

MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 6

1. Prove the Fundamental Theorem of Algebra using Cauchy's theorem as follows. Suppose we have a polynomial with complex coefficients $p(z) = \sum_{k=0}^n a_k z^k$ that has no zeros in \mathbb{C} .

(a) Consider the polynomial $q(z) = \sum_{k=0}^n \bar{a}_k z^k$ whose coefficients are conjugates of the corresponding coefficients of $p(z)$. Show that

$$f(z) = \frac{1}{p(z)q(z)}$$

is an entire function.

(b) Let $R > 0$. Apply Cauchy's theorem to the line integral along the closed curve L in Figure 1b to show that

$$\int_{-R}^R \frac{dx}{|p(x)|^2} + i \int_0^\pi \frac{Re^{i\theta}}{p(Re^{i\theta})q(Re^{i\theta})} d\theta = 0.$$

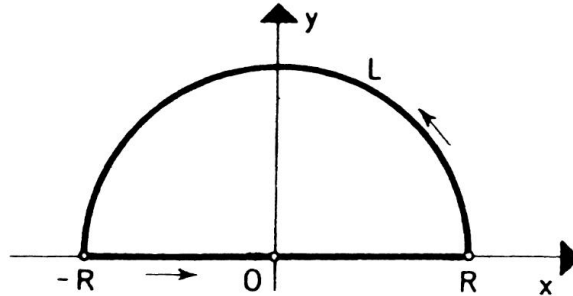


FIGURE 1. L is a closed curve in the counter clockwise direction along the semicircle $\{Re^{it} \mid 0 \leq t \leq \pi\}$ and the line segment $\{z \mid -R \leq \operatorname{Re} z \leq R, \operatorname{Im} z = 0\}$.

(c) Show that as $R \rightarrow \infty$, we get

$$\int_{-\infty}^{\infty} \frac{dx}{|p(x)|^2} = 0.$$

(d) Why is this a contradiction?

2. Let $r > 1$ and let f be analytic in $D(0, r)$. Starting from the integral

$$\int_{\Gamma} \left[2 \pm \left(z + \frac{1}{z} \right) \right] \frac{f(z)}{z} dz,$$

where Γ is the curve $z : [0, 2\pi] \rightarrow \mathbb{C}$, $z(t) = e^{it}$, prove that

$$\frac{2}{\pi} \int_0^{2\pi} f(e^{it}) \cos^2 \left(\frac{t}{2} \right) dt = 2f(0) + f'(0),$$

$$\frac{2}{\pi} \int_0^{2\pi} f(e^{it}) \sin^2 \left(\frac{t}{2} \right) dt = 2f(0) - f'(0).$$

3. Let $a \in \mathbb{C}$ and $r > 0$. Evaluate the integral

$$\frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{e^z}{z(1-z)^3} dz$$

- (a) if $0 \in D(a,r)$ and $1 \notin \overline{D(a,r)}$;
- (b) if $0 \notin \overline{D(a,r)}$ and $1 \in D(a,r)$;
- (c) if $0 \in D(a,r)$ and $1 \in D(a,r)$.

4. (a) Let f and g be entire functions that satisfy

$$|f(z)| < |g(z)|$$

for all $z \in \mathbb{C}$. Show that there exists a constant $\lambda \in \mathbb{C}$ such that

$$f(z) = \lambda g(z)$$

for all $z \in \mathbb{C}$.

(b) Determine all entire functions f that satisfies

$$|f'(z)| < |f(z)|$$

for all $z \in \mathbb{C}$.

5. If f is analytic on $D(0,1)$ and let the power series expansion of f be

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

e below denotes the base of natural logarithms, ie. $e = \exp(1)$.

(a) Suppose for all $z \in D(0,1)$,

$$|f(z)| \leq \frac{1}{1-|z|}.$$

Show that for all $n \in \mathbb{N}$,

$$|a_n| < (n+1)e.$$

(b) Suppose for all $z \in D(0,1)$,

$$|f'(z)| \leq \frac{1}{1-|z|}.$$

Show that for all $n \in \mathbb{N}$,

$$|a_n| < e.$$