MATH 185: COMPLEX ANALYSIS
FALL 2009/10
PROBLEM SET 2

Throughout the problem set, $i = \sqrt{-1}$; and whenever we write $x + yi$, it is implicit that $x, y \in \mathbb{R}$. For $z \in \mathbb{C}$, recall that the *argument* of z, denoted $\arg(z)$, is any $\theta \in \mathbb{R}$ such that $z = |z|e^{i\theta}$. We write $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$.

1. Let $(z_n)_{n=1}^\infty$ be a sequence of complex numbers.

(a) Show that if $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty}|z_n| = |z|$ but that the converse is not true in general.

(b) Is it true that if $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} \arg(z_n) = \arg(z)$?

(c) Show that if $\lim_{n\to\infty}|z_n| = r$ and $\lim_{n\to\infty} \arg(z_n) = \theta$, then $\lim_{n\to\infty} z_n = re^{i\theta}$.

2. Which of the following limits exists? Prove your answers.

$$\lim_{n\to\infty} \left(\frac{1+i}{1-i}\right)^n, \quad \sum_{n=1}^{\infty} i^n \log\left(\frac{n}{n+1}\right), \quad \lim_{z\to 1} \frac{1-\overline{z}}{1-z}.$$

3. Let $\Omega \subseteq \mathbb{C}$ be a region. Let $f : \Omega \to \mathbb{C}$ and $z_0 \in \Omega$.

(a) Suppose $\lim_{z\to z_0} f(z) = w$. Prove that

$$\lim_{z\to z_0} \overline{f(z)} = \overline{w}, \quad \lim_{z\to z_0} \operatorname{Re} f(z) = \operatorname{Re} w, \quad \lim_{z\to z_0} \operatorname{Im} f(z) = \operatorname{Im} w, \quad \lim_{z\to z_0} |f(z)| = |w|.$$

(b) Suppose $\lim_{z\to z_0}|f(z)| = |w|$. For which value of w is it always true that $\lim_{z\to z_0} f(z) = w$? You will need to prove that it is true for that value and false for all other values.

4. The functions $f, g, h : \mathbb{C} \to \mathbb{C}$ are defined as follows

$$f(z) = \begin{cases} \dfrac{\operatorname{Re}(z)}{z} & \text{if } z \neq 0, \\ \alpha & \text{if } z = 0, \end{cases} \qquad g(z) = \begin{cases} \dfrac{z}{|z|} & \text{if } z \neq 0, \\ \beta & \text{if } z = 0, \end{cases} \qquad h(z) = \begin{cases} \dfrac{z\operatorname{Re}(z)}{|z|} & \text{if } z \neq 0, \\ \gamma & \text{if } z = 0, \end{cases}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$ are constants. Show that f, g, h are continuous on $\mathbb{C}^\times$. Are there values of α, β, γ for which f, g, h are continuous on $\mathbb{C}$?

5. Let $f : \mathbb{C}^\times \to \mathbb{C}$ be the reciprocal function

$$f(z) = \frac{1}{z}.$$

Define the sequence of function $(f_n)_{n=1}^\infty$, $f_n : \mathbb{C}^\times \to \mathbb{C}$, by

$$f_n(z) = \frac{1}{nz}.$$

Let $g : \mathbb{C}^\times \to \mathbb{C}$ be the zero function, ie. $g(z) = 0$ for all $z \in \mathbb{C}^\times$. Let $\Omega = \{z \in \mathbb{C} \mid r \leq |z| \leq R\}$ where $0 < r < R < \infty$.

(a) Show that f is continuous but not uniformly continuous on $\mathbb{C}^\times$.

(b) Show that f is uniformly continuous on Ω.

(c) Show that f_n converges pointwise but not uniformly to g on $\mathbb{C}^\times$.

(d) Show that f_n converges uniformly to g on Ω.

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6. Let R_a and R_b be the radii of convergence of

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n z^n$$

respectively.

(a) Show that the radii of convergence of

$$\sum_{n=0}^{\infty} (a_n + b_n) z^n \quad \text{and} \quad \sum_{n=0}^{\infty} a_n b_n z^n$$

are at least $\min(R_a, R_b)$ and $R_a R_b$ respectively.

(b) Suppose $0 < R_a < \infty$ and $p > 0$. Find the radii of convergence of the following power series in terms of R_a and p:

$$\sum_{n=0}^{\infty} a_n^p z^n, \quad \sum_{n=0}^{\infty} n^p a_n z^n, \quad \sum_{n=0}^{\infty} n^n a_n z^n, \quad \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n.$$

7. Use the power series representation of $\exp(z)$ for this problem.

(a) Prove that

$$\left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| \leq \left| e^{|z|} - \sum_{k=0}^{n} \frac{|z|^k}{k!} \right| \leq |z|^{n+1} e^{|z|}$$

for all $n \in \mathbb{N}$. Hence deduce that

$$|e^z - 1| \leq |e^{|z|} - 1| \leq |z| e^{|z|}.$$

(b) Suppose

$$0 < \limsup_{n \to \infty} |a_n|^{1/n} < \alpha < \infty,$$

show that there exists $\beta > 0$ such that

$$\left| \sum_{k=0}^{\infty} \frac{a_n}{n!} z^n \right| \leq \beta e^{\alpha |z|}$$

for all $z \in \mathbb{C}$.