

MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 5 SOLUTIONS

1. Let $a \in \mathbb{R}$ and $z \in \mathbb{C}$.

(a) Evaluate the following integrals

$$\int_0^1 e^{it} \cos at \, dt \quad \text{and} \quad \int_{-1}^1 \frac{dt}{t^2 + i}.$$

SOLUTION. For the first integral, we have

$$\int_0^1 e^{it} \cos at \, dt = \int_0^1 \cos t \cos at \, dt + i \int_0^1 \sin t \cos at \, dt.$$

Since

$$\begin{aligned} \cos t \cos at &= \frac{1}{2} [\cos(t+at) + \cos(t-at)], \\ \sin t \sin at &= \frac{1}{2} [\sin(t+at) + \sin(t-at)], \end{aligned}$$

we obtain

$$\begin{aligned} \int_0^1 e^{it} \cos at \, dt &= \frac{1}{2} \left(\frac{\sin(1+a)t}{1+a} + \frac{\sin(1-a)t}{1-a} \right) \Big|_0^1 - \frac{i}{2} \left(\frac{\cos(1+a)t}{1+a} + \frac{\cos(1-a)t}{1-a} \right) \Big|_0^1 \\ &= \frac{1}{2} \left(\frac{\sin(1+a)}{1+a} + \frac{\sin(1-a)}{1-a} \right) - \frac{i}{2} \left(\frac{\cos(1+a)}{1+a} + \frac{\cos(1-a)}{1-a} - \frac{2}{1-a^2} \right). \end{aligned}$$

For the second integral, we first apply the formula for complex square roots in Chapter 1 and then find a partial fraction decomposition.

$$\frac{1}{t^2 + i} = \frac{1}{\left(t - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \left(t + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)} = \frac{\alpha}{t - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}} + \frac{\beta}{t + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}}$$

and so

$$\alpha = \frac{1}{2\sqrt{2}}(1+i), \quad \beta = -\frac{1}{2\sqrt{2}}(1+i).$$

Therefore

$$\int_{-1}^1 \frac{dt}{t^2 + i} = \frac{1}{2\sqrt{2}}(1+i) \log \left(t - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) \Big|_{-1}^1 - \frac{1}{2\sqrt{2}}(1+i) \log \left(t + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) \Big|_{-1}^1.$$

(b) Show that if $\operatorname{Re} z > -1$, then the integral $\int_0^1 t^z \, dt$ exists and

$$\left| \int_0^1 t^z \, dt \right| \leq \frac{1}{1 + \operatorname{Re} z}.$$

SOLUTION. Note that for $0 \leq t \leq 1$,

$$|t^z| = t^{\operatorname{Re} z}$$

and so

$$\left| \int_0^1 t^z \, dt \right| \leq \int_0^1 |t^z| \, dt = \int_0^1 t^{\operatorname{Re} z} \, dt.$$

Since for $\operatorname{Re} z > -1$ the last integral converges, $\int_0^1 t^z dt$ exists and the required inequality follows from

$$\int_0^1 t^{\operatorname{Re} z} dt = \frac{t^{\operatorname{Re} z+1}}{1 + \operatorname{Re} z} \Big|_0^1 = \frac{1}{1 + \operatorname{Re} z}.$$

(c) Show that if $a < 1$, then

$$\left| \int_{-1}^1 \frac{\cos it}{t^a} dt \right| \leq 2 \int_{-1}^1 \frac{dt}{t^a}$$

and thus the (improper) integral $\int_{-1}^1 t^{-a} \cos it dt$ converges absolutely.

SOLUTION. Note that $\cos it = \cosh t$ and for $-1 \leq t \leq 1$, we have $0 \leq \cosh t < 2$. Hence

$$\left| \int_{-1}^1 \frac{\cos it}{t^a} dt \right| \leq \int_{-1}^1 \left| \frac{\cos it}{t^a} \right| dt = \int_{-1}^1 \frac{\cosh t}{|t^a|} dt \leq 2 \int_{-1}^1 \frac{dt}{t^a}.$$

Since the last integral converges for $a < 1$, we see that $\int_{-1}^1 t^{-a} \cos it dt$ converges absolutely.

2. (a) For $k = 1, 2, 3$, evaluate the following integrals

$$\int_{\Gamma_k} \operatorname{Re}(z) dz, \quad \int_{\Gamma_k} z^2 dz, \quad \int_{\Gamma_k} \frac{dz}{z}$$

along the curves from the point $z_0 = 1$ to $z_1 = i$ in the counter clockwise direction as described in Figure 1.

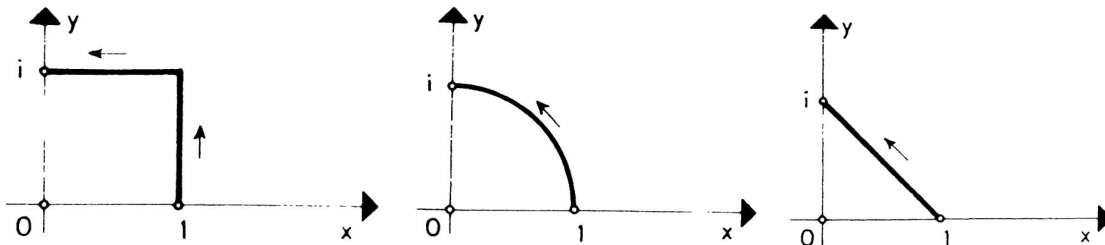


FIGURE 1. *Left:* Γ_1 is along the boundary of the square: $\{x + iy \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$. *Center:* Γ_2 is along the boundary of the circle: $\{e^{it} \mid 0 \leq t \leq \pi/2\}$. *Right:* Γ_3 is along the line segment: $\{(1-t) + it \mid 0 \leq t \leq 1\}$.

SOLUTION. For $k = 1, 2, 3$, we have

$$\int_{\Gamma_1} \operatorname{Re}(z) dz = \int_0^1 1 \cdot i dt + \int_0^1 \operatorname{Re}(i+t) dt = i - \int_0^1 t dt = -\frac{1}{2} + i;$$

and

$$\begin{aligned} \int_{\Gamma_2} \operatorname{Re}(z) dz &= i \int_0^{\pi/2} \operatorname{Re}(e^{it}) e^{it} dt \\ &= i \int_0^{\pi/2} \cos t (\cos t + i \sin t) dt \\ &= i \int_0^{\pi/2} \cos^2 t dt - \int_0^{\pi/2} \cos t \sin t dt \\ &= -\frac{1}{2} + i \frac{\pi}{4}; \end{aligned}$$

and

$$\begin{aligned}\int_{\Gamma_3} \operatorname{Re}(z) dz &= \int_{\Gamma_3} \operatorname{Re}[(1-t) + it](i-1) dt \\ &= (-1+i) \int_0^1 (1-t) dt \\ &= -\frac{1}{2} + \frac{i}{2}.\end{aligned}$$

The hard way to evaluate the second integral would be from definition like above, i.e. for $k = 1, 2, 3$,

$$\int_{\Gamma_1} z^2 dz = -i \int_0^1 t^2 dt + \int_1^0 t^2 dt = -\frac{1}{3} - \frac{i}{3};$$

and

$$\int_{\Gamma_2} z^2 dz = i \int_0^{\pi/2} e^{2it} e^{it} dt = \int_0^{\pi/2} (\cos 3t + i \sin 3t) dt = -\frac{1}{3} - \frac{i}{3};$$

and

$$\begin{aligned}\int_{\Gamma_2} z^2 dz &= \int_0^1 [(1-t) + it]^2 (-1+i) dt \\ &= (-1+i) \left(\int_0^1 dt - 2 \int_0^1 t dt + 2i \int_0^1 0 \cdot t dt - 2i \int_0^1 t^2 dt \right) \\ &= -\frac{1}{3} - \frac{i}{3}.\end{aligned}$$

The answers are all the same. Why? Because of the antiderivative theorem that we proved in lecture — $f(z) = z^2$ has antiderivative $F(z) = z^3/3$, which gives the easy way to evaluate all three integrals in one go:

$$\int_{\Gamma_k} z^2 dz = F(i) - F(1) = -\frac{1}{3} - \frac{i}{3}.$$

The last integral can be evaluated analogously.

- (b) Let $a, b > 0$. By considering a path along the ellipse $\{a \cos t + ib \sin t \mid 0 \leq t \leq 2\pi\}$ or otherwise, show that

$$\int_0^{2\pi} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t} = \frac{2\pi}{ab}.$$

(Hint: Recall from lecture that $\int_{\partial D(0,r)} z^{-1} dz = 2\pi i$ and emulate a trick we used in the proof of Cauchy's theorem.)

SOLUTION. We shall choose the following parametrization for the ellipse Γ

$$z : [0, 2\pi] \rightarrow \mathbb{C}, \quad z(t) = a \cos t + ib \sin t.$$

By the trick that we discussed in lecture, where we added a back and forth path (whose integrals cancel exactly) between Γ and a circle $\partial D(0, r)$, we see that

$$\int_{\Gamma} \frac{dz}{z} = \int_{\partial D(0,r)} \frac{dz}{z} = 2\pi i. \tag{2.1}$$

Now

$$\begin{aligned}\int_{\Gamma} \frac{dz}{z} &= \int_0^{2\pi} \frac{-a \sin t + ib \cos t}{a \cos t + ib \sin t} dt \\ &= \int_0^{2\pi} \frac{(b^2 - a^2) \sin t \cos t}{a^2 \cos^2 t + b^2 \sin^2 t} dt + i \int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt.\end{aligned} \tag{2.2}$$

Comparing (2.1) and (2.2), we see that the real part of (2.2) must be 0 and the imaginary part must equal 2π , i.e.

$$\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2\pi.$$

Dividing by ab then gives the required expression.

(c) Evaluate the following integrals

$$\int_{D_1} |z|\bar{z} dz \quad \text{and} \quad \int_{D_2} \bar{z} dz$$

where D_1 and D_2 are the curves in Figure 2.

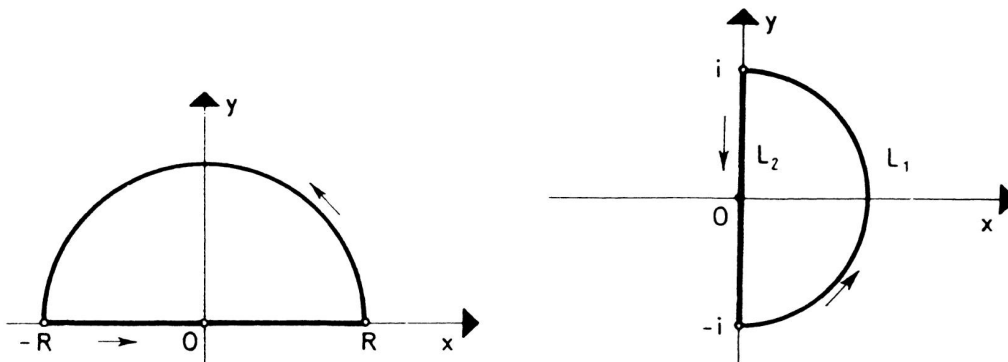


FIGURE 2. *Left:* D_1 is a closed curve in the counter clockwise direction along the semicircle $\{Re^{it} \mid 0 \leq t \leq \pi\}$ and the line segment $\{z \mid -R \leq \operatorname{Re} z \leq R, \operatorname{Im} z = 0\}$. *Right:* D_2 is a closed curve in the counter clockwise direction along the semicircle $\{e^{it} \mid -\pi/2 \leq t \leq \pi/2\}$ and the line segment $\{z \mid -1 \leq \operatorname{Im} z \leq 1, \operatorname{Re} z = 0\}$.

SOLUTION. We have

$$\begin{aligned} \int_{D_1} |z|\bar{z} dz &= \int_{-R}^R |x|x dx + i \int_0^\pi R^3 e^{-it} e^{it} dt \\ &= -\int_{-R}^0 x^2 dx + \int_0^R x^2 dx + i \int_0^\pi R^3 dt \\ &= R^3 \pi i \end{aligned}$$

and

$$\int_{D_2} \bar{z} dz = \int_{-\pi/2}^{\pi/2} e^{-it} i e^{it} dt + \int_1^{-1} (-it) i dt = \pi i.$$

3. Let Γ be a smooth curve.

(a) Suppose $\Gamma = [a, b]$. Prove that for any integrable function $f : [a, b] \rightarrow \mathbb{C}$,

$$\overline{\int_a^b f(t) dt} = \int_a^b \overline{f(t)} dt.$$

What can you deduce about the integrability of \bar{f} if f is integrable?

SOLUTION. Let $f(t) = u(t) + iv(t)$. Then

$$\begin{aligned}\overline{\int_a^b f(t) dt} &= \overline{\int_a^b u(t) dt + i \int_a^b v(t) dt} \\ &= \overline{\int_a^b u(t) dt} - i \overline{\int_a^b v(t) dt} \\ &= \int_a^b [u(t) - iv(t)] dt \\ &= \int_a^b \overline{f(t)} dt.\end{aligned}$$

Hence if f is integrable over $[a, b]$, then so is \overline{f} .

- (b) Let f be a function that is continuous on Γ . Consider $\overline{\Gamma}$, the image of Γ under complex conjugation $z \mapsto \overline{z}$, i.e. Γ reflected about the real axis. Show that the function $z \mapsto \overline{f(\overline{z})}$ is also continuous on $\overline{\Gamma}$ and

$$\overline{\int_{\Gamma} f(z) dz} = \int_{\overline{\Gamma}} \overline{f(\overline{z})} dz.$$

SOLUTION. Note that complex conjugation, $\sigma : \mathbb{C} \rightarrow \mathbb{C}$, $\sigma(z) = \overline{z}$, is a bijective (easy) and continuous function. Continuity follows from the fact that for $\varepsilon > 0$,

$$|\sigma(z) - \sigma(z_0)| = |\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0| < \varepsilon$$

for $|z - z_0| < \delta = \varepsilon$. Since for $\overline{z} \in \overline{\Gamma}$ we have $z \in \Gamma$, and since f is continuous on Γ , the function $z \mapsto \overline{f(\overline{z})}$ is continuous on $\overline{\Gamma}$ being a composition of three continuous functions $\sigma \circ f \circ \sigma$.

For the integral, first note that if we let $w = \overline{z}$, then

$$\int_{\overline{\Gamma}} \overline{f(\overline{z})} dz = \int_{\Gamma} \overline{f(w)} d\overline{w} = \int_{\Gamma} \overline{f(w)} dw. \quad (3.3)$$

Let $\overline{\Gamma}$ be given by $w : [a, b] \rightarrow \mathbb{C}$, $w(t) = x(t) + iy(t)$ and let $f(w(t)) = u(t) + iv(t)$. Then

$$\begin{aligned}\int_{\Gamma} \overline{f(w)} dw &= \int_a^b \overline{f(w(t))w'(t)} dt \\ &= \int_a^b \overline{(u(t) + iv(t))(x(t) + iy'(t))} dt \\ &= \int_a^b (u(t) - iv(t))(x(t) - iy'(t)) dt \\ &= \int_a^b (u(t)x'(t) - v(t)y'(t)) dt - i \int_a^b (v(t)x'(t) + u(t)y'(t)) dt \\ &= \overline{\int_a^b (u(t)x'(t) - v(t)y'(t)) dt + i \int_a^b (v(t)x'(t) + u(t)y'(t)) dt} \\ &= \overline{\int_a^b (u(t) + iv(t))(x(t) + iy'(t)) dt} \\ &= \int_{\Gamma} f(w) dw.\end{aligned} \quad (3.4)$$

Together, (3.3) and (3.4) imply that

$$\int_{\overline{\Gamma}} \overline{f(\overline{z})} dz = \overline{\int_{\Gamma} f(w) dw}.$$

Since w is a dummy variable, we could equally well have written z in place of w on the RHS, taking conjugate of both sides then yields the required expression.

- (c) Suppose Γ is the positively oriented (i.e. going counter clockwise) circle $z : [0, 2\pi] \rightarrow \mathbb{C}$, $z(t) = e^{it}$. Show that

$$\overline{\int_{\Gamma} f(z) dz} = - \int_{\Gamma} \overline{f(z)} \frac{dz}{z^2}.$$

SOLUTION. By the proof of (b), we have

$$\begin{aligned} \overline{\int_{\Gamma} f(z) dz} &= \int_0^{2\pi} \overline{f(e^{it})ie^{it}} dt \\ &= - \int_0^{2\pi} \overline{f(e^{it})}ie^{-it} dt \\ &= - \int_0^{2\pi} \overline{f(e^{it})} \frac{ie^{it}}{e^{2it}} dt \\ &= - \int_{\Gamma} \overline{f(z)} \frac{dz}{z^2}. \end{aligned}$$

4. Let $r > 0$ and Γ be the curve $z : [0, 2\pi] \rightarrow \mathbb{C}$, $z(t) = re^{it}$. Let $f : \overline{D(0, r)} \rightarrow \mathbb{C}$ be continuous.
 (a) For $n \in \mathbb{N}$, let Γ_n be the curve $z_n : [0, 2\pi] \rightarrow \mathbb{C}$, $z_n(t) = (1 - 1/n)re^{it}$. Prove that

$$\int_{\Gamma} f(z) dz = \lim_{n \rightarrow \infty} \int_{\Gamma_n} f(z) dz.$$

(Hint: The result we discussed in class is of the form $\int_{\Gamma} f = \lim_{n \rightarrow \infty} \int_{\Gamma} f_n$. Find a way to use this.)

SOLUTION. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Gamma_n} f(z) dz &= \lim_{n \rightarrow \infty} \int_0^{2\pi} f\left(\left(1 - \frac{1}{n}\right)re^{it}\right) \left(1 - \frac{1}{n}\right)ire^{it} dt \\ &= \int_0^{2\pi} \lim_{n \rightarrow \infty} \left[f\left(\left(1 - \frac{1}{n}\right)re^{it}\right) \left(1 - \frac{1}{n}\right)ire^{it} \right] dt \\ &= \int_0^{2\pi} f(re^{it})ire^{it} dt \\ &= \int_{\Gamma} f(z) dz \end{aligned}$$

where we have used the continuity of integration (i.e. $\lim_{n \rightarrow \infty} \int_{\Gamma} f_n = \int_{\Gamma} \lim_{n \rightarrow \infty} f_n$) that we proved in lecture.

- (b) Show that

$$\lim_{r \rightarrow 0} \int_0^{2\pi} f(re^{it}) dt = 2\pi f(0) \quad \text{and} \quad \lim_{r \rightarrow 0} \int_{\Gamma} \frac{f(z)}{z} dz = 2\pi i f(0).$$

(Hint: Emulate the proof of the aforementioned result.)

SOLUTION. Since the function f is continuous at 0, put $z = re^{it}$, we have that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(re^{it}) - f(0)| < \frac{\varepsilon}{2\pi}$$

for $r < \delta$ and $0 \leq t \leq 2\pi$. Therefore

$$\begin{aligned} \left| \int_0^{2\pi} f(re^{it}) dt - 2\pi f(0) \right| &= \left| \int_0^{2\pi} [f(re^{it}) - f(0)] dt \right| \\ &\leq \int_0^{2\pi} |f(re^{it}) - f(0)| dt \\ &< \frac{\varepsilon}{2\pi} \int_0^{2\pi} dt \\ &= \varepsilon \end{aligned}$$

for $r < \delta$, $0 \leq t \leq 2\pi$. Likewise we have

$$\begin{aligned} \left| \int_{\Gamma} \frac{f(z)}{z} dz - 2\pi i f(0) \right| &= \left| \int_0^{2\pi} \frac{f(re^{it})}{re^{it}} ire^{it} dt - i \int_0^{2\pi} f(0) dt \right| \\ &= \left| i \int_0^{2\pi} [f(re^{it}) - f(0)] dt \right| \\ &\leq \int_0^{2\pi} |f(re^{it}) - f(0)| dt \\ &< \frac{\varepsilon}{2\pi} \int_0^{2\pi} dt \\ &= \varepsilon \end{aligned}$$

for $r < \delta$, $0 \leq t \leq 2\pi$.

5. Let $R > 0$ and $f : D(0, R) \rightarrow \mathbb{C}$ be analytic.

(a) Suppose at least one of the following four conditions is true

- (i) $\operatorname{Re} f'(z) > 0$ for all $z \in D(0, R)$;
- (ii) $\operatorname{Re} f'(z) < 0$ for all $z \in D(0, R)$;
- (iii) $\operatorname{Im} f'(z) > 0$ for all $z \in D(0, R)$;
- (iv) $\operatorname{Im} f'(z) < 0$ for all $z \in D(0, R)$.

Show that f is injective on $D(0, R)$.

SOLUTION. Suppose (i) is satisfied. Let $a, b \in D(0, R)$ be two arbitrary points and $a \neq b$. Let Γ be the line segment connecting a and b , i.e. $z : [0, 1] \rightarrow \mathbb{C}$, $z(t) = a + t(b - a)$. Note that $\Gamma \subseteq D(0, R)$. By Proposition 3.9 in our lectures,

$$f(b) - f(a) = \int_{\Gamma} f'(z) dz = \int_0^1 f'(z(t))(b - a) dt.$$

So

$$\frac{f(b) - f(a)}{b - a} = \int_0^1 f'(z(t)) dt = \int_0^1 \operatorname{Re} f'(z(t)) dt + i \int_0^1 \operatorname{Im} f'(z(t)) dt.$$

Since $\operatorname{Re} f'(z) > 0$ for all $z \in D(0, R)$,

$$\int_0^1 \operatorname{Re} f'(z(t)) dt \neq 0$$

and thus

$$\int_0^1 f'(z(t)) dt \neq 0,$$

which implies that $f(a) \neq f(b)$. For (ii), (iii), (iv), apply the result of (i) to $-f$, $-if$, and if respectively; note that these are all analytic on $D(0, R)$ and satisfy the condition of (i).

(b) Suppose

$$[\operatorname{Re} f'(z)][\operatorname{Im} f'(z)] \neq 0$$

for all $z \in D(0, R)$. Show that f is injective on $D(0, R)$.

SOLUTION. Since $[\operatorname{Re} f'(z)][\operatorname{Im} f'(z)] \neq 0$ for all $z \in D(0, R)$, it must be the case that $\operatorname{Re} f'(z) \neq 0$ and $\operatorname{Im} f'(z) \neq 0$ for all $z \in D(0, R)$. Now suppose there exist $z_1, z_2 \in D(0, R)$ such that $\operatorname{Re} f'(z_1) < 0$ and $\operatorname{Re} f'(z_2) > 0$. Define $\gamma : [0, 1] \rightarrow D(0, R)$ by $\gamma(t) = z_1 + t(z_2 - z_1)$ (this line is contained in $D(0, R)$ since $D(0, R)$ is convex). Observe that γ is clearly differentiable on $[0, 1]$. Also observe that by Theorem **2.9**, since f is analytic on $D(0, R)$, so is f' . From these two observations, conclude that $f' \circ \gamma$ is differentiable on $[0, 1]$, which implies that $f' \circ \gamma$ is continuous on $[0, 1]$. Then, $\operatorname{Re} f' \circ \gamma$ is also continuous on $[0, 1]$.

Then by the Intermediate Value Theorem, since $\operatorname{Re} f' \circ \gamma(0) = \operatorname{Re} f'(\gamma(0)) = \operatorname{Re} f'(z_1) < 0$ and $\operatorname{Re} f' \circ \gamma(1) = \operatorname{Re} f'(\gamma(1)) = \operatorname{Re} f'(z_2) > 0$, there exists some $c \in (0, 1)$ such that $\operatorname{Re} f' \circ \gamma(c) = 0$. But this is a contradiction since for all $z \in D(0, R)$, $\operatorname{Re} f'(z) \neq 0$. Then it must be the case that either $\operatorname{Re} f'(z) > 0$ for all $z \in D(0, R)$ or $\operatorname{Re} f'(z) < 0$ for all $z \in D(0, R)$.

Then by the result of part (a), since either property (i) or (ii) is satisfied, conclude that f is injective on $D(0, R)$.