MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 4

1. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $f = u + iv$ be analytic on Ω.
(a) If $\alpha u + \beta v$ is constant on Ω for some $\alpha, \beta \in \mathbb{C}^\times$, show that f is constant on Ω.
(b) If $u^2 + v^2$ is constant on Ω, show that f is constant on Ω.
(c) If $u = v^2$, show that f is constant on Ω.
(d) If $u = \varphi \circ v$ for some differentiable real function $\varphi : \mathbb{R} \to \mathbb{R}$, show that f is constant on Ω.
(e) Determine all f for which $g = u^2 + iv^2$ (ie. $g(z) := [u(x, y)]^2 + i[v(x, y)]^2$) is also analytic on Ω.

2. Derive the following power series expansions and show that they must converge uniformly and absolutely in the respective given sets.
(a) For all $z \in \mathbb{C}$,
$$e^z = e + e\sum_{n=1}^{\infty} \frac{1}{n!}(z-1)^n.$$
(b) For all $z \in D(1, 1)$,
$$\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n.$$
(c) For all $z \in D(-1, 1)$,
$$\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n.$$

3. For each of the following functions, find a power series expansion about 0 and state its radius of convergence.
$$e(z) = \exp\left(\frac{1}{1-z}\right), \quad f(z) = \sin\left(\frac{1}{1-z}\right), \quad g(z) = \frac{1}{1-z-z^2}, \quad h(z) = \sum_{n=0}^{\infty} \frac{z^n}{1-z^n}.$$
[*Hints*: The solutions for e and f require that you interchange $\sum_{m=0}^{\infty}$ and $\sum_{n=0}^{\infty}$ — you may assume that you could do this; for more information, google *Weierstrass double series theorem.* The solution for g should involve the golden ratio $(1 + \sqrt{5})/2$. The solution for h should involve $\tau(n)$ = number of divisors of n.]

4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, and $h(z) = \sum_{n=0}^{\infty} c_n z^n$ be power series with positive radii of convergence.
(a) Is it possible for f to satisfy
$$f\left(\frac{1}{n^3}\right) = \frac{1}{n^6} \quad \text{and} \quad f\left(\frac{1}{n^2}\right) = \frac{1}{n^6}$$
for all $n \in \mathbb{N}$? If so, what is f?
(b) Is it possible for g to satisfy
$$g\left(\frac{i^n}{n}\right) = \frac{1}{n^4}$$
for all $n \in \mathbb{N}$? If so, what is g?

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(c) Is it possible for h to satisfy

$$h\left(\frac{1}{n}\right) = \frac{i^n}{n}$$

for all $n \in \mathbb{N}$? If so, what is h?

5. Let $m \in \mathbb{N} \cup \{0\}$. Show that the complex differential equation

$$\begin{cases} (1-z^2)f''(z) - zf'(z) + m^2 f(z) = 0, \\ f(0) = 1, \quad f'(0) = im, \end{cases}$$

has a unique solution in $D(0,1)$. [*Hint*: assume first that the solution f has a power series representation about 0, plug it into the differential equation and find what the coefficients are, then show that the radius of convergence is 1.]