_{i=q+1}^n E_i^2 \right) P_q \right\| = \left\| P_q \left(I - \sum_{i=1}^q E_i^2 \right) P_q \right\| \leqslant \frac{1}{4m^2},$$

so we have

$$\begin{aligned}
\|\Phi_{\mathcal{A}}(Y)\| &= \left\| \Phi_q(Y) + \sum_{i=q+1}^n E_i Y E_i \right\| \\
&= \left\| \Phi_q(Y) + \sum_{i=q+1}^n E_i P_q Y P_q E_i \right\| \\
&\leqslant \|\Phi_q(Y)\| + \left\| \sum_{i=q+1}^n P_q E_i Y E_i P_q \right\| \\
&= \|\Phi_q(Y)\| + \|\Phi_{\mathcal{A}_1}(Y)\| \\
&\leqslant \left(1 - \frac{3}{8m^2}\right)\|Y\| + \frac{1}{4m^2}\|Y\| \\
&= \left(1 - \frac{1}{8m^2}\right)\|Y\|. \qquad (7)
\end{aligned}$$

On the other hand, we show that $Y = P_q P P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] B P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] Q P_q \in \mathcal{B}(H)^{\Phi_{\mathcal{A}}}$. In fact, note that $\left\{P_q, P, P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right], P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right], Q\right\} \subseteq \mathcal{A}'$ and $\Phi_{\mathcal{A}}(B) = B$, so we have

$$\begin{aligned}
\Phi_{\mathcal{A}}(Y) &= \sum_{i=1}^n E_i Y E_i = \sum_{i=1}^n E_i P_q P P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] B P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] Q P_q E_i \\
&= P_q P P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] \left(\sum_{i=1}^n E_i B E_i\right) P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] Q P_q \\
&= P_q P P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] \Phi_{\mathcal{A}}(B) P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] Q P_q \\
&= P_q P P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] B P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] Q P_q = Y.
\end{aligned}$$

This contradicts (7) and so $P_q P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] B P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] P_q = 0$. Note that

$$P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] B P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] = \lim_{q\to\infty} P_q P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] B P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] P_q$$

in strong operator topology [17], so

$$P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] B P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] = 0.$$

This contradicts $P^{E_1}\left(\frac{k}{m}, \frac{k+1}{m}\right] B P^{E_1}\left(\frac{j}{m}, \frac{j+1}{m}\right] \neq 0$. So $B \in \mathcal{A}'$. □

Theorem 3.2. *Let* $1 \leqslant n \leqslant \infty$, $\mathcal{A} = \{E_i\}_{i=1}^n \subseteq \mathcal{E}(H)$ *be commutative and* $F = \sum_{i=1}^n E_i^2 < I$. *If* $P = P^F\{1\}$, *where* P^F *is the spectral measure of* F, *then*

$$\mathcal{B}(H)^{\Phi_{\mathcal{A}}} = \left\{ B \in \mathcal{B}(H) | \Phi_{\mathcal{A}}(B) = \sum_{i=1}^n E_i B E_i = B \right\} = P\mathcal{A}'.$$

Proof. Firstly, by the spectral representation theorem [17] we have $PF = FP = P$. Let $B \in \mathcal{B}(H)^{\Phi_{\mathcal{A}}}$. Then as the analysis of theorem 3.1, we have $B \in \mathcal{A}'$. Let $Q = I - P$ and $Q_k = P^F\big(0, 1 - \frac{1}{k}\big]$. Then $Q_k \to Q$ in strong operator topology and $Q_k \in \mathcal{A}'$, so $Q_k B \in \mathcal{B}(H)^{\Phi_{\mathcal{A}}}$. Let Φ_k be the completely positive map which is decided by $\{E_i Q_k\}_{i=1}^n$. Then $\|\Phi_k\| \leqslant 1 - \frac{1}{k}$. Note that $B, Q_k \in \mathcal{A}'$ and $Q_k^2 = Q_k$; thus, we have $\|Q_k B\| = \|\Phi_{\mathcal{A}}(Q_k B)\| = \|\Phi_k(Q_k B)\| \leqslant \big(1 - \frac{1}{k}\big)\|Q_k B\|$, so $Q_k B = 0$. Note that $QB = \lim_{k\to\infty} Q_k B$ in strong operator topology, so $QB = 0$, that is, $(I - P)B = 0$, i.e., $B = PB$; this showed that $\mathcal{B}(H)^{\Phi_{\mathcal{A}}} \subseteq P\mathcal{A}'$. If $B \in P\mathcal{A}'$, note that $P \in \mathcal{A}'$, so $PB = BP = B$. Moreover, $\Phi_{\mathcal{A}}(B) = BF = PBF = BPF = BP = B$, that is, $B \in \mathcal{B}(H)^{\Phi_{\mathcal{A}}}$; thus, we have $P\mathcal{A}' \subseteq \mathcal{B}(H)^{\Phi_{\mathcal{A}}}$ and the theorem is proved. □

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