On supremum of bounded quantum observable

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In this paper, we present a new, necessary, and sufficient condition for which the supremum $A \vee B$ exists with respect to the logic order $\preccurlyeq$. Moreover, we give out a new and much simpler representation of $A \vee B$ with respect to $\preccurlyeq$. Our results have nice physical meanings. © *2009 American Institute of Physics.*
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I. INTRODUCTION

First some basic notations: H is a complex Hilbert space, $S(H)$ is the set of all bounded linear self-adjoint operators on H, $S^+(H)$ is the set of all positive operators in $S(H)$, $P(H)$ is the set of all orthogonal projection operators on H, and $\mathcal{B}(\mathbb{R})$ is the set of all Borel subsets of real number set $\mathbb{R}$. Each element in $P(H)$ is said to be a quantum event on H. Each element in $S(H)$ is said to be a bounded quantum observable on H. For $A \in S(H)$, let $R(A)$ be the range of A, $\overline{R(A)}$ be the closure of $R(A)$, P_A be the orthogonal projection on $\overline{R(A)}$, P^A be the spectral measure of A, null(A) be the null space of A, and N_A be the orthogonal projection on null(A).

Let $A,B \in S(H)$. If for each $x \in H$, $[Ax,x] \leq [Bx,x]$, then we say that $A \leq B$. Equivalently, there exists a $C \in S^+(H)$ such that $A+C=B$. $\leq$ is a partial order on $S(H)$. The physical meaning of $A \leq B$ is that the expectation of A is not greater than the expectation of B for each state of the system. So the order $\leq$ is said to be a numerical order of $S(H)$. But $(S(H),\leq)$ is not a lattice. Nevertheless, as a well known theorem attributed to Kadison, $(S(\mathbb{H}),\leq)$ is an antilattice, that is, for any two elements A and B in $S(\mathbb{H})$, the infimum $A \wedge B$ of A and B exists with respect to $\leq$ if and only if A and B are comparable with respect to $\leq$.[1]

In 2006, Gudder introduced a new order $\preccurlyeq$ on $S(H)$: if there exists a $C \in S(H)$ such that $AC=0$ and $A+C=B$, then we say that $A \preccurlyeq B$.[2] Equivalently, $A \preccurlyeq B$ if and only if for each $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$, $P^A(\Delta) \leq P^B(\Delta)$.[2] The physical meaning of $A \preccurlyeq B$ is that for each $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$, the quantum event $P^A(\Delta)$ implies the quantum event $P^B(\Delta)$. Thus, the order $\preccurlyeq$ is said to be a logic order of $S(H)$.[2] In Ref. 2, it is proven that $(S(H),\preccurlyeq)$ is not a lattice since the supremum of arbitrary A and B may not exist in general. In Ref. 3, it is proven that the infimum $A \wedge B$ of A and B with respect to $\preccurlyeq$ always exists. In Ref. 4, the representation theorems of the infimum $A \wedge B$ of A and B with respect to $\preccurlyeq$ were obtained. More recently, Xu *et al.* in Ref. 5 discussed the existence of the supremum $A \vee B$ of A and B with respect to $\preccurlyeq$ by the technique of the operator block. Moreover, they gave sufficient and necessary conditions for the existence of $A \vee B$ with respect to $\preccurlyeq$. Nevertheless, their conditions are difficult to check since the conditions depend on an operator W, but W is not easy to get. Moreover, their proof is so algebraic that we cannot understand its physical meaning.

In this paper, we present a new, necessary, and sufficient condition for which $A \vee B$ exists with respect to $\preccurlyeq$ in a totally different form. Furthermore, we give a new and much simpler representation of $A \vee B$ with respect to $\preccurlyeq$. Our results have nice physical meanings.

Lemma 1.1: (Ref. 2) Let $A,B \in S(H)$. If $A \preccurlyeq B$, then $A=BP_A$.

Lemma 1.2: (Ref. 2) If $P,Q \in P(H)$, then $P \leq Q$ if and only if $P \preccurlyeq Q$, and P and Q have the

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same infimum $P \wedge Q$ and the supremum $P \vee Q$ with respect to the orders $\leq$ and $\preccurlyeq$. We denote them by $P \wedge Q$ and $P \vee Q$, respectively.

Lemma 1.3: (Ref. 6) Let $A, B \in S(H)$. Then $P^A(\{0\}) = N(A)$, $P_A = P^A(R \setminus \{0\})$, $P_A + N(A) = I$, and $P_A \vee P_B = I - N(A) \wedge N(B)$.

II. SOME ELEMENTARY LEMMAS

Let $A, B \in S(H)$ and they have the following forms:

$$A = \int_{-M}^{M} \lambda dA_\lambda$$

and

$$B = \int_{-M}^{M} \lambda dB_\lambda,$$

where $\{A_\lambda\}_{\lambda \in \mathbb{R}}$ and $\{B_\lambda\}_{\lambda \in \mathbb{R}}$ are the identity resolutions of A and B,[6] respectively, and $M = \max(\|A\|, \|B\|)$. If A has an upper bound F in $S(H)$ with respect to $\preccurlyeq$, then it follows from Lemma 1.1 that $A = FP_A$. Note that $A \in S(H)$, so $FP_A = P_A F$ and thus $AF = FA$. Let F have the following form:

$$F = \int_{-G}^{G} \lambda dF_\lambda,$$

where $\{F_\lambda\}_{\lambda \in \mathbb{R}}$ is the identity resolution of F and $G = \max(\|F\|, M)$. Then we have

$$A = FP_A = \left(\int_{-G}^{G} \lambda dF_\lambda \right) P_A = \int_{-G}^{G} \lambda d(F_\lambda P_A).$$

Lemma 2.1: Let $A \in S(H)$ and $F \in S(H)$ be an upper bound of A with respect to $\preccurlyeq$. Then for each $\Delta \in \mathcal{B}(\mathbb{R})$, we have

$$P^A(\Delta) = \begin{cases} P^F(\Delta)P_A, & 0 \notin \Delta, \\ P^F(\Delta \setminus \{0\})P_A + N(A), & 0 \in \Delta. \end{cases}$$

Proof: We just need to check $P^A(\Delta) = P^F(\Delta)P_A$ when $0 \notin \Delta$; the rest is trivial. Note that if we restrict on the subspace $P_A(H) = \overline{R(A)}$, since $AF = FA$, then $\{F_\lambda P_A\}_{\lambda \in \mathbb{R}}$ is the identity resolution of $F|_{P_A(H)}$.[6] Let f be the characteristic function of Δ. Then the following equality proves the conclusion:

$$P^A(\Delta) = f(A) = f(FP_A) = \int_{-G}^{G} f(\lambda) d(F_\lambda P_A) = \int_{\lambda \in \Delta} d(F_\lambda P_A) = P^F(\Delta)P_A.$$

It follows from Lemma 2.1 immediately:

Lemma 2.2: Let $A, B \in S(H)$ and $F \in S(H)$ be an upper bound of A and B with respect to $\preccurlyeq$. Then for any two Borel subsets Δ_1 and Δ_2 of $\mathbb{R}$, if $\Delta_1 \cap \Delta_2 = \emptyset$, $0 \notin \Delta_1$ and $0 \notin \Delta_2$, we have

$$P^A(\Delta_1)P^B(\Delta_2) = P^F(\Delta_1)P_A P^F(\Delta_2)P_B = P_A P^F(\Delta_1)P^F(\Delta_2)P_B = 0.$$

Lemma 2.3: Let $A, B \in S(H)$ and have the following property: For each pair $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$, whenever $\Delta_1 \cap \Delta_2 = \emptyset$ and $0 \notin \Delta_1$ and $0 \notin \Delta_2$, we have $P^A(\Delta_1)P^B(\Delta_2) = 0$; then the following mapping $E: \mathcal{B}(\mathbb{R}) \to P(H)$ defines a spectral measure:

$$E(\Delta) = \begin{cases} P^A(\Delta) \vee P^B(\Delta), & 0 \notin \Delta, \\ P^A(\Delta \setminus \{0\}) \vee P^B(\Delta \setminus \{0\}) + N(A) \wedge N(B), & 0 \in \Delta. \end{cases}$$

Proof: First, we show that for each $\Delta \in \mathcal{B}(\mathbb{R})$, $E(\Delta) \in P(H)$. It is sufficient to check the case of $0 \in \Delta$. Since $P^A(\Delta \backslash \{0\}) \vee P^B(\Delta \backslash \{0\}) \leq P^A(R \backslash \{0\}) \vee P^B(R \backslash \{0\}) = P_A \vee P_B$, it follows from Lemma 1.3 that $P^A(\Delta \backslash \{0\}) \vee P^B(\Delta \backslash \{0\}) + N(A) \wedge N(B) \in P(H)$ and the conclusion holds.

Second, we have

$$E(\emptyset) = P^A(\emptyset) \vee P^B(\emptyset) = 0 \vee 0 = 0,$$

$$E(R) = P^A(R \setminus \{0\}) \vee P^B(R \setminus \{0\}) + N(A) \wedge N(B) = P_A \vee P_B + N(A) \wedge N(B) = I.$$

Third, if $\Delta_1 \cap \Delta_2 = \emptyset$, there are two cases:

(i) 0 does not belong to any one of Δ_1 and Δ_2. It follows from the definition of E that $E(\Delta_1)E(\Delta_2) = (P^A(\Delta_1) \vee P^B(\Delta_1))(P^A(\Delta_2) \vee P^B(\Delta_2))$. Note that $P^B(\Delta_1)P^A(\Delta_2) = 0$ by the conditions of the lemma and $P^B(\Delta_1)P^B(\Delta_2) = 0$; we have $P^B(\Delta_1)(P^A(\Delta_2) \vee P^B(\Delta_2)) = 0$; similarly, we also have $P^A(\Delta_1)(P^A(\Delta_2) \vee P^B(\Delta_2)) = 0$; thus,

$$E(\Delta_1)E(\Delta_2) = 0.$$

Furthermore, we have

$$\begin{aligned} E(\Delta_1 \cup \Delta_2) &= P^A(\Delta_1 \cup \Delta_2) \vee P^B(\Delta_1 \cup \Delta_2) = P^A(\Delta_1) \vee P^A(\Delta_2) \vee P^B(\Delta_1) \vee P^B(\Delta_2) \\ &= (P^A(\Delta_1) \vee P^B(\Delta_1)) \vee (P^A(\Delta_2) \vee P^B(\Delta_2)) = (P^A(\Delta_1) \vee P^B(\Delta_1)) \\ &\quad + (P^A(\Delta_2) \vee P^B(\Delta_2)) = E(\Delta_1) + E(\Delta_2). \end{aligned}$$

That is, in this case, we proved that

$$E(\Delta_1)E(\Delta_2) = 0,$$

$$E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2).$$

(ii) 0 belongs to one of Δ_1 and Δ_2. Without losing generality, we suppose that $0 \in \Delta_1$, since $\Delta_1 \cap \Delta_2 = \emptyset$, so $0 \notin \Delta_2$; thus we have

$$\begin{aligned} E(\Delta_1)E(\Delta_2) &= (P^A(\Delta_1 \setminus \{0\}) \vee P^B(\Delta_1 \setminus \{0\}) + N(B) \wedge N(A))(P^A(\Delta_2) \vee P^B(\Delta_2)) \\ &= (P^A(\Delta_1 \setminus \{0\}) \vee P^B(\Delta_1 \setminus \{0\}))(P^A(\Delta_2) \vee P^B(\Delta_2)) = 0, \end{aligned}$$

$$\begin{aligned} E(\Delta_1 \cup \Delta_2) &= P^A(\Delta_1 \setminus \{0\} \cup \Delta_2) \vee P^B(\Delta_1 \setminus \{0\} \cup \Delta_2) + (N(B) \wedge N(A)) \\ &= (P^A(\Delta_1 \setminus \{0\}) \vee P^B(\Delta_1 \setminus \{0\}) + (N(B) \wedge N(A))) + (P^A(\Delta_2) \vee P^B(\Delta_2)) \\ &= (P^A(\Delta_1 \setminus \{0\}) \vee P^B(\Delta_1 \setminus \{0\}) + (N(A) \wedge N(B))) + (P^A(\Delta_2) \vee P^B(\Delta_2)) = E(\Delta_1) \\ &\quad + E(\Delta_2). \end{aligned}$$

Thus, it follows from (i) and (ii) that whenever $\Delta_1 \cap \Delta_2 = \emptyset$, we have

$$E(\Delta_1)E(\Delta_2) = 0,$$

$$E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2).$$

Finally, if $\{\Delta_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint Borel sets in $\mathcal{B}(\mathbb{R})$, then it is easy to prove that

$$E\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} E(\Delta_n).$$

Thus, the lemma is proved.

III. MAIN RESULTS AND PROOFS

Theorem 3.1: Let $A,B \in S(H)$ and have the following property: For each pair $\Delta_1,\Delta_2 \in \mathcal{B}(\mathbb{R})$, whenever $\Delta_1 \cap \Delta_2=\emptyset$ and $0 \notin \Delta_1$ and $0 \notin \Delta_2$, we have $P^A(\Delta_1)P^B(\Delta_2)=0$. Then the supremum $A \vee B$ of A and B exists with respect to the logic order $\preccurlyeq$.

Proof: By Lemma 2.3, $E(\cdot)$ is a spectral measure and so it can generate a bounded quantum observable K and K can be represented by $K=\int_{-M}^{M}\lambda dE_\lambda$, where $\{E_\lambda\}=E(-\infty,\lambda]$, $\lambda \in \mathbb{R}$, and $M=\max(\|A\|,\|B\|)$. Moreover, for each $\Delta \in \mathcal{B}(\mathbb{R})$, $P^K(\Delta)=E(\Delta)$.[6] We confirm that K is the supremum $A \vee B$ of A and B with respect to $\preccurlyeq$. In fact, for each $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$, by the definition of E, we knew that $P^K(\Delta)=E(\Delta)=P^A(\Delta)\vee P^B(\Delta) \geq P^A(\Delta)$ and $P^K(\Delta)=E(\Delta)=P^A(\Delta)\vee P^B(\Delta) \geq P^B(\Delta)$. So it follows from the equivalent properties of $\preccurlyeq$ that $A \preccurlyeq K$ and $B \preccurlyeq K$.[2] If K' is another upper bound of A and B with respect to $\preccurlyeq$, then for each $\Delta \in \mathcal{B}(\mathbb{R})$ with $0 \notin \Delta$, we have $P^A(\Delta) \leq P^{K'}(\Delta)$ and $P^B(\Delta) \leq P^{K'}(\Delta)$,[2] so $P^A(\Delta)\vee P^B(\Delta)=E(\Delta)=P^K(\Delta) \leq P^{K'}(\Delta)$; thus we have $K \preccurlyeq K'$ and that K is the supremum of A and B with respect to $\preccurlyeq$ is proved.

It follows from Lemma 2.2 and Theorem 3.1 and their proofs that we have the following theorem immediately.

Theorem 3.2: Let $A,B \in S(H)$. Then the supremum $A \vee B$ of A and B exists with respect to the logic order $\preccurlyeq$ if and only if for each pair $\Delta_1,\Delta_2 \in \mathcal{B}(\mathbb{R})$, whenever $\Delta_1 \cap \Delta_2=\emptyset$ and $0 \notin \Delta_1$ and $0 \notin \Delta_2$, we have $P^A(\Delta_1)P^B(\Delta_2)=0$. Moreover, in this case, we have the following nice representation:

$$A \vee B = \int_{-M}^{M} \lambda dE_\lambda,$$

where $E_\lambda=E(-\infty,\lambda]$, $\lambda \in \mathbb{R}$. and $M=\max(\|A\|,\|B\|)$.

Remark 3.3: Let $A,B \in S(H)$. Note that for each $\Delta \in \mathcal{B}(\mathbb{R})$, $P^A(\Delta)$ is interpreted as the quantum event that the quantum observable A has a value in Δ,[2] and the conditions $\Delta_1 \cap \Delta_2=\emptyset$, $0 \notin \Delta_1$ and $0 \notin \Delta_2$, must have $P^A(\Delta_1)P^B(\Delta_2)=0$ told us that the quantum events $P^A(\Delta_1)$ and $P^B(\Delta_2)$ cannot happen at the same time, so the physical meanings of the supremum $A \vee B$ exists with respect to $\preccurlyeq$ if and only if for each pair $\Delta_1,\Delta_2 \in \mathcal{B}(\mathbb{R})$, whenever $\Delta_1 \cap \Delta_2=\emptyset$ and $0 \notin \Delta_1$ and $0 \notin \Delta_2$, that the quantum observable A takes value in Δ_1 and the quantum observable B takes value in Δ_2 cannot happen at the same time.

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