MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 5

1. Let $a \in \mathbb{R}$ and $z \in \mathbb{C}$.
(a) Evaluate the following integrals

$$\int_0^1 e^{it} \cos at\, dt \quad \text{and} \quad \int_{-1}^1 \frac{dt}{t^2 + i}.$$

(b) Show that if $\operatorname{Re} z > -1$, then the integral $\int_0^1 t^z\, dt$ exists and

$$\left| \int_0^1 t^z\, dt \right| \le \frac{1}{1 + \operatorname{Re} z}.$$

(c) Show that if $a < 1$, then

$$\left| \int_{-1}^1 \frac{\cos it}{t^a}\, dt \right| \le 2 \int_{-1}^1 \frac{dt}{t^a}$$

and thus the (improper) integral $\int_{-1}^1 t^{-a} \cos it\, dt$ converges absolutely.

2. (a) For $k = 1, 2, 3$, evaluate the following integrals

$$\int_{\Gamma_k} \operatorname{Re}(z)\, dz, \quad \int_{\Gamma_k} z^2\, dz, \quad \int_{\Gamma_k} \frac{dz}{z}$$

along the curves from the point $z_0 = 1$ to $z_1 = i$ in the counter clockwise direction as described in Figure 1.

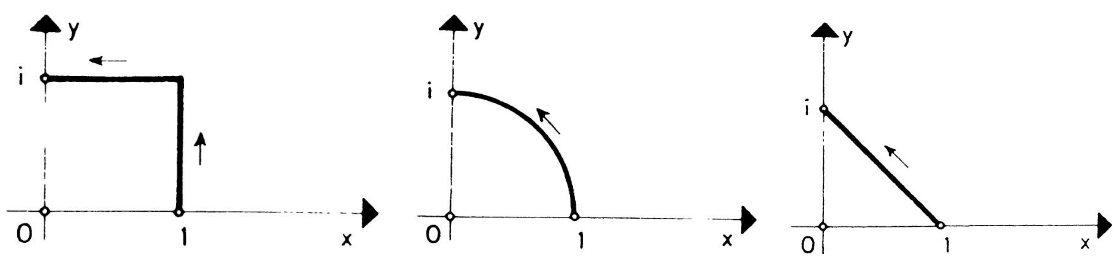

FIGURE 1. *Left*: Γ_1 is along the boundary of the square: $\{x + iy \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. *Center*: Γ_2 is along the boundary of the circle: $\{e^{it} \mid 0 \le t \le \pi/2\}$. *Right*: Γ_3 is along the line segment: $\{(1 - t) + it \mid 0 \le t \le 1\}$.

(b) Let $a, b > 0$. By considering a path along the ellipse $\{a \cos t + ib \sin t \mid 0 \le t \le 2\pi\}$ or otherwise, show that

$$\int_0^{2\pi} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t} = \frac{2\pi}{ab}.$$

(*Hint*: Recall from lecture that $\int_{\partial D(0,r)} z^{-1}\, dz = 2\pi i$ and emulate a trick we used in the proof of Cauchy's theorem.)

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(c) Evaluate the following integrals

$$\int_{D_1} |z|\overline{z}\,dz \quad \text{and} \quad \int_{D_2} \overline{z}\,dz$$

where D_1 and D_2 are the curves in Figure 2.

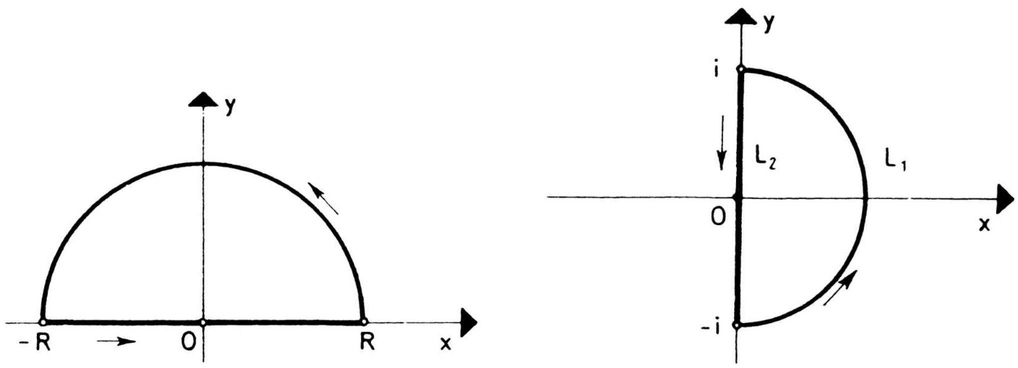

FIGURE 2. *Left*: D_1 is a closed curve in the counter clockwise direction along the semicircle $\{Re^{it} \mid 0 \le t \le \pi\}$ and the line segment $\{z \mid -R \le \operatorname{Re} z \le R,\ \operatorname{Im} z = 0\}$. *Right*: D_2 is a closed curve in the counter clockwise direction along the semicircle $\{e^{it} \mid -\pi/2 \le t \le \pi/2\}$ and the line segment $\{z \mid -1 \le \operatorname{Im} z \le 1,\ \operatorname{Re} z = 0\}$.

3. Let Γ be a smooth curve.

(a) Suppose $\Gamma = [a,b]$. Prove that for any integrable function $f : [a,b] \to \mathbb{C}$,

$$\overline{\int_a^b f(t)\,dt} = \int_a^b \overline{f(t)}\,dt.$$

What can you deduce about the integrability of $\overline{f}$ if f is integrable?

(b) Let f be a function that is continuous on Γ. Consider $\overline{\Gamma}$, the image of Γ under complex conjugation $z \mapsto \overline{z}$, i.e. Γ reflected about the real axis. Show that the function $z \mapsto \overline{f(\overline{z})}$ is also continuous on $\overline{\Gamma}$ and

$$\overline{\int_\Gamma f(z)\,dz} = \int_{\overline{\Gamma}} \overline{f(\overline{z})}\,dz.$$

(c) Suppose Γ is the positiviely oriented (i.e. going counter clockwise) circle $z : [0, 2\pi] \to \mathbb{C}$, $z(t) = e^{it}$. Show that

$$\overline{\int_\Gamma f(z)\,dz} = -\int_\Gamma \overline{f(z)}\,\frac{dz}{z^2}.$$

4. Let $r > 0$ and Γ be the curve $z : [0, 2\pi] \to \mathbb{C}$, $z(t) = re^{it}$. Let $f : \overline{D(0,r)} \to \mathbb{C}$ be continuous.

(a) For $n \in \mathbb{N}$, let Γ_n be the curve $z_n : [0, 2\pi] \to \mathbb{C}$, $z_n(t) = (1 - 1/n)re^{it}$. Prove that

$$\int_\Gamma f(z)\,dz = \lim_{n\to\infty} \int_{\Gamma_n} f(z)\,dz.$$

(*Hint*: The result we discussed in class is of the form $\int_\Gamma f = \lim_{n\to\infty} \int_\Gamma f_n$. Find a way to use this.)

(b) Show that

$$\lim_{r\to 0} \int_0^{2\pi} f(re^{it})\,dt = 2\pi f(0) \quad \text{and} \quad \lim_{r\to 0} \int_\Gamma \frac{f(z)}{z}\,dz = 2\pi i f(0).$$

(*Hint*: Emulate the proof of the aforementioned result.)

5. Let $R > 0$ and $f : D(0, R) \to \mathbb{C}$ be analytic.

(a) Suppose at least one of the following four conditions is true

(i) $\operatorname{Re} f'(z) > 0$ for all $z \in D(0, R)$;

(ii) $\operatorname{Re} f'(z) < 0$ for all $z \in D(0, R)$;

(iii) $\operatorname{Im} f'(z) > 0$ for all $z \in D(0, R)$;

(iv) $\operatorname{Im} f'(z) < 0$ for all $z \in D(0, R)$.

Show that f is injective on $D(0, R)$.

(b) Suppose

$$[\operatorname{Re} f'(z)][\operatorname{Im} f'(z)] \neq 0$$

for all $z \in D(0, R)$. Show that f is injective on $D(0, R)$.