

A representation theorem of infimum of bounded quantum observables

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In 2006, Gudder introduced a logic order on the bounded quantum observable set $S(H)$. In 2007, Pulmannova and Vincekova proved that for each subset \mathcal{D} of $S(H)$, the infimum of \mathcal{D} exists with respect to this logic order. In this paper, we present a representation theorem for the infimum of \mathcal{D} . © 2008 American Institute of Physics. [DOI: 10.1063/1.2963968]

Let H be a complex Hilbert space, $S(H)$ be the set of all bounded linear self-adjoint operators on H , $S^+(H)$ be the set of all positive operators in $S(H)$, and $P(H)$ be the set of all orthogonal projection operators on H . Each element in $P(H)$ is said to be a quantum event, and each element in $S(H)$ is said to be a bounded quantum observable on H . If $A \in S(H)$, $R(A)$ is the range of A , $\overline{R(A)}$ is the closure of $R(A)$, P_A is the orthogonal projection on $\overline{R(A)}$, P^A is the spectral measure of A , and $\text{null}(A)$ is the null space of A .

Let $A, B \in S(H)$. If for each $x \in H$, $[Ax, x] \leq [Bx, x]$, then we say that $A \leq B$. Equivalently, there exists a $C \in S^+(H)$ such that $A + C = B$. \leq is a partial order on $S(H)$. The physical meaning of $A \leq B$ is that the expectation of A is not greater than the expectation of B for each state of the system. So the order \leq is said to be a numerical order of $S(H)$.

In 2006, Gudder¹ introduced the order \leq on $S(H)$: If there exists a $C \in S(H)$ such that $AC = 0$ and $A + C = B$, then we say that $A \leq B$.

Equivalently, $A \leq B$ if and only if for each Borel subset Δ with $0 \notin \Delta$, we have $P^A(\Delta) \leq P^B(\Delta)$. The physical meaning of $A \leq B$ is that the quantum event $P^A(\Delta)$ implies the quantum event $P^B(\Delta)$. Thus, the order \leq is said to be a logic order of $S(H)$.¹

Lemma 1:¹ If $A, B \in S(H)$, the following statements are equivalent:

- (i) $AB = 0$.
- (ii) $\overline{R(A)} \subseteq \text{null}(B)$.
- (iii) $\overline{R(B)} \subseteq \text{null}(A)$.
- (iv) $P_A P_B = 0$.
- (v) $\overline{R(A)} \perp \overline{R(B)}$.

Lemma 2:¹ If $A, B \in S(H)$, the following statements are equivalent:

- (i) $A \leq B$.
- (ii) $Ax = Bx$ for each $x \in \overline{R(A)}$.
- (iii) $A = BP_A$.
- (iv) $AB = A^2$.

Lemma 3:¹ If $P, Q \in P(H)$, then $P \leq Q$ if and only if $P \leq Q$, and P and Q have the same infimum with respect to the orders \leq and \leq . We denote it by $P \wedge Q$.

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For a given order, the infimum problem of bounded quantum observables is to find under what condition the infimum $A \wedge B$ exists for the given order for $A, B \in S(H)$. Moreover, can we give the structure of $A \wedge B$?

For the numerical order \leq on $S(H)$, the problem has been studied in a different content in Refs. 2–6.

In 2007, Pulmannova and Vincekova⁷ proved that for each subset \mathcal{D} of $S(H)$, the infimum exists with respect to the logic order \leq . However, their proof is abstract and there is no information about the structure of the infimum.

In this note, moreover, we find the representation of the infimum.

By Theorem 4.5 in Ref. 7 we can prove the dual result.

Lemma 4: Let $\{A_\alpha\}_{\alpha \in \Gamma}$ be a monotone decreasing net in $S(H)$ with respect to \leq . Then $A = \bigwedge_{\alpha \in \Gamma} A_\alpha$ exists in $(S(H), \leq)$ and $\lim A_\alpha = A$ in the strong operator topology.

Now, we give out the representation theorem of infimum by the following construction:

(I) Consider the two positive operator case. Let $A, B \in S^+(H)$. The resolutions of the identity of A and B are $\{E_\lambda\}_{\lambda \in R}$ and $\{F_\lambda\}_{\lambda \in R}$,⁸ respectively. Let $\Delta E_{n,i} = E_{i/2^n} - E_{(i-1)/2^n}$, $\Delta F_{n,i} = F_{i/2^n} - F_{(i-1)/2^n}$. Then $\Delta E_{n,i}$ and $\Delta F_{n,i}$ are both orthogonal projection operators on H . Take positive integer p such that $\|A\| \leq p, \|B\| \leq p$. Let

$$A_n = \sum_{i=1}^{2^n p} \frac{i}{2^n} \Delta E_{n,i}, \quad B_n = \sum_{i=1}^{2^n p} \frac{i}{2^n} \Delta F_{n,i}.$$

Then $\|A_n - A\| \rightarrow 0, \|B_n - B\| \rightarrow 0$.⁸

Denote

$$D_n = \sum_{i=1}^{2^n p} \frac{i}{2^n} \Delta E_{n,i} \wedge \Delta F_{n,i}.$$

We claim that the strong operator topology limit of $\{D_n\}$ must exist and be the infimum of A and B with respect to the order \leq .

In order to prove the above conclusion, at first, we show that D_n is the infimum of A_n and B_n with respect to \leq .

It follows from the definition that $D_n \leq A_n$ and $D_n \leq B_n$ are clear.

If $D'_n \leq A_n, B_n$, it follows from Lemma 2 that $D'_n = A_n P_{D'_n} = B_n P_{D'_n}$, that is,

$$D'_n = \sum_{i=1}^{2^n p} \frac{i}{2^n} \Delta E_{n,i} P_{D'_n} = \sum_{i=1}^{2^n p} \frac{i}{2^n} \Delta F_{n,i} P_{D'_n}.$$

This shows that for each positive integer i we have

$$\Delta E_{n,i} P_{D'_n} = \Delta F_{n,i} P_{D'_n}.$$

Note that

$$\Delta E_{n,i} P_{D'_n} \leq \Delta E_{n,i},$$

$$\Delta F_{n,i} P_{D'_n} \leq \Delta F_{n,i}.$$

Hence

$$\Delta E_{n,i} P_{D'_n} \leq \Delta E_{n,i} \wedge \Delta F_{n,i}.$$

Thus we have $D'_n \leq D_n$ and so D_n is the infimum of A_n and B_n with respect to \leq .

For each positive integer n , it follows from the definition of D_n that

$$\overline{R(D_n)} = R(D_n) = \sum_{i=1}^{2^n p} R(\Delta E_{n,i} \wedge \Delta F_{n,i}).$$

Moreover, we have

$$R(D_{n+1}) \subseteq R(D_n).$$

Indeed,

$$\begin{aligned} R(D_{n+1}) &= \sum_{i=1}^{2^{n+1}p} R(\Delta E_{n+1,i} \wedge \Delta F_{n+1,i}) = \sum_{i=1}^{2^{n+1}p} R[(E_{i/2^{n+1}} - E_{(i-1)/2^{n+1}}) \wedge (F_{i/2^{n+1}} - F_{(i-1)/2^{n+1}})] \\ &= \sum_{i=1}^{2^n p} \{R[(E_{2i/2^{n+1}} - E_{(2i-1)/2^{n+1}}) \wedge (F_{2i/2^{n+1}} - F_{(2i-1)/2^{n+1}})] \\ &\quad + R[(E_{(2i-1)/2^{n+1}} - E_{(2i-2)/2^{n+1}}) \wedge (F_{(2i-1)/2^{n+1}} - F_{(2i-2)/2^{n+1}})]\} \\ &\subseteq \sum_{i=1}^{2^n p} [R(E_{2i/2^{n+1}} - E_{(2i-1)/2^{n+1}}) + R(E_{(2i-1)/2^{n+1}} - E_{(2i-2)/2^{n+1}})] \wedge [R(F_{2i/2^{n+1}} - F_{(2i-1)/2^{n+1}}) \\ &\quad + R(F_{(2i-1)/2^{n+1}} - F_{(2i-2)/2^{n+1}})] \\ &= \sum_{i=1}^{2^n p} R(E_{2i/2^{n+1}} - E_{(2i-2)/2^{n+1}}) \wedge R(F_{2i/2^{n+1}} - F_{(2i-2)/2^{n+1}}) \\ &= \sum_{i=1}^{2^n p} R[(E_{i/2^n} - E_{(i-1)/2^n}) \wedge R(F_{i/2^n} - F_{(i-1)/2^n})] = \sum_{i=1}^{2^n p} R(\Delta E_{n,i}) \wedge \Delta F_{n,i} = R(D_n). \end{aligned}$$

Now, we prove that there exists a $D \in S(H)$ such that $\{D_n\}$ is strong operator topology convergent to D .

Let $E = \bigwedge_{n=1}^{\infty} P_{D_n}$. Note that $R(D_{n+1}) \subseteq R(D_n)$. So $\{P_{D_n}\}$ is a monotone decreasing sequence in $P(H)$. So $\{P_{D_n}\}$ is strong operator topology convergent to E . On the other hand, it is easy to prove that $D_n = D_n P_{D_n}$. Note that $D_n \leq A_n$. It follows from Lemma 2 that $D_n = A_n P_{D_n}$. Thus we have $D_n = A_n P_{D_n} = D_n P_{D_n}$. Let $x \in H$. It follows from

$$\begin{aligned} \|D_n x - A E x\| &\leq \|D_n x - A_n E x\| + \|(A_n - A) E x\| \\ &= \|D_n P_{D_n} x - A_n E x\| + \|(A_n - A) E x\| \\ &\leq \|(D_n P_{D_n} - A_n P_{D_n}) x\| + \|A_n (P_{D_n} - E) x\| + \|(A_n - A) E x\| \\ &= \|A_n (P_{D_n} - E) x\| + \|(A_n - A) E x\|, \end{aligned}$$

and $\{P_{D_n}\}$ is strong operator topology convergent to E and $\{A_n\}$ is norm topology convergent to A that

$$\lim D_n x = A E x.$$

That is, $\{D_n\}$ is strong operator topology convergent to $A E$. It follows from the fact that $\{D_n\}$ is a bounded linear self-adjoint operator sequence that $A E$ is also a bounded linear self-adjoint operator. So $A E = E A$. Similarly, we have $\lim D_n x = B E x$. We denote $D = A E = B E$.

Note that $A = A E + A(I - E)$ and as A and $A E$ are bounded linear self-adjoint operators, we have that $A(I - E)$ is also a bounded linear self-adjoint operator.

Moreover, note that $A(I - E) A E = A(I - E) E A = 0$. So we have $A E \leq A$. Similarly, we have $A E \leq B$. This shows that $A E \leq A$ and $A E \leq B$.

Now we prove that AE is the infimum of A and B with respect to \leq .
 If $C \in S(H)$ and $C \leq A, B$, then $C = AP_C$. So $\|C\| = \|AP_C\| \leq \|A\| \|P_C\| \leq \|A\| \leq p$. Let

$$C_n = \sum_{i=1}^{2^n p} \frac{i}{2^n} \Delta K_{n,i}, \quad \Delta K_{n,i} = K_{i/2^n} - K_{(i-1)/2^n}.$$

Here $\{K_\lambda\}_{\lambda \in R}$ is the resolution of the identity of C , and we have $\|C_n - C\| \rightarrow 0$.

Note that $C \leq A$ if and only if for each Borel subset Δ of R with $0 \notin \Delta$, $P^C(\Delta) \leq P^A(\Delta)$.¹ So

$$\Delta K_{n,i} = P^C \left(\left(\frac{i-1}{2^n}, \frac{i}{2^n} \right] \right) \leq P^A \left(\left(\frac{i-1}{2^n}, \frac{i}{2^n} \right] \right) = \Delta E_{n,i}.$$

Therefore, we have $C_n \leq A_n$. This shows that if $C \leq A$, then $C_n \leq A_n$. Thus we have $C_n \leq D_n$. By the definition of \leq , there exists $Q_n \in S(H)$ such that $C_n Q_n = 0$ and $C_n + Q_n = D_n$. It is clear that the strong operator topology limit of $\{Q_n\}$ exists. Let Q be the strong operator topology limit of $\{Q_n\}$. Then $Q \in S(H)$, $CQ = 0$, and $C + Q = D$ ($[1, P_{116}]$). So $C \leq D$. Thus, we proved that D is the infimum of A and B with respect to \leq .

The above process showed that the infimum of A and B with respect to \leq is

$$AE = A \bigwedge_{n=1}^{\infty} P_{D_n}.$$

Here

$$D_n = \sum_{i=1}^{2^n p} \frac{i}{2^n} \Delta E_{n,i} \wedge \Delta F_{n,i}.$$

(II) Consider the two bounded linear self-adjoint operator case. Let $A, B \in S(H)$. Then A and B can be decomposed uniquely into $A = A_+ - A_-, B = B_+ - B_-$. Here $A_+ A_- = 0, B_+ B_- = 0$. Now, we show that the infimum of A and B with respect to \leq is $A_+ \wedge B_+ - A_- \wedge B_-$. Here $A_+ \wedge B_+$ is the infimum of A_+ and B_+ with respect to \leq , and $A_- \wedge B_-$ is the infimum of A_- and B_- with respect to \leq .

In fact, let $D = A_+ \wedge B_+ - A_- \wedge B_-$. It follows from $A_+ A_- = 0, B_+ B_- = 0$ and Lemma 1 and the definition of \leq that $D \leq A$ and $D \leq B$. If $C \leq A$ and $C \leq B$, C has been decomposed uniquely into $C = C_+ - C_-$. Here $C_+ C_- = 0$. It follows from $C \leq A$ that $C = AP_C = BP_C$. On the other hand, it is easy to prove $A_+ P_C = P_C A_+$. Note the uniqueness of decomposition of C . We have $C_+ = P_C A_+$. Since $A_+ = A_+ P_C + A_+ (I - P_C)$, $A_+ (I - P_C)$ is also a bounded linear self-adjoint operator. It follows from $A_+ (I - P_C) A_+ P_C = 0$ that $C_+ \leq A_+$. Similarly, we have $C_+ \leq B_+$. Thus, $C_+ \leq A_+ \wedge B_+$. Similarly, $C_- \leq A_- \wedge B_-$. By using Lemma 1 we have $C \leq D$. This showed that $A_+ \wedge B_+ - A_- \wedge B_-$ is the infimum of A and B with respect to \leq .

Thus, we can obtain the representation of the infimum of A and B with respect to \leq by the above conclusion and case (I).

(III) Consider any subset \mathcal{D} of $S(H)$ case. Let \mathcal{F} be the all finitely nonempty subsets of \mathcal{D} . If $F_1, F_2 \in \mathcal{F}$, we define an order \leq in \mathcal{F} by $F_1 \leq F_2$ if and only if $F_2 \subseteq F_1$. Then \mathcal{F} is a directed set with respect to the order \leq . It follows from Lemma 4 that the infimum of \mathcal{D} with respect to \leq is the strong operator topology limit of $\{\bigwedge_{A \in F} A\}_{F \in \mathcal{F}}$. It follows from case (II) that for each $F \in \mathcal{F}$, we can obtain its infimum $\bigwedge_{A \in F} A$ with respect to \leq . Thus, we complete the structure process for the infimum of \mathcal{D} with respect to \leq .

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