# MATH 185: COMPLEX ANALYSIS
# FALL 2009/10
# PROBLEM SET 4 SOLUTIONS

**1.** Let $\Omega \subseteq \mathbb{C}$ be a region. Let $f = u + iv$ be analytic on $\Omega$.

(a) If $\alpha u + \beta v$ is constant on $\Omega$ for some $\alpha, \beta \in \mathbb{C}^\times$, show that $f$ is constant on $\Omega$.

Solution. Taking partial derivatives of

$$\alpha u + \beta v = \text{constant},$$

we get

$$\begin{cases} \alpha u_x + \beta v_x &= 0, \\ \alpha u_y + \beta v_y &= 0. \end{cases}$$

Substituting the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ gives

$$\begin{cases} \alpha u_x - \beta v_y &= 0, \\ \alpha u_y + \beta u_x &= 0. \end{cases}$$

Taking complex conjugate of the second equation (and noting that $u_x$ and $u_y$ are real valued) and writing the resulting system in matrix form yields

$$\begin{bmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $\alpha, \beta \neq 0$,

$$\det \begin{bmatrix} \alpha & -\beta \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} = |\alpha|^2 + |\beta|^2 \neq 0,$$

and thus the system has a unique solution $u_x = 0$, $u_y = 0$. Applying the Cauchy-Riemann equations again, we get $v_x = 0$, $v_y = 0$. Since $\Omega$ is a region, we must have that $f$ is constant-valued on $\Omega$.

(b) If $u^2 + v^2$ is constant on $\Omega$, show that $f$ is constant on $\Omega$.

Solution. Let the constant be $c$. Then

$$u^2 + v^2 = c.$$

If $c = 0$, then $u = 0$ and $v = 0$ and so $f = 0$, thus constant. We will assume that $c \neq 0$. Taking partial derivatives, we get

$$\begin{cases} uu_x + vv_x &= 0, \\ uu_y + vv_y &= 0. \end{cases} \tag{1.1}$$

Substituting the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ into the second equation of (5.12) gives

$$\begin{cases} uu_x + vv_x &= 0, \\ -uv_x + vu_x &= 0. \end{cases} \tag{1.2}$$

Now we multiply the first equation in (1.2) by $u$ on both sides and use the second equation in (1.2) to get

$$0 = u(uu_x + vv_x) = u^2 u_x + v(uv_x) = u^2 u_x + v(vu_x) = (u^2 + v^2)u_x = cu_x.$$

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Similarly, we could have instead substitute the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ into the first equation of (5.12) to get

$$\begin{cases} uv_y - vu_y &= 0, \\ uu_y + vv_y &= 0. \end{cases} \tag{1.3}$$

This time, multiplying the second equation in (4.10) by $v$ on both sides and use the first equation in (4.10) yields

$$0 = v(uu_y + vv_y) = u(vu_y) + v^2 v_y = u(uv_y) + v^2 v_y = (u^2 + v^2)v_y = cv_y.$$

Since $c \neq 0$, we get

$$u_x = 0 \quad \text{and} \quad v_y = 0.$$

Applying the Cauchy-Riemann equations again, we get $v_x = 0$, $u_y = 0$. Since $\Omega$ is a region, we must have that $f$ is constant-valued on $\Omega$.

(c) If $u = v^2$, show that $f$ is constant on $\Omega$.

SOLUTION. Applying the Cauchy-Riemann equations and chain rule, we get

$$2vv_x = v_y, \quad 2vv_y = -v_x.$$

Eliminating $v_x$ from the equations, we get

$$-4v^2 v_y = v_y.$$

Suppose $v_y \not\equiv 0$. Then there must exist a point $z_0 = x_0 + iy_0 \in \Omega$ such that $v_y(x_0, y_0) \neq 0$. So $v(x_0, y_0)^2 = -1/4$ — which is impossible since $v$ is real-valued. Hence we must have $v_y \equiv 0$ and in which case, we apply the Cauchy-Rieman equations again to get

$$v_x \equiv v_y \equiv u_x \equiv u_y \equiv 0.$$

Hence $f$ is constant.

(d) If $u = \varphi \circ v$ for some differentiable real function $\varphi : \mathbb{R} \to \mathbb{R}$, show that $f$ is constant on $\Omega$.

SOLUTION. Note that this is a generalization of (c). Applying the Cauchy-Riemann equations and chain rule, we get

$$\varphi(v)v_x = v_y, \quad \varphi(v)v_y = -v_x.$$

Eliminating $v_x$ from the equations, we get

$$-[\varphi(v)]^2 v_y = v_y.$$

Suppose $v_y \not\equiv 0$. Then there must exist a point $z_0 = x_0 + iy_0 \in \Omega$ such that $v_y(x_0, y_0) \neq 0$. So $[\varphi(v(x_0, y_0))]^2 = -1$ — which is impossible since $\varphi \circ v$ is real-valued. Hence we must have $v_y \equiv 0$ and in which case, we apply the Cauchy-Rieman equations again to get

$$v_x \equiv v_y \equiv u_x \equiv u_y \equiv 0.$$

Hence $f$ is constant.

(e) Determine all $f$ for which $g = u^2 + iv^2$ (ie. $g(z) := [u(x, y)]^2 + i[v(x, y)]^2$) is also analytic on $\Omega$.

SOLUTION. Applying the Cauchy-Riemann equations to $g$ yields $2uu_x = 2vv_y$, $2uu_y = -2vv_x$, ie.

$$\begin{cases} uu_x - vv_y &= 0, \\ uu_y + vv_x &= 0. \end{cases}$$

Adding the two equations gives

$$uu_x + vv_x - vv_y + uu_y = 0$$

Now substitute the Cauchy-Riemann equations for $f$, $u_x = v_y$, $u_y = -v_x$, to get an equation free of partials in $y$,

$$uu_x + vv_x - vu_x - uv_x = 0$$

and observe that

$$uu_x + vv_x - vu_x - uv_x = (u-v)(u_x - v_x) = \frac{1}{2}\frac{\partial}{\partial x}(u-v)^2.$$

Subtracting the two equations gives

$$uu_y + vv_y + vv_x - uu_x = 0.$$

Now substitute the Cauchy-Riemann equations for $f$, $u_x = v_y$, $u_y = -v_x$, to get an equation free of partials in $x$,

$$uu_y + vv_y - vu_y - uv_y = 0$$

and observe that

$$uu_y + vv_y - vu_y - uv_y = (u-v)(u_y - v_y) = \frac{1}{2}\frac{\partial}{\partial y}(u-v)^2.$$

So $(u-v)^2$ must be constant on $\Omega$ and so

$$u - v = \text{constant}$$

on $\Omega$. Applying (a) with $\alpha = 1$, $\beta = -1$ then implies that $f$ must be constant on $\Omega$.

**2.** Derive the following power series expansions and show that they must converge uniformly and absolutely in the respective given sets.

(a) For all $z \in \mathbb{C}$,

$$e^z = e + e\sum_{n=1}^{\infty} \frac{1}{n!}(z-1)^n.$$

SOLUTION. By the Taylor expansion of exp,

$$e^{z-1} = \sum_{n=0}^{\infty} \frac{1}{n!}(z-1)^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}(z-1)^n.$$

Since $e^{z-1} = e^z e^{-1}$, multiplying both sides by $e$ gives the required expansion.

(b) For all $z \in D(1,1)$,

$$\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n.$$

SOLUTION. For $|z-1| < 1$, we may use the formula for geometric series

$$\frac{1}{z} = \frac{1}{1+(z-1)} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n.$$

(c) For all $z \in D(-1,1)$,

$$\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n.$$

SOLUTION. For $|z+1| < 1$, we have

$$-\frac{1}{z} = \frac{1}{1-(z+1)} = 1 + \sum_{n=1}^{\infty}(z+1)^n.$$

Since the convergence of the power series is uniform in $D(-1,1)$, we may differentiate the power series on the RHS term-by-term to get

$$\frac{d}{dz}\left(-\frac{1}{z}\right) = \sum_{n=1}^{\infty} \frac{d}{dz}(z+1)^n$$

and thus

$$\frac{1}{z^2} = \sum_{n=1}^{\infty} n(z+1)^{n-1} = 1 + \sum_{n=0}^{\infty}(n+1)(z+1)^n.$$

**3.** For each of the following functions, find a power series expansion about 0 and state its radius of convergence.

$$e(z) = \exp\left(\frac{1}{1-z}\right), \quad f(z) = \sin\left(\frac{1}{1-z}\right), \quad g(z) = \frac{1}{1-z-z^2}, \quad h(z) = \sum_{n=0}^{\infty} \frac{z^n}{1-z^n}.$$

[*Hints*: The solutions for $e$ and $f$ require that you interchange $\sum_{m=0}^{\infty}$ and $\sum_{n=0}^{\infty}$ — you may assume that you could do this; for more information, google *Weierstrass double series theorem.* The solution for $g$ should involve the golden ratio $(1+\sqrt{5})/2$. The solution for $h$ should involve $\tau(n)$ = number of divisors of $n$.]

(a) $e(z) = \exp(1-z)^{-1}$.

SOLUTION. By the Taylor expansion of $\exp z$ and the binomial expansion of $(1-z)^{\alpha}$, we get

$$\begin{aligned}
\exp\left(\frac{1}{1-z}\right) &= \sum_{n=0}^{\infty} \frac{1}{n!}(1-z)^{-n} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left[\sum_{m=0}^{\infty}(-1)^m \binom{-n}{m} z^m\right] \\
&= \sum_{m=0}^{\infty}\left[\sum_{n=0}^{\infty} \frac{(-1)^m}{n!}\binom{-n}{m}\right] z^m
\end{aligned}$$

as formal power series. Now the required radius of convergence is the distance from 0 to the nearest singular point of the function, which in this case is the point $z = 1$, and so the radius of convergence is 1.

An alternative solution would be to observe that

$$\frac{1}{1-z} = 1 + \frac{z}{1-z},$$

and so

$$e^{1/(1-z)} = e \cdot e^{z/(1-z)}.$$

Let $w = z/(1-z)$. For $|w| < \infty$, we have

$$e \cdot e^w = e \sum_{k=0}^{\infty} \frac{w^k}{k!}. \tag{3.4}$$

By the binomial expansion of $(1-z)^{-k}$,

$$\begin{aligned}
w^k &= z^k(1-z)^{-k} \\
&= z^k\left[\sum_{n=0}^{\infty}(-1)^n \binom{-k}{n} z^n\right] \\
&= z^k\left[1 + \frac{k}{1}z + \frac{k(k+1)}{2!}z^2 + \cdots + \frac{k(k+1)\cdots(k+n-1)}{n!}z^n + \cdots\right].
\end{aligned}$$

Since

$$\frac{k(k+1)\cdots(k+n-1)}{n!} = \frac{(k+n-1)(k+n-2)\cdots(n+1)}{(k-1)!} = \binom{n+k-1}{k-1},$$

we obtain

$$w^k = (z + z^2 + z^3 + \cdots)^k = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} z^{n+k}.$$

Now substitute this into (3.4) to get

$$\begin{aligned} f(z) = e\Bigg[1 &+ \frac{1}{1!}\sum_{n=0}^{\infty} z^{n+1} + \frac{1}{2!}\sum_{n=0}^{\infty}(n+1)z^{n+1} + \frac{1}{3!}\sum_{n=0}^{\infty}\frac{(n+2)(n+3)}{2!}z^{n+3} + \ldots \\ &+ \frac{1}{k!}\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}z^{n+k} + \cdots\Bigg] \\ = e\Bigg[1 &+ z + \left(\frac{1}{1!} + \frac{1}{2!}\right)z^2 + \left(\frac{1}{1!} + 2\frac{1}{2!} + \frac{1}{3!}\right)z^3 + \ldots \\ &+ \left(\frac{1}{1!} + \binom{n-1}{1}\frac{1}{2!} + \binom{n-1}{2}\frac{1}{3!} + \cdots + \binom{n-1}{n-2}\frac{1}{(n-1)!} + \frac{1}{n!}\right)z^n + \ldots\Bigg]. \end{aligned}$$

Of course the two answers are really the same — has to be since power series expansions are unique.

(b) $f(z) = \sin(1-z)^{-1}$.

SOLUTION. By the Taylor expansion of $\sin z$ and the binomial expansion of $(1-z)^{\alpha}$, we get

$$\begin{aligned} \sin\left(\frac{1}{1-z}\right) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(1-z)^{-(2n+1)} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left[\sum_{m=0}^{\infty}(-1)^m\binom{-2n-1}{m}z^m\right] \\ &= \sum_{m=0}^{\infty}\left[\sum_{n=0}^{\infty}\frac{(-1)^{m+n}}{(2n+1)!}\binom{-2n-1}{m}\right]z^m \end{aligned}$$

as formal power series. Now the required radius of convergence is the distance from 0 to the nearest singular point of the function, which in this case is the point $z = 1$, and so the radius of convergence is 1.

(c) $g(z) = (1 - z - z^2)^{-1}$.

SOLUTION. Since

$$\frac{1}{1-z-z^2} = \frac{1}{\left[1 - z\left(\frac{1+\sqrt{5}}{2}\right)\right]\left[1 - z\left(\frac{1-\sqrt{5}}{2}\right)\right]} = \frac{1}{(1-\alpha z)(1-\beta z)}$$

where $\alpha := (1+\sqrt{5})/2$ and $\beta := (1-\sqrt{5})/2$, we get

$$\frac{1}{1-z-z^2} = \left(\sum_{n=0}^{\infty}\alpha^n z^n\right)\left(\sum_{n=0}^{\infty}\beta^n z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\alpha^k\beta^{n-k}\right)z^n.$$

Note that

$$\sum_{k=0}^{n}\alpha^k\beta^{n-k} = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

and so the required power series expansion is[1]

$$\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]z^n$$

[1] The coefficients are really the Fibonacci numbers, i.e. $a_n = a_{n-1} + a_{n-2}$.

The radius of convergence is

$$\frac{2}{1+\sqrt{5}}.$$

(d) $h(z) = \sum_{n=1}^{\infty} z^n(1-z^n)^{-1}$.
Solution. The functions

$$f_n(z) = \frac{z^n}{1-z^n}, \quad n \in \mathbb{N},$$

are analytic functions in $D(0,1)$. For any $0 < r < 1$ and $z \in \overline{D(0,r)}$, we have

$$\left|\frac{z^n}{1-z^n}\right| \leq \frac{r^n}{1-r^n} \leq \frac{r^n}{1-r}$$

and since

$$\sum_{n=1}^{\infty} \frac{r^n}{1-r}$$

is convergent, the series

$$\sum_{n=1}^{\infty} f_n(z)$$

converges uniformly on every closed disc $\overline{D(0,r)}$ by Weierstrass $M$-test and so $f$ is an analytic function on $D(0,1)$.
Now observe that by expanding the denominator as a geometric series,

$$f_n(z) = \frac{z^n}{1-z^n} = z^n + z^{2n} + z^{3n} + \cdots = \sum_{k=1}^{\infty} z^{kn}$$

for each $n \in \mathbb{N}$. We see that the coefficient of $z^m$ in the power series expansion of $f_n(z)$ is 0 if $n$ is not a divisor of $m$ and 1 if $n$ is a divisor of $m$. Let the power series expansion of $f$ be given by

$$f(z) = \sum_{m=0}^{\infty} a_m z^m.$$

It is clear that $a_0 = 0$ and

$$\sum_{m=1}^{\infty} a_m z^m = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} z^{kn}.$$

By the preceding reasoning $a_m$ must be the sum of a number of 1's equal to the number of positive divisors of the number $m$, i.e. the arithmetic function $\tau(m)$ for those of you have taken/are taking Math **115**. Specifically

$$\tau(1) = 1, \quad \tau(2) = 2, \quad \tau(3) = 2, \quad \tau(4) = 3, \ldots.$$

So we have

$$f(z) = \sum_{n=1}^{\infty} \tau(n) z^n. \tag{3.5}$$

Since $f$ is an analytic function $D(0,1)$, the radius of convergence of the power series in (3.5) about 0 must be at least 1 by a theorem from the lectures. However for $z = 1$,

$$\sum_{n=1}^{\infty} \tau(n)$$

is divergent since

$$\lim_{n\to\infty} \tau(n) \neq 0.$$

Hence the radius of convergence of the power series in (3.5) is at most 1.

**4.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, and $h(z) = \sum_{n=0}^{\infty} c_n z^n$ be power series with positive radii of convergence.

(a) Is it possible for $f$ to satisfy

$$f\left(\frac{1}{n^3}\right) = \frac{1}{n^6} \quad \text{and} \quad f\left(\frac{1}{n^2}\right) = \frac{1}{n^6} \tag{4.6}$$

for all $n \in \mathbb{N}$? If so, what is $f$?

Solution. Since $f$ is continuous at 0, we have that

$$f(0) = \lim_{n\to\infty} f\left(\frac{1}{n^3}\right) = \lim_{n\to\infty} \frac{1}{n^6} = 0.$$

The first equality says that the two power series[2]

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \tilde{f}(z) = z^2$$

agree on the set

$$\tilde{S} = \{n^{-3} \mid n \in \mathbb{Z}^\times\} \cup \{0\} \subseteq \mathbb{C}.$$

Since $\tilde{S}$ has an accumulation point 0, the Uniqueness Theorem for power series (ie. Corollary **2.13**) implies that $f = \tilde{f}$, ie.

$$f(z) = z^2. \tag{4.7}$$

However, we also have that

$$f(0) = \lim_{n\to\infty} f\left(\frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{1}{n^6} = 0.$$

The second equality says that the two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \hat{f}(z) = z^3$$

agree on the set

$$\hat{S} = \{n^{-2} \mid n \in \mathbb{Z}^\times\} \cup \{0\} \subseteq \mathbb{C}.$$

Since $\hat{S}$ has an accumulation point 0, the Uniqueness Theorem for power series (ie. Corollary **2.13**) implies that $f = \hat{f}$, ie.

$$f(z) = z^3. \tag{4.8}$$

Since (4.7) and (4.8) cannot be simultaneously satisfied. It is not possible for $f$ to satisfy (4.6).

(b) Is it possible for $g$ to satisfy

$$g\left(\frac{i^n}{n}\right) = \frac{1}{n^4}$$

for all $n \in \mathbb{N}$? If so, what is $g$?

Solution. Let $g(z) = z^4$. Then

$$g\left(\frac{i^n}{n}\right) = \frac{i^{4n}}{n^4} = \frac{1}{n^4}$$

for all $n \in \mathbb{N}$. Since

$$S = \{i^n/n \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{C}$$

has an accumulation point 0, the Uniqueness Theorem for power series implies this is the only possibility.

[2] Note that $\tilde{f}$ is a power series with all its coefficients 0 except the 2nd, which is 1. Similar observations apply below and to (b) and (c).

(c) Is it possible for $h$ to satisfy

$$h\left(\frac{1}{n}\right) = \frac{i^n}{n} \tag{4.9}$$

for all $n \in \mathbb{N}$? If so, what is $h$?

SOLUTION. Since $h$ is continuous at 0, we have that

$$h(0) = \lim_{n\to\infty} h\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{i^n}{n} = 0.$$

The condition for $n \equiv 0 \bmod 4$, ie.

$$h\left(\frac{1}{4n}\right) = \frac{1}{4n}$$

says that the two power series

$$h(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad \tilde{h}(z) = z$$

agree on the set

$$S_0 = \{(4n)^{-1} \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{C}.$$

Since $S_0$ has an accumulation point 0, the Uniqueness Theorem for power series implies that $h = \tilde{h}$, ie.

$$h(z) = z \quad \text{for all } z \in D(0, r) \tag{4.10}$$

where $r > 0$ is the radius of convergence of $h$.

However, the condition for $n \equiv 2 \bmod 4$, ie.

$$h\left(\frac{1}{4n+2}\right) = -\frac{1}{4n+2}$$

says that the two power series

$$h(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad \hat{h}(z) = -z$$

agree on the set

$$S_2 = \{(4n+2)^{-1} \mid n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{C}.$$

Since $S_2$ has an accumulation point 0, the Uniqueness Theorem for power series implies that $h = \hat{h}$, ie.

$$h(z) = -z \quad \text{for all } z \in D(0, r). \tag{4.11}$$

Since (4.10) and (4.11) cannot be simultaneously satisfied. It is not possible for $h$ to satisfy (4.9).

**5.** Let $m \in \mathbb{N} \cup \{0\}$. Show that the complex differential equation

$$\begin{cases} (1 - z^2) f''(z) - z f'(z) + m^2 f(z) = 0, \\ f(0) = 1, \quad f'(0) = im, \end{cases}$$

has a unique solution in $D(0, 1)$. [*Hint*: assume first that the solution $f$ has a power series representation about 0, plug it into the differential equation and find what the coefficients are, then show that the radius of convergence is 1.]

SOLUTION. As suggested in the hint, the trick is to solve the differential equation with formal power series and worry about convergence later. So suppose

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then as formal power series,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}, \quad f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}.$$

So $(1-z^2)f''(z) - zf'(z) + m^2 f(z) = 0$ yields (after relabelling the indices of $f''$ and $f'$) the following recursion formula:

$$(n+2)(n+1)a_{n+1} - n(n-1)a_n - na_n + m^2 a_n = 0$$

for all $n = 0, 1, 2, \ldots$. Hence

$$a_{n+2} = \frac{n^2 - m^2}{(n+1)(n+2)} a_n \tag{5.12}$$

for all $n = 0, 1, 2, \ldots$. Since $a_0 = f(0) = 1$, $a_1 = f'(0) = im$, we may use (5.12) to determine all other coefficients

$$a_{2n} = (-1)^n \frac{m^2(m^2-4)\cdots(m^2-4n^2)}{(2n)!}, \quad a_{2n+1} = (-1)^n \frac{im(m^2-1)(m^2-9)\cdots(m^2-(2n-1)^2)}{(2n+1)!}.$$

Note that by (5.12),

$$\limsup_{n\to\infty} \sqrt[2n]{|a_{2n}|} = \lim_{n\to\infty} \frac{a_{2(n+1)}}{a_{2n}} = \lim_{n\to\infty} \frac{(2n)^2 - m^2}{(2n+1)(2n+2)} = 1,$$

$$\limsup_{n\to\infty} \sqrt[2n+1]{|a_{2n+1}|} = \lim_{n\to\infty} \frac{a_{2(n+1)+1}}{a_{2n+1}} = \lim_{n\to\infty} \frac{(2n+1)^2 - m^2}{(2n+2)(2n+3)} = 1,$$

and so the two series

$$\sum_{n=0}^{\infty} a_{2n} z^{2n} \quad \text{and} \quad \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1}$$

both have radii of convergence 1. Hence by Problem **6**(a) in Homework **2**, the sum of these two power series,

$$\sum_{n=0}^{\infty} a_n z^n$$

must also have radius of convergence 1. By a result in the lecture,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

defines an analytic function in $D(0,1)$. The required uniqueness follows from the uniqueness of power series representation.