On fixed points of Lüders operation

Liu Weihua and Wu Junde[a)]
Department of Mathematics, Zhejiang University, Hangzhou 310027, People's Republic of China

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In this paper, we give a concrete example of a Lüders operation $L_{\mathcal{A}}$ with $n=3$, such that $L_{\mathcal{A}}(B)=B$ does not imply that B commutes with all E_1, E_2, and E_3 in $\mathcal{A}$, this example answers an open problem of Professor Gudder. © *2009 American Institute of Physics.* [doi:10.1063/1.3253574]

Let H be a complex Hilbert space, $\mathcal{B}(H)$ be the bounded linear operator set on H, $\mathcal{E}(H)=\{A:0\leq A\leq I\}$, $\mathcal{A}=\{E_i\}_{i=1}^{n}\subseteq\mathcal{E}(H)$, and $\Sigma_{i=1}^{n}E_i=I$, where $1\leq n\leq\infty$. The famous *Lüders operation* $L_{\mathcal{A}}$ is a map which is defined on $\mathcal{B}(H)$ by

$$L_{\mathcal{A}}:A\rightarrow\sum_{i=1}^{n}E_i^{1/2}AE_i^{1/2}.$$

A question related to a celebrated theorem of Lüders operation is whether $L_{\mathcal{A}}(A)=A$ for some $A\in\mathcal{E}(H)$ implies that A commutes with all E_i for $i=1,2,\ldots,n$.[1] The answer to this question is positive for $n=2$ (Ref. 2) and negative for $n=5$.[1] In this paper it is shown, by using a simple derivation of the example of Arias–Gheondea–Gudder in Ref. 1, that the answer is negative as well for $n=3$, a question raised by Gudder in 2005.[3]

First, we denote $\mathcal{B}(H)^{L_{\mathcal{A}}}=\{B\in\mathcal{B}(H):L_{\mathcal{A}}(B)=B\}$ is the fixed point set of $L_{\mathcal{A}}$, $\mathcal{A}'$ is the commutant of $\mathcal{A}$.

Lemma 1: (Ref. 1) If $\mathcal{B}(H)^{L_{\mathcal{A}}}=\mathcal{A}'$, then $\mathcal{A}'$ is injective.

Lemma 2: (Ref. 1) Let F_2 be the free group generated by two generators g_1 and g_2 with identity e, $\mathbb{C}$ be the complex numbers set, and $H=l_2(F_2)$ be the separable complex Hilbert space,

$$H=l_2(F_2)=\left\{f\middle|f:F_2\rightarrow\mathbb{C},\quad\sum|f(x)|^2<\infty\right\}.$$

For $x\in F_2$ define $\delta_x:F_2\rightarrow C$ by $\delta_x(y)$ equals 0 for all $y\neq x$ and 1 when $y=x$. Then $\{\delta_x|x\in F_2\}$ is an orthonormal basis for H. Define the unitary operators U_1 and U_2 on H by $U_1(\delta_x)=\delta_{g_1x}$ and $U_2(\delta_x)=\delta_{g_2x}$. Then the von Neumann algebra $\mathcal{N}$ which is generated by U_1 and U_2 and its commutant $\mathcal{N}'$ are not injective.

Now, we follow Lemma 1 and Lemma 2 to prove our main result:

Let $\mathbb{C}_1$ be the unit circle in $\mathbb{C}$ and h be a Borel function defined on $\mathbb{C}_1$ as the following: $h(e^{i\theta})=\theta$ for $\theta\in[0,2\pi)$. Then $A_1=h(U_1)$ and $A_2=h(U_2)$ are two positive operators in $\mathcal{N}$. Taking the real and imagined parts of $U_1=V_1+iV_2$ and $U_2=V_3+iV_4$, $\mathcal{N}$ is generated by the self-adjoint operators $\{V_1,V_2,V_3,V_4\}$. Since functions cos and sin are two Borel functions, we have $V_1=\frac{1}{2}(U_1+U_1^*)=\cos(A_1)$, $V_2=\sin(A_1)$, $V_3=\cos(A_2)$, and $V_4=\sin(A_2)$. Thus $\mathcal{N}$ is contained in the von Neumann algebra which is generated by A_1 and A_2.

On the other hand, it is clear that the von Neumann algebra which is generated by A_1 and A_2 is contained in $\mathcal{N}$. So $\mathcal{N}$ is the von Neumann algebra which is generated by A_1 and A_2. Let $E_1=A_1/2\|A_1\|$, $E_2=A_2/2\|A_2\|$, and $E_3=I-E_1-E_2$. Then $\mathcal{A}=\{E_1,E_2,E_3\}\subseteq\mathcal{E}(H)$ and $E_1+E_2+E_3=I$.

Now, we define the Lüders operation on $\mathcal{B}(H)$ by

[a)]Author to whom correspondence should be addressed. Electronic mail: wjd@zju.edu.cn.

$$L_{\mathcal{A}}(B) = \sum_{i=1}^{3} E_i^{1/2} B E_i^{1/2}.$$

It is clear that the von Neumann algebra which is generated by $\{E_1, E_2, E_3\}$ is $\mathcal{N}$. It follows from Lemma 1 and Lemma 2 that $B(H)^{L_{\mathcal{A}}} \not\supseteq \mathcal{A}'$, thus there exists a $D \in B(H)^{L_{\mathcal{A}}} \backslash \mathcal{A}'$. The real part or the imaginary part D_1 of D also satisfies $D_1 \in B(H)^{L_{\mathcal{A}}} \backslash \mathcal{A}'$. If $D_2 = \|D_1\| I - D_1$, then $D_2 \geq 0$. Thus $D_3 = D_2 / \|D_2\| \in \mathcal{E}(H)$ and $D_3 \in B(H)^{L_{\mathcal{A}}} \backslash \mathcal{A}'$. We have proven the following theorem which answered the question in Ref. 3.

Theorem 1: Let $H = l_2(F_2)$, $\mathcal{A} = \{E_i\}_{i=1}^3$ be defined as above. Then there is a $B \in \mathcal{E}(H)$, such that $L_{\mathcal{A}}(B) = B$, but B does not commute with all E_1, E_2, and E_3.

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