MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 2

Throughout the problem set, $i = \sqrt{-1}$; and whenever we write $x + yi$, it is implicit that $x, y \in \mathbb{R}$. For $z \in \mathbb{C}$, recall that the *argument* of z, denoted $\arg(z)$, is any $\theta \in \mathbb{R}$ such that $z = |z|e^{i\theta}$. We write $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$.

1. Let $(z_n)_{n=1}^\infty$ be a sequence of complex numbers.

(a) Show that if $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty}|z_n| = |z|$ but that the converse is not true in general.

SOLUTION. We will first prove the inequality

$$||u| - |v|| \le |u - v|$$

for $u, v \in \mathbb{C}$. Since $u = (u - v) + v$, we have $|u| \le |u - v| + |v|$ and so

$$|u| - |v| \le |u - v|.$$

Since $v = (v - u) + u$, we have $|v| \le |v - u| + |u|$ and so

$$|v| - |u| \le |u - v|.$$

Hence

$$-|u - v| \le |u| - |v| \le |u - v|$$

as required.

Since $\lim_{n\to\infty} z_n = z$, we have that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|z_n - z| < \varepsilon$ whenver $n > N$. Now using the inequality that we proved, we see that

$$||z_n| - |z|| \le |z_n - z| < \varepsilon$$

whenever $n > N$. Hence $\lim_{n\to\infty}|z_n| = |z|$ as required.

The converse is not true. Let $z_n = (-1)^n$. Then $\lim_{n\to\infty}|z_n| = 1$ but $\lim_{n\to\infty} z_n$ does not exist.

(b) Is it true that if $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} \arg(z_n) = \arg(z)$?

SOLUTION. No. Let $z_n = -1 + (-1)^n/n$. Then $\lim_{n\to\infty} = -1$ but since $\arg(z_{2n}) = \pi - \tan^{-1} 1/2n$ and $\arg(z_{2n+1}) = -\pi + \tan^{-1} 1/(2n+1)$, $\lim_{n\to\infty} \arg(z_n)$ does not exist. [*Note*: I take $\arg(z)$ to be the angle that z makes with the positive real axis; if you use some other conventions, you could construct a similar counter example].

(c) Show that if $\lim_{n\to\infty}|z_n| = r$ and $\lim_{n\to\infty} \arg(z_n) = \theta$, then $\lim_{n\to\infty} z_n = re^{i\theta}$.

SOLUTION. Let $z_n = x_n + iy_n$. Then

$$x_n = \operatorname{Re} z_n = |z_n| \cos\arg(z_n), \quad y_n = \operatorname{Im} z_n = |z_n| \sin\arg(z_n).$$

Since cos and sin are continuous functions,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} |z_n| \times \lim_{n\to\infty} \cos\arg(z_n) = r\cos\theta,$$
$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} |z_n| \times \lim_{n\to\infty} \sin\arg(z_n) = r\sin\theta.$$

Hence

$$\lim_{n\to\infty} z_n = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

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2. Which of the following limit exists? Prove your answer.

$$\lim_{n\to\infty}\left(\frac{1+i}{1-i}\right)^n, \qquad \sum_{n=1}^{\infty} i^n \log\left(\frac{n}{n+1}\right), \qquad \lim_{z\to 1}\frac{1-\overline{z}}{1-z}.$$

Solution. Note that

$$\left(\frac{1+i}{1-i}\right)^n = \begin{cases} 1 & \text{if } n \equiv 0 \bmod 4, \\ i & \text{if } n \equiv 1 \bmod 4, \\ -1 & \text{if } n \equiv 2 \bmod 4, \\ -i & \text{if } n \equiv 3 \bmod 4. \end{cases}$$

Since limit of a sequence, if exists, must be unique (Why?), $\lim_{n\to\infty}[(1+i)/(1-i)]^n$ doesn't exist.

Note that the usual way of summing a geometric progression yields

$$\sum_{n=1}^{m} i^n = \frac{1-i^{m+1}}{1-i} = \begin{cases} 1 & \text{if } m \equiv 0 \bmod 4, \\ 1+i & \text{if } m \equiv 1 \bmod 4, \\ i & \text{if } m \equiv 2 \bmod 4, \\ 0 & \text{if } m \equiv 3 \bmod 4, \end{cases}$$

and so

$$\left|\sum_{n=1}^{m} i^n\right| \le \sqrt{2} \quad \text{for all } m \in \mathbb{N}.$$

Since we also have

$$\left|\log\left(\frac{n}{n+1}\right)\right| = \left|-\log\left(\frac{n}{n+1}\right)\right| = \left|\log\left(1+\frac{1}{n}\right)\right|$$

converges monotonically to 0 as $n \to \infty$, the series converges.

Note that for $z = 1-x$ where $x \in \mathbb{R}$,

$$\lim_{z\to 1}\frac{1-\overline{z}}{1-z} = \lim_{x\to 0}\frac{1-1+x}{1-1+x} = 1,$$

but for $z = 1+iy$ where $y \in \mathbb{R}$,

$$\lim_{z\to 1}\frac{1-\overline{z}}{1-z} = \lim_{y\to 0}\frac{1-1-iy}{1-1+iy} = -1.$$

Since the limit of a function, if exists, must be unique (Why?), $\lim_{z\to 1}(1-\overline{z})/(1-z)$ doesn't exist.

3. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $f : \Omega \to \mathbb{C}$ and $z_0 \in \Omega$.

(a) Suppose $\lim_{z\to z_0} f(z) = w$. Prove that

$$\lim_{z\to z_0} \overline{f(z)} = \overline{w}, \quad \lim_{z\to z_0} \operatorname{Re} f(z) = \operatorname{Re} w, \quad \lim_{z\to z_0} \operatorname{Im} f(z) = \operatorname{Im} w, \quad \lim_{z\to z_0} |f(z)| = |w|.$$

Solution. All we need to show is that the following functions $\psi, \rho, \iota, \mu : \mathbb{C} \to \mathbb{C}$ are all continuous on $\mathbb{C}$,

$$\psi(z) = \overline{z}, \quad \rho(z) = \operatorname{Re} z, \quad \iota(z) = \operatorname{Im} z, \quad \mu(z) = |z|,$$

and the required limits then follows from composing these functions with f, i.e. $\psi \circ f, \rho \circ f, \iota \circ f, \mu \circ f$. But the required continuity (in fact uniform continuity) on $\mathbb{C}$ follows easily from the equalities/inequalities

$$|\overline{z} - \overline{z}_0| = |\overline{z - z_0}| = |z - z_0|,$$
$$|\operatorname{Re} z - \operatorname{Re} z_0| = |\operatorname{Re}(z - z_0)| \le |z - z_0|,$$
$$|\operatorname{Im} z - \operatorname{Im} z_0| = |\operatorname{Im}(z - z_0)| \le |z - z_0|,$$
$$||z| - |z_0|| \le |z - z_0|.$$

(b) Suppose $\lim_{z\to z_0}|f(z)| = |w|$. For which value of w is it always true that $\lim_{z\to z_0} f(z) = w$? You will need to prove that it is true for that value and false for all other values.

SOLUTION. It is always true if $w = 0$: For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - 0| = |f(z)| = ||f(z)| - |0|| < \varepsilon$ whenever $|z - z_0| < \delta$. We will show that this is the only case in general. Suppose $\lim_{z\to z_0}|f(z)| = |w| > 0$. If $\lim_{z\to z_0} f(z) = v$ for some $v \in \mathbb{C}^\times$, then by part (a), we must have $|v| = |w|$. We shall examine when equality is attained in

$$||f(z)| - |w|| = ||f(z)| - |v|| \le |f(z) - v|$$

(since then $\varepsilon > ||f(z)| - |w|| = |f(z) - v|$ for $|z - z_0| < \delta(\varepsilon)$). The equality would imply

$$(|f(z)| - |v|)^2 = (|f(z)| - v)\overline{(|f(z)| - v)} = |f(z)|^2 + |v|^2 - 2\operatorname{Re} f(z)\overline{v},$$

and thus

$$\operatorname{Re} f(z)\overline{v} = |f(z)v| = |f(z)\overline{v}|,$$

which implies that

$$\operatorname{Re} f(z)\overline{v} \ge 0 \quad \text{and} \quad \operatorname{Im} f(z)\overline{v} = 0.$$

In other words $f(z)\overline{v}$ have to be real and nonngative for all $z \in \mathbb{C}$, which is clearly impossible[1] for a general complex function f.

4. The functions $f, g, h : \mathbb{C} \to \mathbb{C}$ are defined as follows

$$f(z) = \begin{cases} \dfrac{\operatorname{Re}(z)}{z} & \text{if } z \ne 0, \\ \alpha & \text{if } z = 0, \end{cases} \qquad g(z) = \begin{cases} \dfrac{z}{|z|} & \text{if } z \ne 0, \\ \beta & \text{if } z = 0, \end{cases} \qquad h(z) = \begin{cases} \dfrac{z\operatorname{Re}(z)}{|z|} & \text{if } z \ne 0, \\ \gamma & \text{if } z = 0, \end{cases}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ are constants. Show that f, g, h are continuous on $\mathbb{C}^\times$. Are there values of α, β, γ for which f, g, h are continuous on $\mathbb{C}$?

SOLUTION. As noted in the solution of Problem **3**(a), $z \mapsto |z|$ and $z \mapsto \operatorname{Re} z$ are both continuous functions on $\mathbb{C}$; clearly so is the identity function $z \mapsto z$. Hence the product of two continuous functions $z \mapsto z\operatorname{Re} z$ is continuous on $\mathbb{C}$ and the quotient of continuous functions f, g, h are continuous when the denominator is non-zero, i.e. on $\mathbb{C}^\times$. For f, g, h to be continuous at $z = 0$, we must have

$$\lim_{z\to 0} \frac{\operatorname{Re} z}{z} = \alpha, \quad \lim_{z\to 0} \frac{z}{|z|} = \beta, \quad \lim_{z\to 0} \frac{z\operatorname{Re} z}{|z|} = \gamma.$$

Note that the first two limits do not exist:

$$\lim_{\substack{z\to 0\\ z\in i\mathbb{R}}} \frac{\operatorname{Re} z}{z} = 0 \ne 1 = \lim_{\substack{z\to 0\\ z\in\mathbb{R}}} \frac{\operatorname{Re} z}{z}, \qquad \lim_{\substack{z\to 0\\ z\in\mathbb{R}, z>0}} \frac{z}{|z|} = 1 \ne -1 \lim_{\substack{z\to 0\\ z\in\mathbb{R}, z<0}} \frac{z}{|z|}.$$

The last limit is 0 since

$$\left|\frac{z\operatorname{Re} z}{|z|} - 0\right| = |\operatorname{Re} z| \le |z| = |z - 0|$$

and we may pick $\delta = \varepsilon$.

[1] This is possible if $f(z) = v$ for all $z \in \mathbb{C}$ or when f is real-valued with constant sign but the problem did not impose such assumptions on f.

5. Let $f : \mathbb{C}^\times \to \mathbb{C}$ be the reciprocal function

$$f(z) = \frac{1}{z}.$$

Define the sequence of function $(f_n)_{n=1}^{\infty}$, $f_n : \mathbb{C}^\times \to \mathbb{C}$, by

$$f_n(z) = \frac{1}{nz}.$$

Let $g : \mathbb{C}^\times \to \mathbb{C}$ be the zero function, ie. $g(z) = 0$ for all $z \in \mathbb{C}^\times$. Let $\Omega = \{z \in \mathbb{C} \mid r \le |z| \le R\}$ where $0 < r < R < \infty$.

(a) Show that f is continuous but not uniformly continuous on $\mathbb{C}^\times$.

Solution. Let $\alpha \in \mathbb{C}^\times$ be fixed. We need to show that

$$\lim_{z \to \alpha} f(z) = f(\alpha).$$

Let $\varepsilon > 0$ be given. We want $\delta > 0$ so that when $|z - \alpha| < \delta$,

$$\left| \frac{1}{z} - \frac{1}{\alpha} \right| = \frac{1}{|z\alpha|} |z - \alpha| < \varepsilon.$$

Note that the denominator becomes large when z is near 0, so we will first want to pick δ to prevent that. The standard trick to do this is to pick

$$\delta \le \frac{|\alpha|}{2} \tag{5.1}$$

since when

$$|z - \alpha| < \frac{|\alpha|}{2},$$

then by the triangle inequality,

$$|z| > \frac{|\alpha|}{2}$$

and so

$$\left| \frac{1}{z} - \frac{1}{\alpha} \right| = \frac{1}{|z\alpha|} |z - \alpha| < \frac{2}{|\alpha|^2} |z - \alpha|.$$

However, we also want the last term above to be not more than ε and so we need to pick

$$\delta \le \frac{\varepsilon |\alpha|^2}{2}. \tag{5.2}$$

Now for (5.1) and (5.2) to be both satisfied, we let

$$\delta = \min\left\{ \frac{|\alpha|}{2}, \frac{\varepsilon |\alpha|^2}{2} \right\}.$$

Hence when $|z - \alpha| < \delta$,

$$\left| \frac{1}{z} - \frac{1}{\alpha} \right| = \frac{1}{|z\alpha|} |z - \alpha| < \frac{2}{|\alpha|^2} |z - \alpha| \le \varepsilon.$$

To see that f is not uniformly continuous on $\mathbb{C}^\times$, let $\varepsilon = 1/2$. For any $\delta > 0$, pick $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \delta$$

and let

$$z = \frac{1}{n}, \quad w = \frac{1}{n+1}.$$

Then

$$|z - w| = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n} < \delta$$

but

$$|f(z) - f(w)| = \left|\frac{1}{z} - \frac{1}{w}\right| = 1 > \varepsilon.$$

Hence for $\varepsilon = 1/2$, there is no $\delta > 0$ such that will satisfy the requirement for uniform continuity.

(b) Show that f is uniformly continuous on Ω.

SOLUTION. Let $\varepsilon > 0$ be given. We want $\delta > 0$ so that when $z, w \in \Omega$ satisfies $|z - w| < \delta$, we will have

$$\left|\frac{1}{z} - \frac{1}{w}\right| = \frac{1}{|zw|}|z - w| < \varepsilon.$$

Note that $z, w \in \Omega$ implies that $|z| \geq r$ and $|w| \geq r$. So

$$\frac{1}{|zw|} \leq \frac{1}{r^2}.$$

Hence we just need to choose a δ so that whenever $|z - w| < \delta$, we will have

$$\frac{1}{r^2}|z - w| < \varepsilon.$$

It is clear that

$$\delta = r^2\varepsilon \tag{5.3}$$

(c) Show that f_n converges pointwise but not uniformly to g on $\mathbb{C}^{\times}$.

SOLUTION. Let $\varepsilon > 0$ be given. Let $\alpha \in \mathbb{C}^{\times}$ be fixed. We want $N \in \mathbb{N}$ so that when $n > N$,

$$|f_n(\alpha) - g(\alpha)| = \left|\frac{1}{n\alpha} - 0\right| = \left|\frac{1}{n\alpha}\right| = \frac{1}{n|\alpha|} < \varepsilon.$$

It is clear that

$$N = \left\lceil \frac{1}{\varepsilon|\alpha|} \right\rceil \tag{5.4}$$

will achieve this. As a sanity check, observe that when $n > N$,

$$|f_n(\alpha) - g(\alpha)| = \frac{1}{n|\alpha|} < \frac{1}{N|\alpha|} \leq \varepsilon.$$

To see that f_n does not converge uniformly to g on $\mathbb{C}^{\times}$, let $\varepsilon = 1/2$. Note that for $z = 1/n \in \mathbb{C}^{\times}$,

$$|f_n(z) - g(z)| = \left|\frac{1}{nz} - 0\right| = 1 > \varepsilon.$$

Hence for $\varepsilon = 1/2$, there is no $N \in \mathbb{N}$ that could give us $|f_n(z) - g(z)| < 1/2$ for all $z \in \mathbb{C}^{\times}$ and all $n > N$.

(d) Show that f_n converges uniformly to g on Ω.

SOLUTION. Let $\varepsilon > 0$ be given. We want $N \in \mathbb{N}$ so that when $n > N$,

$$|f_n(z) - g(z)| = \left|\frac{1}{nz} - 0\right| = \left|\frac{1}{nz}\right| = \frac{1}{n|z|} < \varepsilon$$

for all $z \in \Omega$. Note that $z \in \Omega$ implies that $|z| \geq r$. So

$$\frac{1}{|z|} \leq \frac{1}{r}.$$

Hence we just need to choose an N so that whenever $n > N$, we will have

$$\frac{1}{nr} < \varepsilon.$$

It is clear that

$$N = \left\lceil \frac{1}{\varepsilon r} \right\rceil \tag{5.5}$$

will achieve this. As a sanity check, observe that when $n > N$,

$$|f_n(z) - g(z)| = \frac{1}{n|z|} < \frac{1}{Nr} \le \varepsilon.$$

Remark. Notice that in (5.2) and (5.4), the choice of δ and N are dependent on the point $\alpha \in \mathbb{C}^\times$ that we fixed at the beginning; but in (5.3) and (5.5), the choice of δ and N are independent of any particular point of Ω.

6. Let R_a and R_b be the radii of convergence of

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n z^n$$

respectively.

(a) Show that the radii of convergence of

$$\sum_{n=0}^{\infty} (a_n + b_n) z^n \quad \text{and} \quad \sum_{n=0}^{\infty} a_n b_n z^n$$

are at least $\min(R_a, R_b)$ and $R_a R_b$ respectively.

Solution. The triangle inequality implies that for any $m \in \mathbb{N}$,

$$\sum_{n=0}^{m} |(a_n + b_n) z^n| \le \sum_{n=0}^{m} |a_n z^n| + \sum_{n=0}^{m} |b_n z^n|.$$

We have shown in the lectures that a power series is absolutely convergent within its radius of convergence and so if $|z| \le \min(R_a, R_b)$, then

$$\sum_{n=0}^{m} |a_n z^n| \le \sum_{n=0}^{\infty} |a_n z^n| =: M_a < \infty$$

and

$$\sum_{n=0}^{m} |b_n z^n| \le \sum_{n=0}^{\infty} |b_n z^n| =: M_b < \infty.$$

Hence

$$\sum_{n=0}^{m} |(a_n + b_n) z^n| \le M_a + M_b \tag{6.6}$$

for all $m \in \mathbb{N}$. Since the LHS of $\sum_{n=0}^{\infty} a_n b_n z^n$ is a monotone increasing sequence bounded by the RHS, it must converge. In other words, $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ is absolutely convergent and thus convergent. Let R_c be its radius of convergence. We have shown in the lecture that the set of points for which $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ converge must be a subset of $\overline{D(0, R_c)}$, so

$$D(0, \min(R_a, R_b) \subseteq \overline{D(0, R_c)}$$

and so we must have

$$R_c \ge \min(R_a, R_b).$$

Let R_d be the radius of convergence of $\sum_{n=0}^{\infty} a_n b_n z^n$. Recall from Math **104** the fact that the limit superior of a product is bounded by the product of the limit superiors. So

$$\begin{aligned}\limsup_{n\to\infty} |a_n b_n|^{1/n} &= \limsup_{n\to\infty} |a_n|^{1/n} |b_n|^{1/n} \\ &\le \limsup_{n\to\infty} |a_n|^{1/n} \limsup_{n\to\infty} |b_n|^{1/n}.\end{aligned}$$

Hence $R_d \ge R_a R_b$.

(b) Suppose $0 < R_a < \infty$ and $p > 0$. Find the radii of convergence of the following power series in terms of R_a and p:

$$\sum_{n=0}^{\infty} a_n^p z^n, \quad \sum_{n=0}^{\infty} n^p a_n z^n, \quad \sum_{n=0}^{\infty} n^n a_n z^n, \quad \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n.$$

SOLUTION. As we have discussed in the lecture, if $(x_n)_{n=1}^{\infty}$ is such that $x_n \in \mathbb{R}$ and $x_n > 0$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} x_{n+1}/x_n$ if the RHS exists.

$$\begin{aligned}
\limsup_{n\to\infty} |a_n^p|^{1/n} &= \limsup_{n\to\infty} |a_n|^{p/n} = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^p = \frac{1}{R_a^p},\\
\limsup_{n\to\infty} |n^p a_n|^{1/n} &= \lim_{n\to\infty} n^{p/n} \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{(n+1)^p}{n^p} \times \frac{1}{R_a} = \frac{1}{R_a},\\
\limsup_{n\to\infty} |n^n a_n|^{1/n} &= \limsup_{n\to\infty} n|a_n|^{1/n} \geq \left(\liminf_{n\to\infty} n\right) \times \left(\limsup_{n\to\infty} |a_n|^{1/n}\right) = \infty,\\
\limsup_{n\to\infty} |a_n/n!|^{1/n} &= \lim_{n\to\infty} (1/n!)^{1/n} \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \times \frac{1}{R_a} = 0.
\end{aligned}$$

So the radius of convergence of g is R and the radius of convergence of h is ∞.

7. Use the power series representation of $\exp(z)$ for this problem.

(a) Prove that

$$\left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| \leq \left| e^{|z|} - \sum_{k=0}^{n} \frac{|z|^k}{k!} \right| \leq |z|^{n+1} e^{|z|}$$

for all $n \in \mathbb{N}$. Hence deduce that

$$|e^z - 1| \leq |e^{|z|} - 1| \leq |z| e^{|z|}.$$

SOLUTION. Noting that the power series representation of e^z converges absolutely and uniformly to e^z on $\overline{D(0,R)}$ for any $R > 0$ (Theorem **2.4** in lecture), we may write

$$\left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| = \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k!} \right| \leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} = \left| e^{|z|} - \sum_{k=0}^{n} \frac{|z|^k}{k!} \right|$$

and also

$$\left| e^{|z|} - \sum_{k=0}^{n} \frac{|z|^k}{k!} \right| = \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} = |z|^{n+1} \sum_{m=0}^{\infty} \frac{|z|^m}{(n+m+1)!} \leq |z|^{n+1} \sum_{m=0}^{\infty} \frac{|z|^m}{m!} = |z|^{n+1} e^{|z|}.$$

The special case is obtained with $n = 0$.

(b) Suppose

$$0 < \limsup_{n\to\infty} |a_n|^{1/n} < \alpha < \infty,$$

show that there exists $\beta > 0$ such that

$$\left| \sum_{k=0}^{\infty} \frac{a_n}{n!} z^n \right| \leq \beta e^{\alpha |z|}$$

for all $z \in \mathbb{C}$.

SOLUTION. Since

$$\limsup_{n\to\infty} |a_n|^{1/n} < \alpha,$$

there exists an $N \in \mathbb{N}$ such that

$$|a_n| \leq \alpha^n$$

whenever $n > N$. Let

$$\beta := \max\{|a_n|\alpha^{-n} \mid n = 0, \dots, N\} + 1.$$

So

$$|a_n| \le \beta\alpha^n$$

for all $n \le N$ (and clearly also for all $n > N$). Hence

$$\left|\sum_{k=0}^{\infty} \frac{a_n}{n!} z^n\right| \le \sum_{n=0}^{\infty} \frac{|a_n|}{n!} |z|^n \le \beta \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n |z|^n = \beta e^{\alpha|z|}.$$