MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 3 SOLUTIONS

For a real-valued function of two real variables, $u : \Omega_{\mathbb{R}} \to \mathbb{R}$, we say that u is *twice continuously differentiable* if all second-order partial derivatives $u_{xx}, u_{yy}, u_{xy}, u_{yx}$ exist and are continuous on $\Omega_{\mathbb{R}}$. The set of all twice continuously differentiable functions on $\Omega_{\mathbb{R}}$ is denoted $C^2(\Omega_{\mathbb{R}})$.

1. We mentioned Tauberian theorems in class. Here is an example of an easy one (easy relative to other Tauberian theorems). Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence 1 and suppose

$$\lim_{n\to\infty} na_n = 0.$$

(a) Show that

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} n|a_n|}{m} = 0.$$

(*Hint:* Problem **4**(a), Problem Set **3**, Math **104**, Spring 2009.)

Solution. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} na_n = 0$, there exists $N_1 \in \mathbb{N}$ such that

$$n|a_n| < \varepsilon/2$$

whenever $n > N_1$. Now by the Archimedean property, there exists $N_2 \in \mathbb{N}$ such that

$$\frac{|a_1| + 2|a_2| + \cdots + N_1|a_{N_1}|}{N_2} < \frac{\varepsilon}{2}.$$

Hence

$$\frac{|a_1| + 2|a_2| + \cdots + N_1|a_{N_1}|}{m} < \frac{\varepsilon}{2}$$

whenever $m > N_2$. Now for $m > \max\{N_1, N_2\}$,

$$\begin{aligned}\frac{\sum_{n=0}^{m} n|a_n|}{m} &= \frac{\sum_{n=0}^{N_1} n|a_n|}{m} + \frac{\sum_{n=N_1+1}^{m} n|a_n|}{m} \\ &< \frac{\varepsilon}{2} + \frac{m - N_1}{m}\frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Hence

$$\lim_{m\to\infty} \frac{\sum_{n=0}^{m} n|a_n|}{m} = 0.$$

(b) Define a function f by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all } |z| < 1.$$

Let x be a real variable and suppose the following left limit exists

$$\lim_{x\to 1^-} f(x) = A.$$

Show that the series $\sum_{n=0}^{\infty} a_n$ converges to A.

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Solution. Let $x \in \mathbb{R}$ and $0 \le x < 1$. We have

$$\begin{aligned}\left|\sum_{n=0}^{m} a_n - f(x)\right| &= \left|\sum_{n=0}^{m} a_n(1-x^n) - \sum_{n=m+1}^{\infty} a_n x^n\right| \\ &\le (1-x)\sum_{n=0}^{m} |a_n|(1+x+\cdots+x^{n-1}) + \sum_{n=m+1}^{\infty} |a_n| x^n \\ &< m(1-x)\frac{\sum_{n=0}^{m} n|a_n|}{m} + \sum_{n=m+1}^{\infty} n|a_n|\frac{x^n}{n}.\end{aligned}$$

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} n a_n = 0$, there exists $M_1 \in \mathbb{N}$ such that

$$m|a_m| < \varepsilon/2$$

whenever $m > M_1$. By (a), there exists $M_2 \in \mathbb{N}$ such that

$$\frac{\sum_{n=0}^{m} n|a_n|}{m} < \frac{\varepsilon}{2}$$

for $m > M_2$. Hence when $m > \max\{M_1, M_2\}$,

$$\begin{aligned}\left|\sum_{n=0}^{m} a_n - f(x)\right| &< m(1-x)\frac{\varepsilon}{2} + \frac{\varepsilon}{2m}\sum_{n=m+1}^{\infty} x^n \\ &= \frac{\varepsilon}{2}\left[m(1-x) + \frac{1}{m}\left(\frac{x^{m+1}}{1-x}\right)\right] \\ &< \frac{\varepsilon}{2}\left[m(1-x) + \frac{1}{m(1-x)}\right].\end{aligned}$$

Then for $1 - x = 1/m$ and $m > \max\{M_1, M_2\}$, we get

$$\left|\sum_{n=0}^{m} a_n - f\left(1 - \frac{1}{m}\right)\right| < \varepsilon,$$

which implies that

$$\lim_{m\to\infty} \sum_{n=0}^{m} a_n = \lim_{m\to\infty} f\left(1 - \frac{1}{m}\right).$$

Since $\lim_{x\to 1^-} f(x) = A$ exists, we must have

$$\lim_{m\to\infty} f\left(1 - \frac{1}{m}\right) = A$$

and so we get

$$\sum_{n=0}^{\infty} a_n = A.$$

2. Recall that $\mathbb{C}$ is both a real vector space of dimension 2 and a complex vector space of dimension 1. A function $\varphi : \mathbb{C} \to \mathbb{C}$ is called $\mathbb{R}$-*linear* if φ is a linear transformation of real vector spaces, ie.

$$\varphi(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1\varphi(z_1) + \lambda_2\varphi(z_2) \qquad \text{for all } \lambda_1, \lambda_2 \in \mathbb{R} \text{ and } z_1, z_2 \in \mathbb{C}. \tag{2.1}$$

It is called $\mathbb{C}$-*linear* if φ is a linear transformation of complex vector spaces, ie.

$$\varphi(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1\varphi(z_1) + \lambda_2\varphi(z_2) \qquad \text{for all } \lambda_1, \lambda_2 \in \mathbb{C} \text{ and } z_1, z_2 \in \mathbb{C}. \tag{2.2}$$

(a) Prove that if φ is $\mathbb{C}$-linear, then it is $\mathbb{R}$-linear. Give an example to show that the converse is false.

Solution. This is obvious since $\mathbb{R} \subset \mathbb{C}$ and so (2.1) is a special case of (2.2). For a counterexample to the converse, consider the complex conjugate function, $\varphi : \mathbb{C} \to \mathbb{C}$,

$\varphi(z) = \overline{z}$. For $\lambda_1, \lambda_2 \in \mathbb{R}$,

$$\begin{aligned}\varphi(\lambda_1 z_1 + \lambda_2 z_2) &= \overline{\lambda_1 z_1 + \lambda_2 z_2} \\ &= \bar{\lambda}_1 \bar{z}_1 + \bar{\lambda}_2 \bar{z}_2 \\ &= \lambda_1 \bar{z}_1 + \lambda_2 \bar{z}_2 \\ &= \lambda_1 \varphi(z_1) + \lambda_2 \varphi(z_2)\end{aligned}$$

and so φ is $\mathbb{R}$-linear. However, for $\lambda_1 = i$, $z_1 = 1$, $\lambda_2 = z_2 = 0$, we see that

$$\varphi(i) = -i \neq i = i\varphi(1)$$

and so it is not $\mathbb{C}$-linear.

(b) Let $\varphi : \mathbb{C} \to \mathbb{C}$. Prove that the following statements are equivalent.

(i) φ is $\mathbb{R}$-linear.

(ii) φ satisfies

$$\varphi(z) = \varphi(1)x + \varphi(i)y \tag{2.3}$$

for all $z = x + iy \in \mathbb{C}$.

(iii) φ satisfies

$$\varphi(z) = \left[\frac{\varphi(1) - i\varphi(i)}{2}\right] z + \left[\frac{\varphi(1) + i\varphi(i)}{2}\right] \bar{z} \tag{2.4}$$

for all $z = x + iy \in \mathbb{C}$.

(iv) φ is given by

$$\varphi(x + iy) = (ax + by) + i(cx + dy) \tag{2.5}$$

for some $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$.

Solution. (i) ⇒ (ii): If we let $\lambda_1 = x$, $z_1 = 1$, $\lambda_2 = y$, $z_2 = i$ in (2.1), we get

$$\varphi(z) = \varphi(x + yi) = x\varphi(1) + y\varphi(i) = \varphi(1)x + \varphi(i)y$$

as required.

(ii) ⇒ (iii): Note that if $z = x + iy$, then $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$. Hence

$$\begin{aligned}\varphi(z) &= \varphi(1)x + \varphi(i)y \\ &= \varphi(1)\left[\frac{z + \bar{z}}{2}\right] + \varphi(i)\left[\frac{z - \bar{z}}{2i}\right] \\ &= \left[\frac{\varphi(1) - i\varphi(i)}{2}\right] z + \left[\frac{\varphi(1) + i\varphi(i)}{2}\right] \bar{z}.\end{aligned}$$

(iii) ⇒ (iv): Let $a = \operatorname{Re}\varphi(1)$, $c = \operatorname{Im}\varphi(1)$, $b = \operatorname{Re}\varphi(i)$, $d = \operatorname{Im}\varphi(i)$. Then

$$\begin{aligned}\varphi(x + iy) &= \left[\frac{\varphi(1) - i\varphi(i)}{2}\right](x + iy) + \left[\frac{\varphi(1) + i\varphi(i)}{2}\right](x - iy) \\ &= \varphi(1)x + \varphi(i)y \\ &= (a + ic)x + (b + id)y \\ &= (ax + by) + i(cx + dy).\end{aligned}$$

(iv) ⇒ (i): With respect to the standard basis $\mathcal{B} = \{1, i\}$ of $\mathbb{C}$ as a real vector space, (2.5) implies that φ has the matrix representation

$$[\varphi]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and is thus a $\mathbb{R}$-linear function.

(c) Let $\varphi : \mathbb{C} \to \mathbb{C}$. Prove that the following statements are equivalent.

(i) φ is $\mathbb{C}$-linear.

(ii) φ is $\mathbb{R}$-linear and $\varphi(i) = i\varphi(1)$.

(iii) φ satisfies

$$\varphi(z) = \varphi(1)z \tag{2.6}$$

for all $z \in \mathbb{C}$.

(iv) φ is given by

$$\varphi(x+iy) = (ax - cy) + i(cx + ay) \tag{2.7}$$

for some $\left[\begin{smallmatrix} a & c \\ -c & a \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$.

Solution. (i) $\Rightarrow$ (ii): That φ is $\mathbb{R}$-linear follows from (a). Set $\lambda_1 = i$, $z_1 = 1$, $\lambda_2 = z_2 = 0$ in (2.2) to get

$$\varphi(i) = i\varphi(1). \tag{2.8}$$

(ii) $\Rightarrow$ (iii): Using (2.3) and (2.8), we get

$$\varphi(z) = \varphi(x+iy) = x\varphi(1) + y\varphi(i) = x\varphi(1) + yi\varphi(1) = (x+iy)\varphi(1) = z\varphi(1).$$

(iii) $\Rightarrow$ (iv): Let $a = \operatorname{Re}\varphi(1)$, $c = \operatorname{Im}\varphi(1)$ and (2.6) becomes

$$\varphi(x+iy) = \varphi(1)(x+iy) = (a+ic)(x+iy) = (ax-cy) + i(cx+ay).$$

(iv) $\Rightarrow$ (i): By (2.7), there exists $a, c \in \mathbb{R}$ such that

$$\varphi(z) = \varphi(x+iy) = (ax-cy) + i(cx+ay) = (a+ic)(x+iy) = \alpha z$$

where $\alpha := a + ci$. It is clear that $\varphi(z) = \alpha z$ satisfies (2.2).

3. Let $\Omega \subseteq \mathbb{C}$ be a region and let $f : \Omega \to \mathbb{C}$. We will call f *complex differentiable* at $z \in \Omega$ if it is differentiable as defined in the lectures, ie. the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \tag{3.9}$$

exists. We will call f *real differentiable* at $z \in \Omega$ if there exists a $\mathbb{R}$-linear function $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - \varphi(h)}{h} = 0. \tag{3.10}$$

(a) Prove that if f is complex differentiable at $z \in \Omega$, then f is real differentiable at z.

Solution. Let the limit in (3.9) be α. Then

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \alpha.$$

Hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that when $|h| < \delta$,

$$\left| \frac{f(z+h) - f(z)}{h} - \alpha \right| < \varepsilon,$$

that is

$$\left| \frac{f(z+h) - f(z) - \alpha h}{h} \right| < \varepsilon.$$

Let $\varphi : \mathbb{C} \to \mathbb{C}$ be defined by $\varphi(z) = \alpha z$ for all $z \in \mathbb{C}$. This is $\mathbb{C}$-linear and is thus $\mathbb{R}$-linear by (a). Hence (3.10) holds with this choice of φ.

(b) Give an example to show that the converse of (a) is false.

Solution. Let $g : \Omega \to \mathbb{C}$ be $g(z) = \bar{z}$. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be defined by $\varphi(x+iy) = x - iy$. Note that φ is $\mathbb{R}$-linear by (2.5) with $a = 1, b = c = 0, d = -1$. It easy to see that g is real differentiable at any $z \in \Omega$ with respect to φ since

$$\lim_{h\to 0} \frac{g(z+h) - g(z) - \varphi(h)}{h} = \lim_{h\to 0} \frac{\overline{z+h} - \bar{z} - \bar{h}}{h} = 0.$$

Now g is not complex differentiable since if we write $h = \xi + i\eta$ and let $h \to 0$ along the lines $\eta = 0$ and $\xi = 0$, we get

$$\lim_{\xi\to 0} \frac{g(z+\xi) - g(z)}{\xi} = \lim_{\xi\to 0} \frac{\bar{z} + \xi - \bar{z}}{\xi} = 1$$

and

$$\lim_{\eta\to 0} \frac{g(z+i\eta) - g(z)}{i\eta} = \lim_{\xi\to 0} \frac{\bar{z} - i\eta - \bar{z}}{i\eta} = -1.$$

So the limit

$$\lim_{h\to 0} \frac{g(z+h) - g(z)}{h}$$

cannot exist.

(c) Let f be real differentiable at $z \in \Omega$. If the $\mathbb{R}$-linear function $\varphi : \mathbb{C} \to \mathbb{C}$ in (3.10) is also $\mathbb{C}$-linear, prove that f is complex differentiable at z. In this case, how is φ related to the limit in (3.9)?

Solution. If the φ is also $\mathbb{C}$-linear, then by by Problem **2**(c), we have

$$\varphi(h) = \varphi(1)h.$$

Then

$$\begin{aligned}\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h\to 0} \frac{f(z+h) - f(z) - \varphi(h) + \varphi(h)}{h} \\ &= \lim_{h\to 0} \frac{f(z+h) - f(z) - \varphi(h)}{h} + \lim_{h\to 0} \frac{\varphi(h)}{h} \\ &= 0 + \lim_{h\to 0} \frac{\varphi(1)h}{h} \\ &= \varphi(1).\end{aligned}$$

Hence the limit in (3.9) exists and so f is complex differentiable at z. Note that this also gives the relation between φ and the limit in (3.9).

(d) Let f be real differentiable at $z \in \Omega$. Show that if the limit

$$\lim_{h\to 0} \left| \frac{f(z+h) - f(z)}{h} \right| \tag{3.11}$$

exists[1], then either f or $\bar{f}$ must be complex differentiable at z. Give an example to show that f is not necessarily complex differentiable at z. Here the function $\bar{f} : \Omega \to \mathbb{C}$ is defined by $\bar{f}(z) = \overline{f(z)}$ for all $z \in \Omega$.

Solution. Since f is real differentiable, there exists an $\mathbb{R}$-linear φ for which (3.10) is satisfied. By triangle inequality,

$$0 \le \left| \left| \frac{f(z+h) - f(z)}{h} \right| - \left| \frac{\varphi(h)}{h} \right| \right| = \left| \frac{f(z+h) - f(z) - \varphi(h)}{h} \right|.$$

Since the limit of the RHS is 0, by Sandwich Lemma, the limit $\lim_{h\to 0} |\varphi(h)|/|h|$ exists (and equals the limit in (3.11)). Since φ is $\mathbb{R}$-linear, by (2.4), we have

$$\varphi(h) = \lambda h + \mu \bar{h}$$

where

$$\lambda := \frac{\varphi(1) - i\varphi(i)}{2} \quad \text{and} \quad \mu := \frac{\varphi(1) + i\varphi(i)}{2}. \tag{3.12}$$

Now

$$\left| \frac{\varphi(h)}{h} \right|^2 = \left| \frac{\lambda h + \mu \bar{h}}{h} \right|^2 = |\lambda|^2 + |\mu|^2 + 2 \operatorname{Re} \left(\lambda \bar{\mu} \frac{h}{\bar{h}} \right).$$

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[1] Note the difference between (3.9) and (3.11).

Since the limit of the LHS exists as $h \to 0$, the limit

$$\lim_{h\to 0} \operatorname{Re}\left(\lambda\bar{\mu}\frac{h}{\bar{h}}\right)$$

must also exist. Write $h = \xi + i\eta$. First we let $h \to 0$ along the lines $\eta = 0$ and $\xi = 0$ respectively, we get

$$\lim_{\xi\to 0} \operatorname{Re}\left[\lambda\bar{\mu}\left(\frac{\xi}{\xi}\right)\right] = \lim_{\eta\to 0} \operatorname{Re}\left[\lambda\bar{\mu}\left(\frac{i\eta}{-i\eta}\right)\right]$$

by the uniqueness of limit. This gives $\operatorname{Re}(\lambda\bar{\mu}) = -\operatorname{Re}(\lambda\bar{\mu})$ and thus

$$\operatorname{Re}(\lambda\bar{\mu}) = 0. \tag{3.13}$$

Now we let $h \to 0$ along the lines $\xi = \eta$ and $\xi = -\eta$ respectively, we get

$$\lim_{\xi\to 0} \operatorname{Re}\left[\lambda\bar{\mu}\left(\frac{\xi+\xi i}{\xi-\xi i}\right)\right] = \lim_{\eta\to 0} \operatorname{Re}\left[\lambda\bar{\mu}\left(\frac{-\eta+\eta i}{-\eta-\eta i}\right)\right]$$

by the uniqueness of limit. This gives $\operatorname{Re}(i\lambda\bar{\mu}) = -\operatorname{Re}(i\lambda\bar{\mu})$, ie. $-\operatorname{Im}(\lambda\bar{\mu}) = \operatorname{Im}(\lambda\bar{\mu})$, and thus

$$\operatorname{Im}(\lambda\bar{\mu}) = 0. \tag{3.14}$$

By (3.13) and (3.14),

$$\lambda\bar{\mu} = 0,$$

ie. we must either have $\lambda = 0$ or $\bar{\mu} = 0$. So by (3.12), either

$$\varphi(1) = i\varphi(i) \quad \text{or} \quad \bar{\varphi}(1) = i\bar{\varphi}(i).$$

In the first case, φ is $\mathbb{C}$-linear (by Problem **2**(c)) and therefore f must be complex differentiable (by Problem **3**(c)). In the second case, $\bar{\varphi}$ is $\mathbb{C}$-linear and there $\bar{f}$ must be complex differentiable. In the second case, we also need to observe that if f is real differentiable with respect to φ, then $\bar{f}$ is real differentiable with respect to $\bar{\varphi}$ since

$$\lim_{h\to 0} \frac{f(z+h) - f(z) - \varphi(h)}{h} = 0$$

iff

$$\lim_{h\to 0} \frac{|f(z+h) - f(z) - \varphi(h)|}{|h|} = 0$$

iff

$$\lim_{h\to 0} \frac{|\overline{f(z+h) - f(z) - \varphi(h)}|}{|h|} = 0$$

iff

$$\lim_{h\to 0} \frac{\bar{f}(z+h) - \bar{f}(z) - \bar{\varphi}(h)}{h} = 0.$$

(e) Show that the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \sqrt{|z^2 - \bar{z}^2|}$$

satisfies the Cauchy-Riemann equation at $z = 0$ but is not differentiable at $z = 0$.

SOLUTION. Note that f is identically zero on the real and imaginary axes and so trivially satisfies the Cauchy-Riemann equation at $z = 0$, i.e.

$$f_x(0) = \lim_{\xi\to 0, \xi\in\mathbb{R}} \frac{f(0+\xi) - f(0)}{\xi} = \lim_{\xi\to 0, \xi\in\mathbb{R}} \frac{0-0}{\xi} = 0,$$

$$f_y(0) = \lim_{\eta\to 0, \eta\in\mathbb{R}} \frac{f(0+i\eta) - f(0)}{\eta} = \lim_{\eta\to 0, \eta\in\mathbb{R}} \frac{0-0}{\eta} = 0$$

and so $f_y(0) = -f_x(0)$. To see that it is not differentiable, take $h = r(\cos\theta + i\sin\theta)$ and take limit as $r \to 0$, we get

$$\begin{aligned}\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} &= \lim_{r\to 0}\frac{f(0+re^{i\theta})-f(0)}{re^{i\theta}}\\ &= \lim_{r\to 0}\frac{\sqrt{|4\cos\theta\sin\theta|}-0}{\cos\theta+i\sin\theta}\\ &= \frac{\sqrt{|4\cos\theta\sin\theta|}}{\cos\theta+i\sin\theta}.\end{aligned}$$

Since the last expression depends on θ, taking $\theta = 0$ and $\theta = \pi/4$, we get two different values and thus the limit of the LHS does not exist.

(f) Let $\Omega \subseteq \mathbb{C}$ be a region such that the function

$$f(x+iy) = |x^2 - y^2| + 2i|xy|$$

is analytic on Ω but is not analytic on any larger region Ω' containing Ω. Find all possible Ω with this property.

Solution. The function f is analytic in each of the following regions

$$\begin{aligned}\Omega_1 &= \{z \in \mathbb{C} \mid 0 < \arg(z) < \pi/4\}, & \Omega_2 &= \{z \in \mathbb{C} \mid \pi < \arg(z) < 5\pi/4\},\\ \Omega_3 &= \{z \in \mathbb{C} \mid \pi/2 < \arg(z) < 3\pi/4\}, & \Omega_4 &= \{z \in \mathbb{C} \mid 3\pi/2 < \arg(z) < 7\pi/4\}.\end{aligned}$$

On Ω_1 or Ω_2, we have

$$f(z) = z^2.$$

On Ω_3 or Ω_4, we have

$$f(z) = -z^2.$$

(g) Find constants $a, b, c \in \mathbb{R}$ such that the functions $f, g : \mathbb{C} \to \mathbb{C}$ defined by

$$\begin{aligned}f(x+iy) &= x + ay + i(bx + cy),\\ g(x+iy) &= \cos x(\cosh y + a\sinh y) + i\sin x(\cosh y + b\sinh y)\end{aligned}$$

are analytic on $\mathbb{C}$.

Solution. Applying the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x,$$

we see that $c = 1$ and $b = -a$ in f and so

$$f(z) = (1 - ai)z.$$

Likewise $a = b = -1$ in g and so

$$g(z) = e^{iz}.$$

4. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $f : \Omega \to \mathbb{C}$ be analytic and $u(x,y) = \operatorname{Re} f(x+iy)$, $v(x,y) = \operatorname{Im} f(x+iy)$.

(a) Suppose $u, v \in C^2(\Omega_{\mathbb{R}})$. Show that u and v are *harmonic functions*, i.e. solutions of the Laplace equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0,$$

on $\Omega_{\mathbb{R}}$.

Solution. Taking partial derivatives of the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

gives

$$\begin{aligned}u_{xx} &= v_{yx}, & u_{yx} &= -v_{xx},\\ u_{xy} &= v_{yy}, & u_{yy} &= -v_{xy}.\end{aligned}$$

Since the second order partial derivatives are continuous, we have that $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$. Hence

$$\begin{aligned} u_{xx} + u_{yy} &= v_{yx} - v_{xy} = 0, \\ v_{xx} + v_{yy} &= -u_{yx} + u_{xy} = 0. \end{aligned}$$

(b) Let $a \in \mathbb{R}$. Suppose f is analytic on $D(0,1)$. Which of the following can occur as the real or imaginary part of f?

$$x^2 - axy + y^2, \quad x^3 - x^2 + y^3, \quad x^2 + y^2 - 5x, \quad \frac{x^2 - y^2}{(x^2+y^2)^2}.$$

Solution. Note that all these functions are in $C^2(\mathbb{R}^2)$ and so the result in (a) applies. For $w(x,y) = x^2 - axy + y^2$, we have

$$w_{xx} = 2 \quad \text{and} \quad w_{yy} = 2$$

and so

$$w_{xx} + w_{yy} = 2 + 2 \neq 0.$$

Thus w cannot be the real or imaginary part of an analytic function since it is not harmonic. Likewise for $x^3 - x^2 + y^3$ and $x^2 + y^2 - 5x$. The function $(x^2 - y^2)/(x^2 + y^2)^2$ is not even continuous at 0 and so not a candidate. But on the other hand, if we allow the point 0 to be excluded, then for

$$u(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2},$$

we see that

$$\frac{\partial u}{\partial x} = \frac{2x(3y^2 - x^2)}{(x^2+y^2)^3}, \quad \frac{\partial u}{\partial y} = -\frac{2y(3x^2 - y^2)}{(x^2+y^2)^3}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{6\left(x^4 - 6x^2y^2 + y^4\right)}{(x^2+y^2)^4}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{6\left(x^4 - 6x^2y^2 + y^4\right)}{(x^2+y^2)^4}$$

and thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We want $v(x,y)$ such that the Cauchy-Riemann equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{2x(3y^2 - x^2)}{(x^2+y^2)^3} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{2y(3x^2 - y^2)}{(x^2+y^2)^3}$$

are satisfied and by inspection we see that

$$v(x,y) = \frac{-2xy}{(x^2+y^2)^2}$$

is a possible candidate. Since u and v are both continuously differentiable in $\mathbb{C}^\times$, the function

$$f(x,y) = \frac{x^2 - y^2 - 2ixy}{(x^2+y^2)^2}$$

is analytic in $\mathbb{C}^\times$ by Theorem **2.4** (partial converse of Cauchy-Riemann equations) in the lectures.

5. We may rewrite any complex function f of two real variables x and y as a function of z and $\overline{z}$ via

$$x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}.$$

(a) Considering z and $\overline{z}$ as independent variables, show that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Solution. Treating z and $\overline{z}$ as independent variables, the differentiation rules give, formally

$$\frac{\partial x}{\partial z} = \frac{\partial x}{\partial \overline{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial z} = \frac{-i}{2}, \quad \frac{\partial y}{\partial \overline{z}} = \frac{i}{2}$$

and the chain rule then implies that

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

(b) Show that the Cauchy-Riemann equation may be expressed as

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

This may be interpreted as saying that complex differentiable functions must be independent[2] of $\overline{z}$ and depend only on z.

Solution. By (a),

$$\frac{\partial f}{\partial \overline{z}} = 0$$

iff

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

iff

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}$$

which is one form of the Cauchy-Riemann equations that we have shown in lectures.

(c) Which of the following complex functions of two real variables can be expressed in terms of a polynomial in $z = x + iy$?

$$f_1(x,y) = x^2 - y^2 - ixy, \quad f_2(x,y) = x^2 + y^2 - 2ixy.$$

Solution. By (b), a complex function can be expressed in terms of a polynomial in z (independent of $\overline{z}$) iff it satisfies the Cauchy-Riemann equations. Write $f_1 = u + iv$, since

$$u_x = 2x \neq -x = v_y,$$

f_1 cannot be expressed in terms of z only. Now write $f_2 = u + iv$, since

$$u_x = 2x \neq -2x = v_y,$$

f_2 cannot be expressed in terms of z only either.

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[2] In fact you may also view this as a reason why there isn't a 'quarternion analysis' similar to complex analysis. For a quarternion $q = x + yi + zj + wk$, its quarternionic conjugate $\overline{q} = x - yi - zj - wk$ can always be expressed in terms of q:

$$\overline{q} = -\frac{1}{2}(q + iqi + jqj + kqk),$$

and so we don't have functions dependent on q but not on $\overline{q}$.